Applications of the method of fundamental solutions for some inverse problems

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Applications to inverse problems

- Cauchy problems for elliptic equations
- Cauchy problems for heat equations
- Boundary identifications for elliptic equations
- Boundary identifications for heat equations
- Inverse source problems for heat equations
- ……

Convergence results

- Direct problems for Laplace's equation
  - Dirichlet problem on a circle domain(Bogomolny-1985; Katsurada+Okamoto-1988, 1989)
  - Dirichlet problem on an annular domain(Smyrlis+Karageorghis-2001, 2005)
  - Mixed boundary value problem on a simply connected domain (Z. C. Li-2009)
  - Dirichlet problem on an annular shaped domain (Z. C. Li-2009)
- Cauchy problems for Laplace's equation
  - Cauchy problem on an annular domain without regularization (Ohe+Ohnaka-2004)
  - For noisy data by a regularized MFS (Wei and Zhou, in press)
  - Irregular domain (?)
- Other inverse problems (?)

The Cauchy problem for Laplace's equation

$$\Delta u = 0, \quad in \quad \Omega_1 = \{x \mid \rho < |x| < r_1\} \quad or \quad \Omega_2 = \{x \mid r_2 < |x| < \rho\}$$

$$u|_\Gamma = f \qquad\qquad \Gamma = \{x \in R^2 \,|\, |x| = \rho > 0\}$$

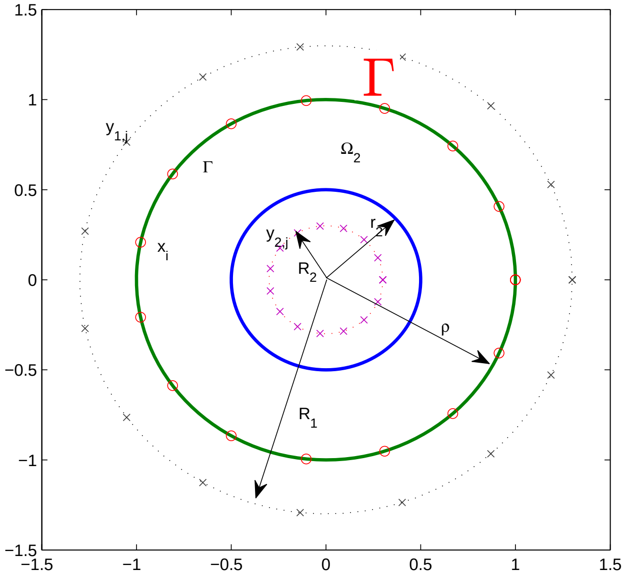
$$\frac{\partial u}{\partial \nu}|_\Gamma = g$$

Unknown: f, g

Known: f^δ, g^δ

Suppose: $|f^\delta(x) - f(x)| \le \delta$

$$|g^\delta(x) - g(x)| \le \delta$$



The method of fundamental solutions

- $G(x,y) = -\frac{1}{2\pi}\log|x-y|$ be a fundamental solution for Laplace's equation

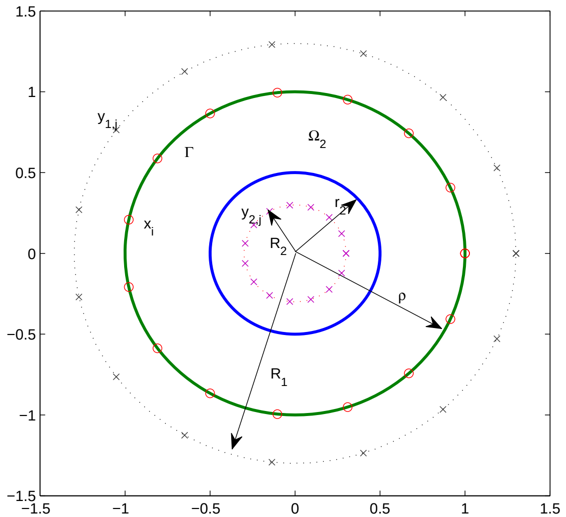
$$y_{1,j} = (R_1\cos\theta_j,\ R_1\sin\theta_j),\ j=0,1,\cdots,N-1, \quad \text{for outer source points,}$$
$$y_{2,j} = (R_2\cos\theta_j,\ R_2\sin\theta_j),\ j=0,1,\cdots,N-1, \quad \text{for inner source points,}$$
$$x_i = (\rho\cos\theta_i,\ \rho\sin\theta_i), j=0,1,\cdots,N-1, \quad \text{for collocation points}$$

