Hopf algebras and representations — Spring 2011

Problem sheet 8 — Model solutions

Ex 1

In this exercise we need the following auxiliar results which should be clear: for any linear map $T: V\otimes V \to V \otimes V$ we have for any $i \neq j \neq k \neq i$

$$\begin{cases} S_{ik}\circ T_{ij} \circ S_{ik} = T_{kj} \\ S_{jk} \circ T_{ij} \circ S_{jk} = T_{ik} \end{cases} \iff \begin{cases} T_{ij}\circ S_{ik} = S_{ik}\circ T_{kj} \\ T_{ij}\circ S_{jk} = S_{jk}\circ T_{ik} \end{cases}$$

Now

$$\check{R}_{12}\circ \check{R}_{23}\circ \check{R}_{12} = S_{12}\circ \underbrace{R_{12}\circ S_{23}}_{= S_{23}\circ R_{13}} \circ \underbrace{R_{23}\circ S_{12}}_{= S_{12}\circ R_{13}} \circ R_{12}$$

where $R_{13}\circ S_{12} = S_{12}\circ R_{23}$, so

$$= S_{12}\circ S_{23}\circ S_{12}\circ R_{23}\circ R_{13}\circ R_{12}$$

Similarly

$$\check{R}_{23}\circ \check{R}_{12}\circ \check{R}_{23} = S_{23}\circ \underbrace{R_{23}\circ S_{12}}_{= S_{12}\circ R_{13}} \circ \underbrace{R_{12}\circ S_{23}}_{= S_{23}\circ R_{13}} \circ R_{23}$$

where $R_{13}\circ S_{23} = S_{23}\circ R_{12}$, so

$$= S_{23}\circ S_{12}\circ S_{23}\circ R_{12}\circ R_{13}\circ R_{23}$$

The claim follows once we note that

$$S_{12}\circ S_{23}\circ S_{12} = S_{23}\circ S_{12}\circ S_{23} .$$

Ex 2

(a) Let $(a,b,c)\in\{0,1,2\}^3$ be such that the set $\{a,b,c\}$ is one of the sets $\{0\}$, $\{0,1\}$ or $\{0,1,2\}$. Let's use a shorthand notation that abc corresponds to $u_a\otimes u_b\otimes u_c$ where $\{u_a,u_b,u_c\}\subset\{v_1,v_2,\dots,v_d\}$ is such that if $u_s=v_i$ and $u_t=v_j$ then $s<t \iff i<j$. Hence we have to check the following cases and their permutations

# permut.	1	3	3	6
a representative of the case	000	001	011	012

Denote by LHS and RHS the left- and right-hand side of YBE.

000: $LHS = q^3\, u_0\otimes u_0\otimes u_0 = RHS$

001: $LHS = \check{R}_{12}\circ\check{R}_{23}\circ\check{R}_{12}(u_0\otimes u_0\otimes u_1) = \check{R}_{12}\circ\check{R}_{23}(q\, u_0\otimes u_0\otimes u_1)$

$= \check{R}_{12}(q\, u_0\otimes u_1\otimes u_0) = q\, u_1\otimes u_0\otimes u_0$

$RHS = \check{R}_{23}\circ\check{R}_{12}\circ\check{R}_{23}(u_0\otimes u_0\otimes u_1) = \check{R}_{23}\circ\check{R}_{12}(u_0\otimes u_1\otimes u_0)$

$= \check{R}_{23}(u_1\otimes u_0\otimes u_0) = q\, u_1\otimes u_0\otimes u_0$

010: $LHS = \check{R}_{12}\circ\check{R}_{23}(u_1\otimes u_0\otimes u_0) = q(u_0\otimes u_1\otimes u_0 + (q-q^{-1})u_1\otimes u_0\otimes u_0)$

$RHS = \check{R}_{23}\circ\check{R}_{12}(u_0\otimes u_0\otimes u_1 + (q-q^{-1})u_0\otimes u_1\otimes u_0)$

$= q(u_0\otimes u_1\otimes u_0) + (q-q^{-1})u_1\otimes u_0\otimes u_0)$

100: $LHS = \check{R}_{12}\circ\check{R}_{23}(u_0\otimes u_1\otimes u_0 + (q-q^{-1})u_1\otimes u_0\otimes u_0)$

$= \check{R}_{12}(u_0\otimes u_0\otimes u_1 + (q-q^{-1})u_0\otimes u_1\otimes u_0 + (q-q^{-1})q\, u_1\otimes u_0\otimes u_0)$

$= q\, u_0\otimes u_0\otimes u_1 + q^2(q-q^{-1})\, u_1\otimes u_0\otimes u_0 + q(q-q^{-1})\, u_0\otimes u_1\otimes u_0$

$RHS = \check{R}_{23}\circ\check{R}_{12}(q\, u_1\otimes u_0\otimes u_0) = \check{R}_{23}(q\, u_0\otimes u_1\otimes u_0 + q(q-q^{-1})u_1\otimes u_0\otimes u_0)$

