Hopf algebras and representations
Spring 2011

Problem sheet 12
Model solutions

Ex 1

(a) Take an invariant subspace $V \subset W_e(\mu; a, b)$, $V \neq \{0\}$. Since K has at least one eigenvector in V, some $w_k \in V$.

If $b \neq 0$, each w_j can be obtained from w_k by applying F several times. Hence $V = W_e(\mu; a, b)$ and $W_e(\mu; a, b)$ is irreducible.

If $b = 0$, then $\{w_k, w_{k+1}, \ldots, w_{e-1}\} \subset V$ by applying F to w_k. By applying E to w_k, we get $\{w_0, w_1, \ldots, w_k\} \subset V$ unless one of the coefficient $c_j = 0$, where c_j is defined by $E.w_j = c_j w_j$. That is, unless

$$\mu q^{1-j} - \mu^{-1} q^{j-1} = 0 \quad \text{for some } j \in \{1, 2, \ldots, e-1\}$$

$$\iff \mu \in \{\pm 1, \pm q, \pm q^2, \ldots, \pm q^{e-2}\}$$

(b) As said in the problem sheet, we can think

$$\left(V \text{ is } \tilde{\mathcal{U}}_q(sl_2)\text{-module} \right) \underset{\text{restrict}}{\overset{\text{extend}}{\rightleftarrows}} \left(\begin{array}{l} V \text{ is } \mathcal{U}_q(sl_2)\text{-module} \\ \text{where } E^e, F^e, K^e - 1 \\ \text{act as zero} \end{array} \right)$$

Therefore $V' \subset V$ is $\tilde{\mathcal{U}}_q(sl_2)$-module if and only if it is $\mathcal{U}_q(sl_2)$-module. There are no extra requirements, since $E^e, F^e, K^e - 1$ already act as zero.

(c) In W_d^ε, $d < e$, E^e and F^e act as zero. If $K^e - 1$ acts as zero, then

$$w_j = K^e . w_j = \varepsilon^e \, (q^{d-1-2j})^e \, w_j$$
$$= \varepsilon^e \, (\underbrace{q^e}_{=\pm 1})^{d-1} \, w_j$$

Therefore we have the following cases when W_d^ε defines $\tilde{\mathcal{U}}_q(sl_2)$-module:

- e odd, $q^e = +1$: $\varepsilon = +1$
- e odd, $q^e = -1$: $\varepsilon = (-1)^{d-1}$
- e even, $q^e = -1$: ε anything, d odd

Note that e even, $q^e = 1$ contradicts with the definition of e.

In $W_e(\mu; a, b)$, $K^e - 1$ acts as zero if

$$w_j = K^e . w_j = \mu^e \cdot (q^{-2j})^e \, w_j = \mu^e w_j .$$

Therefore μ is a e-th root of unity. The set $\{\pm 1, \pm q, \dots, \pm q^{e-1}\}$ contains all e-th roots of $+1$ and -1. By irreducibility $\mu = \pm q^{e-1} = \pm q^{-1}$ if one of those numbers is a e-th root of $+1$. E^e and F^e act as zero only if $a = 0 = b$. Hence $W_e(\mu; 0, 0)$ is irreducible $\tilde{\mathcal{U}}_q(sl_2)$-module, when

- e odd, $q^e = +1$: $\mu = q^{-1}$
- e odd, $q^e = -1$: $\mu = -q^{-1}$

This is not possible when e is even and $q^e = -1$, because $(\pm q^{-1})^e = q^{-e} = -1$.

Ex 2

$$R.(w_0 \otimes w_0) = \frac{1}{e} \sum_{i,j=0}^{e-1} q^{-2ij}\, q^{i+j} \quad w_0 \otimes w_0$$

$$= \Big\{ \sum_{j=0}^{e-1} \underbrace{\Big[\frac{1}{e} \sum_{i=0}^{e-1} q^{i(1-2j)} \Big]}_{\delta_{j,\frac{e+1}{2}}} \cdot q^{j} \Big\} \; w_0 \otimes w_0$$

$$= q^{\frac{e+1}{2}} \; w_0 \otimes w_0$$

$$R.(w_0 \otimes w_1) = \frac{1}{e} \sum_{i,j=0}^{e-1} q^{-2ij}\, q^{i-j} \cdot w_0 \otimes w_1$$

$$= \Big\{ \sum_{j=0}^{e-1} \Big[\frac{1}{e} \sum_{i=0}^{e-1} q^{i(1-2j)} \Big] q^{-j} \Big\} w_0 \otimes w_1 = q^{-\frac{e+1}{2}} \; w_0 \otimes w_1$$

$$R.(w_1 \otimes w_0) = \frac{1}{e} \sum_{i,j=0}^{e-1} q^{-2ij}\, q^{-i+j} \quad w_1 \otimes w_0$$

$$+ (q - q^{-1}) \cdot \frac{1}{e} \sum_{i,j=0}^{e-1} q^{2(i-j)-2ij}\, q^{-i+j} \quad w_0 \otimes w_1$$

