Problem sheet 7

Exercise 1: *The q-binomial coefficients at roots of unity*
Let $q \in \mathbb{C}$ be a primitive p^{th} root of unity, that is, $q^p = 1$ and $q, q^2, q^3, \ldots, q^{p-1} \neq 1$. Show that the values of the q-binomial coefficients are then described as follows: if the quotients and remainders modulo p of n and k are $n = p\,D(n) + R(n)$ and $k = p\,D(k) + R(k)$ with $D(n), D(k) \in \mathbb{N}$ and $R(n), R(k) \in \{0, 1, 2, \ldots, p-1\}$, then

$$\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_q = \binom{D(n)}{D(k)} \times \left[\!\left[\begin{matrix} R(n) \\ R(k) \end{matrix} \right]\!\right]_q .$$

In particular $\left[\!\left[\begin{matrix} n \\ k \end{matrix} \right]\!\right]_q$ is non-zero only if the remainders satisfy $R(k) \leq R(n)$.

Exercise 2: *Representative forms in a representation of the Laurent polynomial algebra*
Let $A = \mathbb{C}[t, t^{-1}] \cong \mathbb{C}[\mathbb{Z}]$ be the algebra of Laurent polynomials

$$A = \left\{ \sum_{n=-N}^{N} c_n t^n \;\middle|\; N \in \mathbb{N}, c_{-N}, c_{-N+1}, \ldots, c_{N-1}, c_N \in \mathbb{C} \right\}.$$

Define the $s \in A^*$ and $g_z \in A^*$, for $z \in \mathbb{C} \setminus \{0\}$, by the formulas

$$\langle g_z, t^n \rangle = z^n \qquad \langle s, t^n \rangle = n.$$

(a) Show that $s \in A^\circ$ and $g_z \in A^\circ$.

Let us equip A with the Hopf algebra structure such that $\Delta(t) = t \otimes t$.

(b) Let $z \in \mathbb{C} \setminus \{0\}$. Consider the finite dimensional A-module V with basis $u_1, u_2, \ldots, u_n$ such that

$$t.u_j = z\,u_j + u_{j-1} \quad \forall\, j > 1 \qquad \text{and} \qquad t.u_1 = z u_1.$$

Define the representative forms $\lambda_{i,j} \in A^\circ$ by $a.u_j = \sum_{i=1}^{n} \langle \lambda_{i,j}, a \rangle\, u_i$. Show that we have the following equalities in the Hopf algebra A°:

$$\lambda_{i,j} = \begin{cases} 0 & \text{if } i > j \\ g_z & \text{if } i = j \\ \frac{z^{i-j}}{(j-i)!}\, s\,(s-1) \cdots (s+i-j+1)\, g_z & \text{if } i < j \end{cases} .$$

Exercise 3: *Taking the restricted dual is a contravariant functor*
Let A and B be two algebras and $f : A \to B$ a homomorphism of algebras, and let f^* be its transpose map $B^* \to A^*$.

(a) Show that for any $\varphi \in B^\circ$ we have $f^*(\varphi) \in A^\circ$.

(b) Show that $f^*|_{B^\circ} : B^\circ \to A^\circ$ is a homomorphism of coalgebras.

Exercise 4: *The restricted dual of the binomial Hopf algebra*
Given two Hopf algebras $(A_i, \mu_i, \Delta_i, \eta_i, \epsilon_i, \gamma_i)$, $i = 1, 2$, we can form the tensor product of Hopf algebras by equipping $A_1 \otimes A_2$ with the structural maps

$$\mu = (\mu_1 \otimes \mu_2) \circ (\mathrm{id}_{A_1} \otimes S_{A_2,A_1} \otimes \mathrm{id}_{A_2}) \qquad \Delta = (\mathrm{id}_{A_1} \otimes S_{A_1,A_2} \otimes \mathrm{id}_{A_2}) \circ (\Delta_1 \otimes \Delta_2)$$

$$\eta = \eta_1 \otimes \eta_2 \qquad \epsilon = \epsilon_1 \otimes \epsilon_2 \qquad \gamma = \gamma_1 \otimes \gamma_2.$$

Let $A = \mathbb{C}[x]$ be the algebra of polynomials in the indeterminate x, equipped with the unique Hopf algebra structure such that $\Delta(x) = 1 \otimes x + x \otimes 1$ (the binomial Hopf algebra). Show that we have an isomorphism of Hopf algebras

$$A^\circ \cong A \otimes \mathbb{C}[\mathbb{C}],$$

that is, the restricted dual of A is isomorphic to the tensor product of the Hopf algebra A with the Hopf algebra of the additive group of complex numbers.