Problem sheet 2

In the first two exercises vector spaces can be over any field $\mathbb{K}$.

Exercise 1: *A construction of the tensor product*
Recall that we defined the tensor product of two vector spaces V and W as a vector space denoted by $V \otimes W$, equipped with a bilinear map $V \times W \to V \otimes W$ denoted by $(v, w) \mapsto v \otimes w$, which satisfies a certain universal property (see lecture notes). We saw that the tensor product is unique (up to isomorphism) if it exists. The purpose of this exercise is to show existence by an explicit construction, under the simplifying assumption that V and W are function spaces (it is easy to see that this can be done without loss of generality).

For any set X, denote by $\mathbb{K}^X$ the vector space of $\mathbb{K}$ valued functions on X, with addition and scalar multiplication defined pointwise. Assume that $V \subset \mathbb{K}^X$ and $W \subset \mathbb{K}^Y$ for some sets X and Y. For $f \in \mathbb{K}^X$ and $g \in \mathbb{K}^Y$, define $f \otimes g \in \mathbb{K}^{X \times Y}$ by

$$\big(f \otimes g\big)(x, y) \;=\; f(x)\, g(y).$$

Also set

$$V \otimes W \;=\; \mathrm{span}\,\{f \otimes g \;\big|\; f \in V,\; g \in W\},$$

so that the map $(f, g) \mapsto f \otimes g$ is a bilinear map $V \times W \to V \otimes W$.

(a) Show that if $(f_i)_{i \in I}$ is a linearly independent collection in V and $(g_j)_{j \in J}$ is a linearly independent collection in W, then the collection $(f_i \otimes g_j)_{(i,j) \in I \times J}$ is linearly independent in $V \otimes W$.

(b) Show that if $(f_i)_{i \in I}$ is a collection that spans V and $(g_j)_{j \in J}$ is collection that spans W, then the collection $(f_i \otimes g_j)_{(i,j) \in I \times J}$ spans $V \otimes W$.

(c) Conclude that if $(f_i)_{i \in I}$ is a basis of V and $(g_j)_{j \in J}$ is a basis of W, then $(f_i \otimes g_j)_{(i,j) \in I \times J}$ is a basis of $V \otimes W$. Conclude furthermore that $V \otimes W$, equipped with the bilinear map $\phi(f, g) = f \otimes g$ from $V \times W$ to $V \otimes W$, satisfies the universal property defining the tensor product.

Exercise 2: *The relation between* $\mathrm{Hom}(V, W)$ *and* $W \otimes V^*$
Let V, W be vector spaces, and recall that we set

$$\mathrm{Hom}(V, W) = \{T : V \to W \;\text{ linear}\} \qquad \text{and} \qquad V^* = \mathrm{Hom}(V, \mathbb{K}).$$

(a) For $w \in W$ and $\varphi \in V^*$, we associate to $w \otimes \varphi$ the following map $V \to W$

$$v \;\mapsto\; \varphi(v)\, w.$$

Show that the linear extension of this defines an injective linear map $W \otimes V^* \to \mathrm{Hom}(V, W)$.

(b) Show that if both V and W are finite dimensional, then the injective map in (a) is an isomorphism

$$W \otimes V^* \;\cong\; \mathrm{Hom}(V, W).$$

Show that under this identification, the rank of a tensor $t \in W \otimes V^*$ is the same as the rank of a matrix of the corresponding linear map $T \in \mathrm{Hom}(V, W)$.

(c) Recall that we have defined the dual of a representation (cf. *Problem sheet 1: Exercise 2*) and the tensor product of two representations. When V and W are representations of a group G, we have also defined a representation on $\mathrm{Hom}(V, W)$, by

$$(g.T)(v) \;=\; g.\big(T(g^{-1}.v)\big) \qquad \text{when } g \in G,\; T \in \mathrm{Hom}(V, W),\; v \in V.$$

Check that under the identification in (b), the two definitions agree.

In the remaining two exercises, vector spaces are again over the field $\mathbb{C}$ of complex numbers, as usually in this course.

Exercise 3: *Irreducible representations of abelian groups*

(a) Let G be an abelian (=commutative) group. Show that any irreducible representation of G is one dimensional. Conclude that (isomorphism classes of) irreducible representations can be identified with group homomorphisms $G \to \mathbb{C}^*$.

(b) Let $C_n \cong \mathbb{Z}/n\mathbb{Z}$ be the cyclic group of order n, i.e. the group with one generator c and relation $c^n = e$. Find all irreducible representations of C_n.

Exercise 4: *An example of tensor products and complete reducibility with* D_4
The group D_4 is the group with two generators, r and m, and relations $r^4 = e$, $m^2 = e$, $rmrm = e$. Recall that we have seen four one-dimensional irreducible representations of D_4 (cf. *Problem sheet 1: Exercise 4*). Let V be the two dimensional irreducible representation of D_4 given by

$$r \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad m \mapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Consider the four-dimensional representation $V \otimes V$, and show by an explicit choice of basis for $V \otimes V$ that it is isomorphic to a direct sum of the four one-dimensional representations.