Problem sheet 3

Hint to all exercises: Use characters!

Exercise 1: *Characters of the group of symmetries of the square*
Let D_4 be the group with two generators, r and m, and relations $r^4 = e$, $m^2 = e$, $rmrm = e$.

(a) List all irreducible representations of D_4, and compute their characters.

(b) Let V be the two-dimensional irreducible representation of D_4 introduced in *Problem sheet 1: Exercise 4*. Verify using character theory that the representation $V \otimes V$ is isomorphic to the direct sum of four one dimensional representations (see also *Problem sheet 2: Exercise 4*).

Exercise 2: *The standard representation of S_4*
Consider the symmetric group S_4 on four elements, and define a four-dimensional representation V with basis e_1, e_2, e_3, e_4 by

$$\sigma.e_j \;=\; e_{\sigma(j)} \qquad \text{for } \sigma \in S_4,\; j = 1, 2, 3, 4.$$

(a) Compute the character of V.

(b) Show that the subspace spanned by $e_1 + e_2 + e_3 + e_4$ is a trivial subrepresentation of V and show that the complementary subrepresentation to it is an irreducible three-dimensional representation of S_4.

(c) Find the entire character table of S_4, that is, characters of all irreducible representations. (*Hint:* You should already know a couple of irredubibles. Try taking tensor products of these, and using orthonormality of irreducible characters.)

Exercise 3: *Example of tensor products of representations of S_3*
Recall that there are three irreducible representations of S_3, the trivial representation U, the alternating representation U' and the two-dimensional irreducible representation V found in *Problem sheet 1: Exercise 3*. Consider the representation $V^{\otimes n}$, the n-fold tensor product of V with itself. Find the multiplicities of U, U' and V when $V^{\otimes n}$ is written as a direct sum of irreducible representations.

Exercise 4: *The center of the group algebra*
Recall that the center of an algebra A is the set $Z \subset A$ of elements that commute with the whole algebra, i.e.

$$Z \;=\; \{ z \in A \mid za = az \;\; \forall a \in A \} .$$

Let G be a finite group and $A = \mathbb{C}[G]$ its group algebra, i.e. the vector space with basis $(e_g)_{g \in G}$ equipped with the product $e_g \, e_h = e_{gh}$ (extended bilinearly).

(a) Show that the element

$$a \;=\; \sum_{g \in G} \alpha(g) \, e_g \;\in A$$

is in the center of the group algebra if and only if $\alpha(g) = \alpha(hgh^{-1})$ for all $g, h \in G$.

(b) Suppose that $\alpha : G \to \mathbb{C}$ is a function which is constant on each conjugacy class, and suppose furthermore that α is orthogonal (with respect to the inner product $(\psi, \phi) = |G|^{-1} \sum_{g \in G} \overline{\psi(g)} \phi(g)$) to the characters of all irreducible representations of G. Show that for any representation $\rho : G \to \mathrm{GL}(V)$ the map $\sum_g \alpha(g) \rho(g) : V \to V$ is the zero map. Conclude that α has to be zero.

(c) Using (b) and the results from the lectures, show that the number of irreducible representations of the group G is equal to the number of conjugacy classes of G

Recall: In the lectures we had shown that the number of irreducible representations of a finite group is at most the number of conjugacy classes of the group, and this exercise shows that the numbers are in fact equal.