Problem sheet 5

Exercise 1: *Grouplike and primitive elements*
Let $(A, \mu, \eta, \Delta, \epsilon, \gamma)$ be a Hopf algebra. An element $a \in A$ is said to be grouplike if $a \neq 0$ and $\Delta(a) = a \otimes a$. An element $x \in A$ is said to be primitive if $\Delta(x) = x \otimes 1_A + 1_A \otimes x$. Show that:

(a) When $a \in A$ is grouplike, we have $\epsilon(a) = 1$ and a is invertible and $\gamma(a) = a^{-1}$.

(b) When $x \in A$ is primitive, we have $\epsilon(x) = 0$ and $\gamma(x) = -x$.

Exercise 2: *Opposite and/or co-opposite bialgebras*
Suppose that $A = (A, \mu, \Delta, \eta, \epsilon)$ is a bialgebra. Let $\mu^{\mathrm{op}} = \mu \circ S_{A,A}$ be the opposite product and $\Delta^{\mathrm{cop}} = S_{A,A} \circ \Delta$ be the (co-)opposite coproduct. Show that all of the following are bialgebras:

the opposite bialgebra $A^{\mathrm{op}} = (A, \mu^{\mathrm{op}}, \Delta, \eta, \epsilon)$
the co-opposite bialgebra $A^{\mathrm{cop}} = (A, \mu, \Delta^{\mathrm{cop}}, \eta, \epsilon)$
the opposite co-opposite bialgebra $A^{\mathrm{op,cop}} = (A, \mu^{\mathrm{op}}, \Delta^{\mathrm{cop}}, \eta, \epsilon)$.

Exercise 3: *Opposite and/or co-opposite Hopf algebras*
Suppose that $(A, \mu, \Delta, \eta, \epsilon)$ is a bialgebra, which admits an antipode $\gamma : A \to A$.

(a) Show that $A^{\mathrm{op,cop}} = (A, \mu^{\mathrm{op}}, \Delta^{\mathrm{cop}}, \eta, \epsilon, \gamma)$ is a Hopf algebra, called the the opposite co-opposite Hopf algebra to $A = (A, \mu, \Delta, \eta, \epsilon, \gamma)$.

(b) Show that the following conditions are equivalent

- the opposite bialgebra A^{op} admits an antipode $\tilde{\gamma}$
- the co-opposite bialgebra A^{cop} admits an antipode $\tilde{\gamma}$
- the antipode $\gamma : A \to A$ is invertible, with inverse $\gamma^{-1} = \tilde{\gamma}$.

Exercise 4: *A lemma for construction of antipode*
Let $B = (B, \mu, \Delta, \eta, \epsilon)$ be a bialgebra. Suppose that as an algebra B is generated by a collection of elements $(g_i)_{i \in I}$. Suppose furthermore that we are given a linear map $\gamma : B \to B$, which is a homomorphism of algebras from $B = (B, \mu, \eta)$ to $B^{\mathrm{op}} = (B, \mu^{\mathrm{op}}, \eta)$, and which satisfies

$$\big(\mu \circ (\gamma \otimes \mathrm{id}_B) \circ \Delta\big)(g_i) \;=\; \epsilon(g_i)\, 1_B \;=\; \big(\mu \circ (\mathrm{id}_B \otimes \gamma) \circ \Delta\big)(g_i) \qquad \text{for all } i \in I.$$

Show that $(B, \mu, \Delta, \eta, \epsilon, \gamma)$ is a Hopf algebra.

Exercise 5: *A building block of quantum groups*
Let $q \in \mathbb{C} \setminus \{0\}$. Let H_q be the algebra with three generators a, a', b and relations

$$a\, a' \;=\; a'\, a \;=\; 1 \qquad , \qquad a\, b \;=\; q\, b\, a.$$

Because of the first relation we can write $a' = a^{-1}$ in H_q. The collection $(b^m\, a^n)_{m \in \mathbb{N},\, n \in \mathbb{Z}}$ is a vector space basis for H_q. We wish to put a Hopf algebra structure on H_q such that the coproducts of a and b are given by

$$\Delta(a) \;=\; a \otimes a \qquad \text{and} \qquad \Delta(b) \;=\; a \otimes b + b \otimes 1.$$

(a) Show that there is a unique bialgebra structure on H_q with the values of the coproduct above.

(b) Show, for example using the result of *Exercise 4*, that there is a unique Hopf algebra structure on H_q with the values of the coproduct above.

(c) Compute $\epsilon(b^m a^n)$ and $\gamma(b^m a^n)$, for $m \in \mathbb{N}$, $n \in \mathbb{Z}$, in the Hopf algebra H_q.

(d) Show that the elements $x = a \otimes b$ and $y = b \otimes 1$ in $H_q \otimes H_q$ satisfy the relation $x\, y \;=\; q\, y\, x$. Then compute $\Delta(b^m a^n)$, for $m \in \mathbb{N}$, $n \in \mathbb{Z}$, in the Hopf algebra H_q.
(*Hint*: The q-binomial formula of *Problem sheet 4: Exercise 1* may be helpful.)