Hopf algebras and representations
Spring 2011

Problem sheet 11
Model solutions

Ex 1

It is enough to prove the formula for simple tensor x. Therefore, let $x = a \otimes \psi \in \mathcal{D}$, $c \in A$ and $\phi \in \mathcal{D}^\circ$. Now $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} = \sum_{(a),(\psi)} (a_{(1)} \otimes \psi_{(2)}) \otimes (a_{(2)} \otimes \psi_{(1)})$.

Let's use notation $(\phi)' = \iota_A \circ (\phi|A)$. Write now

$$LHS = \sum_{(\phi),(x)} \langle \phi_{(1)}, x_{(2)} \rangle \, x_{(1)} (\phi_{(2)})'$$

$$= \sum_{(\phi),(a),(\psi)} \langle \phi_{(1)}, a_{(2)} \otimes \psi_{(1)} \rangle \overbrace{(a_{(1)} \otimes \psi_{(2)})(1 \otimes \phi_{(2)}|A)}^{= a_{(1)} \otimes \psi_{(2)} (\phi_{(2)}|A)}$$

evaluate second component at c

$$\longmapsto \sum_{(\phi),(a),(\psi),(c)} \langle \phi_{(1)}, a_{(2)} \otimes \psi_{(1)} \rangle \, a_{(1)} \langle \psi_{(2)}, c_{(1)} \rangle \langle \phi_{(2)}, \iota_A(c_{(2)}) \rangle$$

$$= \sum_{(a),(\psi),(c)} \langle \phi, (a_{(2)} \otimes \psi_{(1)})(c_{(2)} \otimes 1^*) \rangle \langle \psi_{(2)}, c_{(1)} \rangle \, a_{(1)}$$

$$= \sum_{(a),(\psi),(c)} \langle \phi, (a_{(2)} c_{(3)}) \otimes \psi_{(2)} \rangle \langle \psi_{(1)}, c_{(4)} \rangle \langle \psi_{(3)}, S^{-1}(c_{(2)}) \rangle \cdot \langle \psi_{(4)}, c_{(1)} \rangle \, a_{(1)}$$

$$= \sum_{(a),(\psi),(c)} \langle \phi, (a_{(2)} c_{(2)}) \otimes \psi_{(2)} \rangle \langle \psi_{(1)}, c_{(3)} \rangle \langle \psi_{(3)}, \underbrace{\mu \circ (S^{-1} \otimes id) \circ \Delta^{cop}(c_{(1)})}_{= 1\, \varepsilon(c_{(1)})} \rangle \cdot a_{(1)}$$

$$= \sum_{(a),(\psi),(c)} \langle \phi, (a_{(2)} c_{(1)}) \otimes \psi_{(2)} \rangle \langle \psi_{(1)}, c_{(2)} \rangle \, a_{(1)}$$

where in the last stage we used counitality of A and A^*.

$$\mathrm{RHS} = \sum_{(\phi),(x)} \langle \phi_{(2)}, x_{(1)} \rangle (\phi_{(1)})' x_{(2)}$$

$$= \sum_{(\phi),(a),(\psi)} \langle \phi_{(2)}, a_{(1)} \otimes \psi_{(2)} \rangle (1 \otimes \phi_{(1)}|_A)(a_{(2)} \otimes \psi_{(1)})$$

$$\xmapsto{\text{evaluate at } c} \sum_{(\phi),(a),(\psi),(c)} \sum_{(\phi_{(1)}|_A)} \langle \phi_{(2)}, a_{(1)} \otimes \psi_{(2)} \rangle \langle (\phi_{(1)}|_A)_{(1)}, a_{(4)} \rangle$$

$$\cdot \langle (\phi_{(1)}|_A)_{(3)}, \mathcal{S}^{-1}(a_{(2)}) \rangle \langle (\phi_{(1)}|_A)_{(2)}, c_{(1)} \rangle$$

$$\cdot \langle \psi_{(1)}, c_{(2)} \rangle\, a_{(3)}$$

$$= \sum_{(\phi),(a),(\psi),(c)} \langle \phi_{(2)}, a_{(1)} \otimes \psi_{(2)} \rangle \langle \phi_{(1)}, \iota_A(a_{(4)} c_{(1)} \mathcal{S}^{-1}(a_{(2)})) \rangle$$

$$\langle \psi_{(1)}, c_{(2)} \rangle\, a_{(3)}$$

$$= \sum_{(a),(\psi),(c)} \langle \phi, (a_{(4)} c_{(1)} \mathcal{S}^{-1}(a_{(2)}) a_{(1)}) \otimes \psi_{(2)} \rangle \langle \psi_{(1)}, c_{(2)} \rangle\, a_{(3)}$$

$$= \sum_{(a),(\psi),(c)} \langle \phi, (a_{(2)} c_{(1)}) \otimes \psi_{(2)} \rangle \langle \psi_{(1)}, c_{(2)} \rangle\, a_{(3)}$$

$= \mathrm{LHS}$ evaluated at c

Ex 2

(a) We have to check that relations of $\mathcal{U}_q(sl_2)$ are satisfied by the linear maps given. Clearly K is invertible on W_d^ε. Since the eigenvalues differ by factor q^{-2} for w_j and w_{j+1}, the relations $KEK^{-1} = q^2 E$ and $KFK^{-1} = q^{-2}F$ are satisfied. The last relation follows from a calculation

$$(EF - FE).w_j = \varepsilon\left([j+1]_q [d-j-1]_q - [j]_q [d-j]_q \right) w_j$$

$$= \varepsilon \frac{1}{q-q^{-1}} \left(q^{d-1-2j} - q^{-d+1+2j} \right) w_j$$

$$= \frac{1}{q-q^{-1}} (K - K^{-1}).w_j$$

where we used a special case of Problem sheet 10, Ex 2 (c) $[j+1]_q[k-1]_q + [k]_q[-j]_q + [1]_q[j+1-k]_q = 0$ or just a direct calculation. Hence W_d^ε is a module.

