Hopf algebras and representations — Problem sheet 6

Spring 2011 — Model solutions

Ex 1

(a) Let b be a grouplike element. Suppose that we can write b as a linear combination of grouplike elements a_i, $i = 1, 2, \ldots, n$,

$$b = \sum_{i=1}^{n} \alpha_i a_i$$

where $\alpha_i \in \mathbb{C}$. We can assume that (a_i) is lin. ind. (otherwise we can reduce the above sum). Then

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j \, a_i \otimes a_j = b \otimes b = \Delta(b) = \sum_{i=1}^{n} \alpha_i \, \Delta(a)$$
$$= \sum_{i=1}^{n} \alpha_i \, a_i \otimes a_j$$

Since $(a_i \otimes a_j)$ is lin. ind, either $\alpha_i = 0$ or $\alpha_j = 0$ for any pair (i,j) and either $\alpha_i = 0$ or $\alpha_i = 1$ for any i. Hence $\alpha_i = 1$ for exactly one i and $\alpha_i = 0$ otherwise. Therefore $b = a_i$ for some i and the claim follows.

(b) Let $f : A \to \mathbb{C}$ be linear and let $a, b \in A$. Now

$$\Big[\langle f, a \rangle \langle f, b \rangle = \langle f, ab \rangle \Big] \Leftrightarrow \Big[\langle f \otimes f, a \otimes b \rangle = \langle \mu^*(f), a \otimes b \rangle \Big]$$

because by defition

$$\begin{cases} \langle f \otimes f, a \otimes b \rangle = \langle f, a \rangle \langle f, b \rangle \\ \langle \mu^*(f), a \otimes b \rangle = \langle f, ab \rangle \end{cases}$$

The claim follows.

Ex 2

If there is an invertible $u \in A$ such that $\wp(\wp(x)) = uxu^{-1}$ for all $x \in A$, then clearly $\wp(x) \neq 0$ for $x \neq 0$ and therefore $\wp$ is injective. In addition, $\wp$ is surjective: for $x \in A$, let $y = \wp(u^{-1}xu)$. Then $\wp(y) = \wp(\wp(u^{-1}xu)) = uu^{-1}xuu^{-1} = x$. Hence $\wp$ is invertible.

Therefore it is enough to check that $\tilde{\wp}(x) = u^{-1}\wp(x)u$ is one-sided inverse:

$$\tilde{\wp}(\wp(x)) = u^{-1}\wp(\wp(x))u = u^{-1}uxu^{-1}u = x$$

Ex 3

(a) If we have a finite dimensional repr. of A, $x \mapsto X$, $y \mapsto Y$, then $0 = tr(XY - YX) = tr(id_V) = \dim(V)$. $\Rightarrow V = \{0\}$.

(b) If A had a Hopf algebra structure, then $\varepsilon : A \to \mathbb{C}$ would define a one-dimensional representation, which doesn't exist by (a).

Ex 4

(a) Let $[x,y] \in C_P$.

$$(\Delta \otimes id) \circ \Delta([x,y]) = (\Delta \otimes id)\Big(\sum_{z \in [x,y]} [x,z] \otimes [z,y]\Big)$$

$$= \sum_{z \in [x,y]} \sum_{w \in [x,z]} [x,w] \otimes [w,z] \otimes [z,y]$$

$$= \sum_{w \in [x,y]} \sum_{z \in [w,y]} [x,w] \otimes [w,z] \otimes [z,y]$$

$$= (id \otimes \Delta) \circ \Delta([x,y])$$

$$(\varepsilon \otimes id)\Delta([x,y]) = \sum_{z \in [x,y]} \overbrace{\varepsilon([x,z])}^{=\delta_{x,z}} [z,y] = [x,y]$$

Also $(id \otimes \varepsilon)\Delta([x,y]) = [x,y]$ and hence $(C_P, \Delta, \varepsilon)$ is a coalgebra.

(b) $\zeta([x,y]) = 1 \quad \forall [x,y] \in I_P$ and $\zeta : C_P \to \mathbb{C}$ is a linear map. The unit in the convolution algebra is $\varepsilon \cdot 1_{\mathbb{C}} = \varepsilon$.

Let $[x,y] \in I_P$

$$(m * \zeta)([x,y]) = \sum_{z \in [x,y]} m([x,z])\zeta([z,y])$$

$$= \sum_{z \in [x,y]} m([x,z]) = \begin{cases} 1, & x = y \\ 0, & x \prec y \end{cases}$$

$$= \varepsilon([x,y])$$

Similarly we can define a right inverse by

$$\begin{cases} \tilde{m}([x,x]) = 1 & \forall x \\ \tilde{m}([x,y]) = -\sum_{z : x \prec z \preceq y} \tilde{m}([z,y]) & \forall x, y \succ x \end{cases}$$

Now $\tilde{m} = m * \zeta * \tilde{m} = m \implies \exists\, \zeta^{-1}$ and $\zeta^{-1} = m$.

(c) Let $\hat{f} \in A_p$ be any element such that for any x $\hat{f}([p,x]) = f(x)$ (for example define it to be zero on other basis vectors). Let $\hat{g} = \hat{f} * \zeta$. Now

$$\hat{g}([p,x]) = (\hat{f} * \zeta)([p,x]) = \sum_{y \in [p,x]} \hat{f}([p,y])$$

$$= \sum_{y \preceq x} f(y) = g(x)$$

Since $\hat{f} = \hat{f} * \zeta * m = \hat{g} * m$,

$$f(x) = \sum_{y \preceq x} g(y)\, m([y,x]).$$