An approximate solution (the MFS solution):

$$u_N(x) = q + \sum_{s=1}^{2}\sum_{j=0}^{N-1}\lambda_{s,j}G(x,y_{s,j})$$

where the invariant condition is satisfied

$$\sum_{s=1}^{2}\sum_{j=0}^{N-1}\lambda_{s,j} = 0$$



The coefficients q and $\{\lambda_{s,j}\}$ can be determined by the collocation method

$$u_N(x_i) = q + \sum_{s=1}^{2} \sum_{j=0}^{N-1} \lambda_{s,j} G(x_i, y_{s,j}) = f(x_i), \quad i = 0, 1, \cdots, N-1,$$

$$\frac{\partial u_N}{\partial \nu}(x_i) = \sum_{s=1}^{2} \sum_{j=0}^{N-1} \lambda_{s,j} \frac{\partial G}{\partial \nu}(x_i, y_{s,j}) = g(x_i), \quad i = 0, 1, \cdots, N-1$$

That is

$$A\Lambda = A \begin{pmatrix} q \\ \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & l^T & l^T \\ l & L_1 & L_2 \\ \mathbf{0} & M_1 & M_2 \end{pmatrix} \begin{pmatrix} q \\ \Lambda_1 \\ \Lambda_2 \end{pmatrix} = b = \begin{pmatrix} 0 \\ f \\ g \end{pmatrix},$$

where

$$\begin{aligned} \Lambda_s &= \left(\lambda_{s,0}, \lambda_{s,2}, \cdots, \lambda_{s,N-1}\right)^T \in R^N, s = 1, 2, \\ f &= (f(x_0), f(x_1), \cdots, f(x_{N-1}))^T \in R^N, \\ g &= (g(x_0), g(x_1), \cdots, g(x_{N-1}))^T \in R^N. \end{aligned}$$

Lemma: Suppose that f and g can be expanded into the Fourier series as

$$f(\rho e^{i\theta}) = \sum_{n\in Z} \eta_n e^{in\theta}, \quad g(\rho e^{i\theta}) = \sum_{n\in Z} \xi_n e^{in\theta}.$$

Assume that

$$|\eta_n| \le D\tau^{|n|}, \quad |\xi_n| \le D\tau^{|n|}, \quad n \in Z, \tag{1}$$

for some $D > 0$ and $0 < \tau < 1$. Then there exist positive constants $C_1 = C_1(r,\rho)$, $C_2 = C_2(r,\rho,a)$, $C_3 = C_3(\rho,a)$, $C_4 = C_4(\rho)$ such that

$$\sup_{0\le\theta\le 2\pi} \left| u_N(re^{i\theta}) - u(re^{i\theta}) \right| < \begin{cases} \left(C_1 + C_2 (\frac{\rho}{r})^N\right) D a^N, & a\rho < r < \rho, \\ (C_3 + C_4 N)\, D a^N, & r = \rho, \\ \left(C_1 + C_2 \left(\frac{r}{\rho}\right)^N\right) D a^N, & \rho < r < \rho/a, \end{cases} \tag{2}$$

for sufficiently large N. Here,

$$\begin{aligned} a &= \max\left\{K_1, K_2, \sqrt{\tau}\right\} < 1, \\ K_s &= \min\{\rho/R_s, R_s/\rho\} < 1, \quad s = 1, 2. \end{aligned}$$

[1] T. Ohe and K. Ohnaka. Uniqueness and convergence of numerical solution of the Cauchy problem for the Laplace equation by a charge simulation method. Japan J. Indust. Appl. Math., 21(3):339--359, 2004..