W_d^ε is irreducible, because any submodule $V \neq \{0\}$ contains at least one w_k. Other w_j can be obtained from w_k by applying E and F many times. Hence $V = W_d^\varepsilon$. Here it is important that $[j]_q [d-j]_q \neq 0$ for any $1 \leq j \leq d-1$.

(b) The argument of lecture notes works in this case: If V is an irreducible module, then there is at least one eigenvector v of K, $K.v = \lambda v$, $\lambda \neq 0$. Now since $K E^n. v = \lambda q^{2n} E^n. v$ and since λq^{2n} are distinct for $n \in \{0, 1, \dots, e-1\}$, there has to be a non-zero vector $w_0 = E^n. v$ s.t. $E. w_0 = 0$. Now by defining $w_{j+1} = F. w_j$ leads to a module of the form of the one in (a) by the same calculation as in the lecture notes.

Ex 3

(a) Let $v \in V$, $v \neq 0$ s.t. $F.v = 0$. Let $V' = \mathrm{span}(\{v, E.v, \dots, E^{e-1}.v\})$. Then $V' \subset V$, $\dim(V') \leq e < \dim(V)$ and V' is stable at least under K and E. To show that $E.(E^{e-1}.v)$ is in V', you might want to use that E^e is $c_E \,\mathrm{id}_V$ on V for some $c_E \in \mathbb{C}$. Use Problem shee 10 Ex 3 to write that

$$F.(E^k.v) = E^k.\underbrace{(F.v)}_{=0} + \frac{[k]_q}{q-q^{-1}}(q^{k-1}K^{-1} - q^{1-k}K)E^{k-1}.v$$

$$\in V'$$

Hence V' is a submodule which contradicts with V being irreducible.

(b) Take any eigenvector v of K. By the assumption $v, F.v, F^2.v, \dots, F^{e-1}.v$ are all non-zero and hence they are lin. ind. (after all, they are eigenvectors of K with distinct eigenvalues). Denote their linear span by V'. Then $V' \subset V$, $\dim(V') = e < \dim(V)$ and V' is stable under K and F. Note that $F^e = c_F \,\mathrm{id}_V$ on V and $c_F \neq 0$. Hence $v = \frac{1}{c_F} F^e.v$. Now for any $1 \leq k \leq e$

$$EF^k.v = \left(C - \frac{1}{(q-q^{-1})^2}(q^{-1}K + qK^{-1})\right)F^{k-1}.v$$

$$\in V'$$

Here C is the quadratic Casimir, which acts as a constant since it is central. Hence V' is a submodule which contradicts with V being irreducible.

(c) Should be clear from (a) and (b).

Ex 4

(a) We have to check that the relations of $U_q(sl_2)$ are satisfied on the given module. As before the other relations easily satisfied and we concentrate on checking $EF-FE = \frac{1}{q-q^{-1}}(K-K^{-1})$

$$(EF-FE).w_0$$
$$= \left(ab + \frac{1}{q-q^{-1}}(\mu-\mu^{-1}) - ab\right) w_0 = \frac{1}{q-q^{-1}}(K-K^{-1}).w_0$$

$$(EF-FE).\ w_j\ ,\ 1 \le j \le e-2$$
$$= \frac{1}{q-q^{-1}}\left\{ [j+1](\mu q^{-j} - \mu^{-1}q^{j}) - [j](\mu q^{1-j} - \mu^{-1}q^{j-1})\right\} w_j$$
$$= \frac{1}{q-q^{-1}}\left\{ \mu q^{-j} \underbrace{([j+1]-q[j])}_{=q^{-j}[1]=q^{-j}} - \mu^{-1}q^{j}\underbrace{([j+1]-q^{-1}[j])}_{=q^{j}[1]=q^{j}}\right\} w_j$$

By Prob. sheet 10 Ex 2

$$= \frac{1}{q-q^{-1}}(K-K^{-1}).w_j$$

$$(EF-FE).w_{e-1}$$
$$= -\frac{[e-1]}{q-q^{-1}}(\mu q^{2-e} - \mu^{-1}q^{e-2})\, w_{e-1}$$
$$= \frac{q^e}{q-q^{-1}}(\mu q^{2-e} - \mu^{-1}q^{e-2})\, w_{e-1} = \frac{1}{q-q^{-1}}(K-K^{-1}).w_{e-1}$$

(b) Suppose that $V \subset W_e(\mu; a,b)$ is a submodule, $V \neq \{0\}$. Then K has at least one eigenvector in V and that vector is some w_{k_0}. By applying F several times, we see that $\{w_{k_0}, w_{k_0+1}, \dots, w_{e-1}\} \subset V$. Therefore it is impossible that $W_e(\mu; a,b) = V \oplus V'$ where $V \neq \{0\} \neq V'$ are submodules. Namely, $\{w_{e-1}\} \subset V \cap V' \neq \{0\}$.