$= q\, u_0\otimes u_0\otimes u_1 + q(q-q^{-1})\, u_0\otimes u_1\otimes u_0 + q^2(q-q^{-1})\, u_1\otimes u_0\otimes u_0$

011: $LHS = \check{R}_{12}\circ\check{R}_{23}(u_1\otimes u_0\otimes u_1) = q\, u_1\otimes u_1\otimes u_0$

$RHS = \check{R}_{23}\circ\check{R}_{12}(q\, u_0\otimes u_1\otimes u_1) = q\, u_1\otimes u_1\otimes u_0$

101: $LHS = \check{R}_{12}\circ\check{R}_{23}(u_0\otimes u_1\otimes u_1 + (q-q^{-1})u_1\otimes u_0\otimes u_1)$

$= q\, u_1\otimes u_0\otimes u_1 + q(q-q^{-1})\, u_1\otimes u_1\otimes u_0$

$RHS = \check{R}_{23}\circ\check{R}_{12}(u_1\otimes u_1\otimes u_0)$

$= q\, u_1\otimes u_0\otimes u_1 + q(q-q^{-1})\, u_1\otimes u_1\otimes u_0$

$110:$ $LHS = \check{R}_{12} \circ \check{R}_{23} (q\, u_1 \otimes u_1 \otimes u_0) = \check{R}_{12} (q u_1 \otimes u_0 \otimes u_1 + q(q-q^{-1}) u_1 \otimes u_1 \otimes u_0)$

$= q\, u_0 \otimes u_1 \otimes u_1 + q(q-q^{-1}) u_1 \otimes u_0 \otimes u_1 + q^2 (q-q^{-1}) u_1 \otimes u_1 \otimes u_0$

$RHS = \check{R}_{23} \circ \check{R}_{12} (u_1 \otimes u_0 \otimes u_1 + (q-q^{-1}) u_1 \otimes u_1 \otimes u_0)$

$= \check{R}_{23} (u_0 \otimes u_1 \otimes u_1 + (q-q^{-1}) u_1 \otimes u_0 \otimes u_1 + q(q-q^{-1}) u_1 \otimes u_1 \otimes u_0$

$= q\, u_0 \otimes u_1 \otimes u_1 + q^2 (q-q^{-1}) u_1 \otimes u_1 \otimes u_0 + q(q-q^{-1}) u_1 \otimes u_0 \otimes u_1$

$012:$ $LHS = \check{R}_{12} \circ \check{R}_{23} (u_1 \otimes u_0 \otimes u_2) = u_2 \otimes u_1 \otimes u_0$

$RHS = \check{R}_{23} \circ \check{R}_{12} (u_0 \otimes u_2 \otimes u_1) = u_2 \otimes u_1 \otimes u_0$

$021:$ $LHS = \check{R}_{12} \circ \check{R}_{23} (u_2 \otimes u_0 \otimes u_1) = \check{R}_{12} (u_2 \otimes u_1 \otimes u_0)$

$= u_1 \otimes u_2 \otimes u_0 + (q-q^{-1}) u_2 \otimes u_1 \otimes u_0$

$RHS = \check{R}_{23} \circ \check{R}_{12} (u_0 \otimes u_1 \otimes u_2 + (q-q^{-1}) u_0 \otimes u_2 \otimes u_1)$

$= u_1 \otimes u_2 \otimes u_0 + (q-q^{-1}) u_2 \otimes u_1 \otimes u_0$

etc.

(b) Let $L = (\check{R} - q\, id_{V\otimes V}) \circ (\check{R} + q^{-1} id_{V \otimes V})$. Then

$L(u_0 \otimes u_0) = (\check{R} - q\, id_{V\otimes V}) ((q+q^{-1}) u_0 \otimes u_0) = 0$

$L(u_0 \otimes u_1) = (\check{R} - q\, id_{V\otimes V}) (u_1 \otimes u_0 + q^{-1} u_0 \otimes u_1)$

$= u_0 \otimes u_1 + (q-q^{-1}) u_1 \otimes u_0 - q\, u_1 \otimes u_0 + q^{-1} u_1 \otimes u_0 - u_0 \otimes u_1$

$= 0$

$L(u_1 \otimes u_0) = (\check{R} - q\, id_{V\otimes V}) (u_0 \otimes u_1 + q\, u_1 \otimes u_0)$

$= u_1 \otimes u_0 - q\, u_0 \otimes u_1 + q\, u_0 \otimes u_1 + q(q-q^{-1}) u_1 \otimes u_0$

$- q^2 u_1 \otimes u_0$

$= 0$

Hence $L = 0$.