The regularized method of fundamental solutions for noisy Cauchy data

A regularized solution (the regularized MFS solution):

$$u_{\alpha,N}^{\delta}(x) = q_{\alpha}^{\delta} + \sum_{s=1}^{2} \sum_{j=0}^{N-1} \left(\lambda_{\alpha}^{\delta}\right)_{s,j} G(x, y_{s,j})$$

where the unknown coefficients q_{α}^{δ}, $\lambda_{\alpha}^{\delta}$ are determined by solving the following minimization problem

$$\min_{\Lambda} \|A\Lambda - b^{\delta}\|_2^2 + \alpha\|\Lambda\|_2^2$$

and $b^{\delta} = (0, (f^{\delta})^T, (g^{\delta})^T)^T$

$$f^{\delta} = \left(f^{\delta}(x_0), f^{\delta}(x_1), \cdots, f^{\delta}(x_{N-1})\right)^T \in R^N,$$

$$g^{\delta} = \left(g^{\delta}(x_0), g^{\delta}(x_1), \cdots, g^{\delta}(x_{N-1})\right)^T \in R^N.$$

It is known that the solution is

$$\Lambda_{\alpha}^{\delta} = \left(\alpha I + A^T A\right)^{-1} A^T b^{\delta}$$

Convergence analysis

$$|u^{\delta}_{\alpha,N}(re^{i\theta}) - u| \le |u^{\delta}_{\alpha,N}(re^{i\theta}) - u_N(re^{i\theta})| + |u_N(re^{i\theta}) - u|$$

$$\downarrow \qquad\qquad\qquad \downarrow N \to \infty$$

$$? \qquad\qquad\qquad 0$$

$$\left|u^{\delta}_{\alpha,N}(re^{i\theta}) - u_N(re^{i\theta})\right|^2$$
$$= |q^{\delta}_{\alpha} - q + \sum_{s=1}^{2}\sum_{j=0}^{N-1}((\lambda^{\delta}_{\alpha})_{s,j} - \lambda_{s,j})G(re^{i\theta}, y_{s,j})|^2$$
$$\le \{|q^{\delta}_{\alpha} - q|^2 + \sum_{s=1}^{2}\sum_{j=0}^{N-1}\left|(\lambda^{\delta}_{\alpha})_{s,j} - \lambda_{s,j}\right|^2\}\{1 + \sum_{s=1}^{2}\sum_{j=0}^{N-1}\Big(G(re^{i\theta}, y_{s,j})\Big)^2\}.$$

$$\begin{aligned} |G(re^{i\theta}, y_{s,j})| &\le \frac{1}{2\pi}\max\{|\log(R_1+r)|, |\log(R_1-r)|, |\log(R_2+r)|, |\log(r-R_2)|\} \\ &= \tilde{C}_5, \quad \text{for} \quad R_2 < r < R_1 \end{aligned}$$

$$1 + \sum_{s=1}^{2}\sum_{j=0}^{N-1} G(re^{i\theta}, y_{s,j})^2 \le 1 + 2N\tilde{C}_5^2 \le C_5^2 N,$$

Let the singular system of A be

$$(\mu_j, w_j, v_j), \quad j = 1, 2 \cdots, n, \quad n = 2N + 1$$

then

$$\Lambda = \sum_{j=1}^{n} \frac{1}{\mu_j}(b, v_j) w_j,$$

$$\Lambda_\alpha^\delta = \sum_{j=1}^{n} \frac{\mu_j}{\alpha + \mu_j^2}(b^\delta, v_j) w_j.$$

$$\begin{aligned}\left\|\Lambda_\alpha^\delta - \Lambda\right\|_2^2 &\leq 2\sum_{j=1}^{n}\left[\frac{\mu_j^2}{\alpha+\mu_j^2}\frac{(b^\delta - b, v_j)}{\mu_j}\right]^2 + 2\sum_{j=1}^{n}\left[\left(\frac{\mu_j^2}{\alpha+\mu_j^2} - 1\right)\frac{1}{\mu_j}(b, v_j)\right]^2 \\ &= \frac{n}{2\alpha}\delta^2 + \frac{4\alpha^2}{\mu_{min}^6} N E^2.\end{aligned}$$

where $\mu_{min} = \min\{\mu_1, \mu_2, \cdots, \mu_n\}$ and $E = \frac{2D}{1-\tau}$

$$\left|u_{\alpha,N}^\delta(re^{i\theta}) - u_N(re^{i\theta})\right| \leq \left(\sqrt{\frac{2N+1}{2\alpha}}\delta + \frac{2\alpha}{\mu_{min}^3}\sqrt{N}E\right) C_5 \sqrt{N}$$

