Problem sheet 6

Exercise 1: *More on grouplike elements*
Recall that for $C = (C, \Delta, \epsilon)$ a coalgebra, an element $a \in C$ is said to be grouplike if $a \neq 0$ and $\Delta(a) = a \otimes a$.

(a) Show that the grouplike elements in a coalgebra are linearly independent.

(b) Let $A = (A, \mu, \eta)$ be an algebra and consider its restricted dual $A^\circ = (\mu^*)^{-1}(A^* \otimes A^*)$ with the coproduct $\Delta = \mu^*|_{A^\circ}$ and counit $\epsilon = \eta^*|_{A^\circ}$. Show that for a linear map $f : A \to \mathbb{C}$ the following are equivalent:

- The function f is a homomorphism of algebras.
  (*Remark:* This has the interpretation that f is a one-dimensional representation of A.)
- The element f is grouplike in A°.

Exercise 2: *A sufficient condition for invertibility of antipode*
Suppose that $A = (A, \mu, \Delta, \eta, \epsilon, \gamma)$ is a Hopf algebra, where there is an invertible element $u \in A$ such that the square of the antipode of any element $x \in A$ can be written as

$$\gamma(\gamma(x)) \; = \; u \, x \, u^{-1}.$$

Show (for example using the results of *Problem sheet 5: Exercise 3*) that the antipode $\gamma : A \to A$ is an invertible linear map with inverse given by

$$\gamma^{-1}(x) \; = \; u^{-1} \, \gamma(x) \, u \qquad \forall \, x \in A.$$

Exercise 3: *Representations of the canonical commutation relations of quantum mechanics*
Let A be the algebra with two generators x and y, and one relation

$$x \, y - y \, x \; = \; 1 \qquad \text{(“canonical commutation relation”)}.$$

(a) Show that there are no finite-dimensional representations of A except from the zero vector space $V = \{0\}$.

(b) Conclude that it is impossible to equip A with a Hopf algebra structure.

Exercise 4: *The incidence coalgebra and incidence algebra of a poset*
A partially ordered set (poset) is a set P together with a binary relation $\preceq$ on P which is reflexive ($x \preceq x$ for all $x \in P$), antisymmetric (if $x \preceq y$ and $y \preceq x$ then $x = y$) and transitive (if $x \preceq y$ and $y \preceq z$ then $x \preceq z$). Notation $x \prec y$ means $x \preceq y$ and $x \neq y$. Notation $x \succeq y$ means $y \preceq x$. If $x, y \in P$ and $x \preceq y$, then we call the set

$$[x, y] \; = \; \{z \in P \mid x \preceq z \text{ and } z \preceq y\}$$

an interval in P.

Suppose that P is a poset such that all intervals in P are finite (a locally finite poset). Let I_P be the set of intervals of P, and let C_P be the vector space with basis I_P. Define $\Delta : C_P \to C_P \otimes C_P$ and $\epsilon : C_P \to \mathbb{C}$ by linear extension of

$$\Delta([x, y]) \; = \sum_{z \in [x,y]} [x, z] \otimes [z, y] \quad , \qquad \epsilon([x, y]) \; = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \prec y \end{cases} .$$

(a) Show that $C_P = (C_P, \Delta, \epsilon)$ is a coalgebra (we call C_P the incidence coalgebra of P).

The incidence algebra A_P of the poset P is the convolution algebra associated with the coalgebra C_P and the algebra $\mathbb{C}$. Define $\zeta \in A_P$ by its values on basis vectors $\zeta([x, y]) = 1$ for all intervals $[x, y] \in I_P$.

(b) Show that ζ is invertible in A_P, with inverse m (called the Möbius function of P) whose values on the basis vectors are determined by the recursions

$m([x, x]) = 1$ for all $x \in P$
$m([x, y]) = -\sum_{z \,:\, x \preceq z \prec y} m([x, z])$ for all $x \in P$, $y \succeq x$.

(c) Let $f : P \to \mathbb{C}$ be a function and suppose that there is a $p \in P$ such that $f(x) = 0$ unless $x \succeq p$. Prove the Möbius inversion formula: if

$$g(x) \; = \sum_{y \preceq x} f(y)$$

then

$$f(x) \; = \sum_{y \preceq x} g(y) \; m([y, x]).$$

(*Hint:* It may be helpful to define a $\hat{f} \in A_P$ with the property $\hat{f}([p, x]) = f(x)$ and $\hat{g} = \hat{f} \star \zeta$.)