$$= \Big\{ \sum_{j=0}^{e-1} \overbrace{\Big[\frac{1}{e} \sum_{i=0}^{e-1} q^{-i(1+2j)} \Big]}^{=\delta_{j,\frac{e-1}{2}}} \cdot q^{j} \Big\} \; w_1 \otimes w_0$$

$$+ (q - q^{-1}) \Big\{ \sum_{j=0}^{e-1} \underbrace{\Big[\frac{1}{e} \sum_{i=0}^{e-1} q^{i(1-2j)} \Big]}_{=\delta_{j,\frac{e+1}{2}}} \cdot q^{-j} \Big\} \; w_0 \otimes w_1$$

$$= q^{\frac{e-1}{2}} \cdot \Big\{ w_1 \otimes w_0 + (q - q^{-1})\, w_0 \otimes w_1 \Big\}$$

$$R.(w_1 \otimes w_1) = \frac{1}{e} \sum_{i,j=0}^{e-1} q^{-2ij}\, q^{-i-j} \quad w_1 \otimes w_1$$

$$= \Big\{ \sum_{j=0}^{e-1} \underbrace{\Big[\frac{1}{e} \sum_{i=0}^{e-1} q^{-i(1+2j)} \Big]}_{=\delta_{j,\frac{e-1}{2}}} \cdot q^{-j} \Big\} \; w_1 \otimes w_1$$

$$= q^{-\frac{e-1}{2}} \quad w_1 \otimes w_1$$

Here we used that

$$\frac{1}{e}\sum_{i=0}^{e-1} q^{is} = \begin{cases} 1, & \text{when } q^s = 1 \\ 0, & \text{otherwise} \end{cases}$$

and that there was only one value of j which contributed the sum.

Therefore $q^{\frac{e+1}{2}}\check{R}$ is the same solution of Yang-Baxter that we already saw in Problem sheet 8 Exercise 2.

Ex 3

(a) To define representations on tensor products we need the coproduct of $\mathcal{U}_q(sl_2)$ which is given by

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1$$

Let $(u_i)_{i=0}^{d-1}$ be the usual basis for W_d^{+1} and let v_0 be basis for W_1^{ε}. Then

$$K.(u_j \otimes v_0) = (K.u_j) \otimes (K.v_0)$$

$$= \varepsilon \, q^{d-1-2j} \, u_j \otimes v_0$$

$$F.(u_j \otimes v_0) = (K^{-1}.u_j) \otimes \underbrace{(F.v_0)}_{=0} + (F.u_j) \otimes v_0$$

$$= u_{j+1} \otimes v_0$$

$$E.(u_j \otimes v_0) = (E.u_j) \otimes (K.v_0) + u_j \otimes \underbrace{(E.v_0)}_{=0}$$

$$= \varepsilon \, [j]_q [d-j]_q \, u_{j-1} \otimes v_0$$

Hence if we set $w_j = u_j \otimes v_0$, we get an isomophism from $W_d^{+} \otimes W_1^{\varepsilon}$ to W_d^{ε}.

(b) Let $l \in \{0, 1, \dots, d_2 - 1\}$ and $s \in \{0, 1, \dots, l\}$. Then

$$K.\left(w_s^{(1)} \otimes w_{l-s}^{(2)}\right) = q^{d_1 - 1 - 2s} \cdot q^{d_2 - 1 - 2l + 2s} \, w_s^{(1)} \otimes w_{l-s}^{(2)}$$

$$= q^{d_1 + d_2 - 2 - 2l} \, w_s^{(1)} \otimes w_{l-s}^{(2)}$$

Hence v is an eigenvector of K with

---

eigenvalue $\lambda_l = q^{d_1 + d_2 - 2 - 2l}$.

Now let's calculate

$$E.(w_s^{(1)} \otimes w_{l-s}^{(2)}) = [s]_q [d_1 - s]_q \, q^{d_2-1-2l+2s} \, w_{s-1}^{(1)} \otimes w_{l-s}^{(2)}$$

$$+ [l-s]_q [d_2 - l + s]_q \, w_s^{(1)} \otimes w_{l-s-1}^{(2)}$$

Denote by $c_s \in \mathbb{C}$ the coefficients in v. That is, $v = \sum_{s=0}^{l} c_s \, w_s^{(1)} \otimes w_{l-s}^{(2)}$. Now $E.v = 0$, if for each $s \in \{0, 1, \dots, l-1\}$ we have

$$[s+1]_q [d_1 - s - 1]_q \, c_{s+1} + [l-s]_q [d_2 - l + s]_q c_s = 0.$$

One can check that these equations are satisfied by the given v.

(c) For each v as above, we get an submodule by applying F until $F^n.v = 0$. From K-eigenvalue of v we can read that the submodule is $d_1 + d_2 - 1 - 2l$ dimensional. Hence the span of $\{v, F.v, \dots, F^{d_1+d_2-2-2l}.v\}$ is isomorphic to $W^{+1}_{d_1+d_2-1-2l}$. The sum of dimensions of these modules is

$$\sum_{l=0}^{d_2-1} d_1 + d_2 - 1 - 2l = d_2(d_1 + d_2 - 1) - 2 \cdot \frac{1}{2} d_2 (d_2 - 1)$$

$$= d_1 d_2$$

Hence the direct sum of these submodules is the whole module.