Ex 3

Exercise 4 (b) suggests that a good candidate for inverse of R is

$$\tilde{R} = \frac{1}{N}\sum_{m,n=0}^{N-1} \omega^{mn}\,\theta^{-m}\otimes\theta^{n} = \frac{1}{N}\sum_{m,n=0}^{N-1}\omega^{-mn}\,\theta^{m}\otimes\theta^{n}.$$

(↑ change of var. $m \to N-m$)

Since A is commutative we need to check that $\tilde{R}R = 1_A \otimes 1_A$.

$$\tilde{R}R = \frac{1}{N^2}\sum_{m,n=0}^{N-1}\sum_{k,l=0}^{N-1}\omega^{-nm+kl}\,\theta^{k+m}\otimes\theta^{l+n}$$

change variables: $i \equiv k+m \pmod N$, $i \in \{0,1,\dots,N-1\}$
$j \equiv l+n \pmod N$, $j \in \{0,1,\dots,N-1\}$

use $\theta^N = 1_A$, $\omega^N = 1$

$$= \frac{1}{N^2}\sum_{m,n=0}^{N-1}\sum_{i,j=0}^{N-1}\omega^{-nm+(i-m)(j-n)}\,\theta^{i}\otimes\theta^{j}$$

$$= \frac{1}{N^2}\sum_{i,j=0}^{N-1}\underbrace{\left[\sum_{m,n=0}^{N-1}\omega^{ij-mj-ni}\right]}_{=N^2\delta_{i0}\delta_{j0}}\theta^{i}\otimes\theta^{j}$$

$$= 1_A \otimes 1_A$$

Therefore (R0) holds. (R1) follows from commutativity and cocommutativity. For (R2), calculate

$$(\Delta\otimes \mathrm{id}_A)(R) = \frac{1}{N}\sum_{n,m=0}^{N-1}\omega^{nm}\,\theta^{m}\otimes\theta^{m}\otimes\theta^{n}$$

$$R_{13}R_{23} = \frac{1}{N^2}\sum_{n,m,k,l=0}^{N-1}\omega^{nm+kl}\,\theta^{m}\otimes\theta^{k}\otimes\theta^{n+l}; \quad n' \equiv n+l \pmod N$$

$$= \frac{1}{N^2}\sum_{n,m,k}^{N-1}\underbrace{\left[\sum_{l=0}^{N-1}\omega^{nm+(k-m)l}\right]}_{=N\omega^{nm}\delta_{k,m}}\theta^{m}\otimes\theta^{k}\otimes\theta^{n}$$

$$= (\Delta\otimes\mathrm{id}_A)(R)$$

Similar calculation gives (R3).

Ex 4

(a) Let $S = S_{A\otimes A}$ and $\tilde{R} = S(R^{-1})$. Since $S : A\otimes A \to A\otimes A$ is an algebra morphism, $\tilde{R}$ is invertible and $\tilde{R}^{-1} = S(R)$. Hence (R0) holds. For (R1)

$$\Delta^{op}(x)\,\tilde{R} = S(\Delta(x))\,S(R^{-1}) = S(\Delta(x)R^{-1})$$
$$\overset{(R1)}{=} S(R^{-1}\Delta^{op}(x)) = \tilde{R}\,\Delta(x)$$

For (R2)

$$(\Delta\otimes id_A)(\tilde{R}) = (\Delta\otimes id_A)(S(R^{-1}))$$
$$= S_{23}\circ S_{12}\,(id_A\otimes\Delta)(R^{-1}) = S_{23}\circ S_{12}\big((id_A\otimes\Delta)(R)\big)^{-1}$$
$$= S_{23}\circ S_{12}\big((R_{12})^{-1}(R_{13})^{-1}\big) = S_{23}\big(\tilde{R}_{12}(R_{23})^{-1}\big)$$
$$= \tilde{R}_{13}\,\tilde{R}_{23}$$

Similarly (R3).

(b) Let $r = (\mathcal{S}\otimes id)(R)$. By a proposition from the lectures $(\varepsilon\otimes id_A)(R) = 1_A$. Hence

$$1_A\otimes 1_A = ((\eta\circ\varepsilon)\otimes id_A)(R) \overset{(H3)}{=} (\mu\otimes id_A)\circ(\mathcal{S}\otimes id_A\otimes id_A)\circ(\Delta\otimes id_A)(R)$$
$$\overset{(R2)}{=} (\mu\otimes id_A)\circ(\mathcal{S}\otimes id_A\otimes id_A)(R_{13}R_{23})$$
$$= (\mu\otimes id_A)(r_{13}R_{23}) = rR$$

Since R is invertible, $r = R^{-1}$.

(c) $$(\mathcal{S}\otimes\mathcal{S})(R) = (id_A\otimes\mathcal{S})\circ(\mathcal{S}\otimes id_A)(R) = (id_A\otimes\mathcal{S})(R^{-1})$$
$$= S\circ(\mathcal{S}\otimes id)(S(R^{-1})) = S(S(R^{-1})^{-1}) = R$$