The estimate of μ_{min}

Define $\omega = \exp(2\pi i/N),\quad F_{j,k} = \frac{1}{\sqrt{N}}\omega^{(j-1)(k-1)},\quad F_N = (F_{j,k})$

$$F = \begin{pmatrix} 1 & 0^T & 0^T \\ 0 & F_N & 0 \\ 0 & 0 & F_N \end{pmatrix}$$ —— unitary

There exists a suitable permutation matrix *H* such that

$$H^{-1}F^{-1}AFH = diag\,[\Phi_0, \Phi_1, \cdots, \Phi_{N-1}]$$

where

$$\Phi_0 = \begin{pmatrix} 0 & \sqrt{N} & \sqrt{N} \\ \sqrt{N} & \phi_{1,0} & \phi_{2,0} \\ 0 & \psi_{1,0} & \psi_{2,0} \end{pmatrix},$$

$$\Phi_i = \begin{pmatrix} \phi_{1,i} & \phi_{2,i} \\ \psi_{1,i} & \psi_{2,i} \end{pmatrix},\ i = 1, 2, \cdots, N-1.$$

From ref. [1], [2], we know

$$\phi_{s,0} = -\frac{1}{2\pi}\left|\log|\rho^N - R_s^N|\right|, \quad s = 1, 2,$$

$$\psi_{1,0} = \frac{N}{4\pi\rho}\left(\sum_{l\equiv 0, l\neq 0}\left(\frac{\rho}{R_1}\right)^{|l|}\right),$$

$$\psi_{2,0} = \frac{N}{4\pi\rho}\left(1 + \sum_{l\equiv 0}\left(\frac{R_2}{\rho}\right)^{|l|}\right),$$

and

$$\phi_{1,i} = \frac{N}{4\pi}\sum_{l\equiv i}\frac{1}{|l|}\left(\frac{\rho}{R_1}\right)^{|l|},$$

$$\phi_{2,i} = \frac{N}{4\pi}\sum_{l\equiv i}\frac{1}{|l|}\left(\frac{R_2}{\rho}\right)^{|l|},$$

$$\varphi_{1,i} = \frac{N}{4\pi\rho}\sum_{l\equiv i}\left(\frac{\rho}{R_1}\right)^{|l|},$$

$$\varphi_{2,i} = -\frac{N}{4\pi\rho}\sum_{l\equiv i}\left(\frac{R_2}{\rho}\right)^{|l|},$$

where the congruence $\equiv$ is taken modulo N.

[2] M. Katsurada and H. Okamoto, A mathematical study of the charge simulation method I, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. 35(1988), 507-518.

Furthermore

$$H^{-1}F^{-1}A^TAFH = diag\left[\Phi_0^T\Phi_0, \Phi_1^T\Phi_1, \cdots, \Phi_{N-1}^T\Phi_{N-1}\right]$$

Let μ_1, μ_2, μ_3 be the singular values of Φ_0, then we have

$$\begin{aligned}
\min\{\mu_1, \mu_2, \mu_3\} &\geq \sqrt{det(\Phi_0^T\Phi_0)/(tr(\Phi_0^T\Phi_0))^2} \\
&= det(\Phi_0)/(tr(\Phi_0^T\Phi_0)) \\
&\cdots \\
&\geq \frac{C_{10}N^2}{C_7N^2 + C_8N + C_9}, \\
&\geq C_{11},
\end{aligned}$$

Let $\mu_{i,1}, \mu_{i,2}$ be the singular values of Φ_i, then we have

$$\begin{aligned}
\min\{\mu_{i,1}, \mu_{i,2}\} &\geq |det(\Phi_i)|/(tr(\Phi_i^T\Phi_i))^{1/2} \\
&\cdots \\
&\geq \frac{C_{13}}{\sqrt{C_{12}}}\beta^{2N}, \ i = 1, 2, \cdots, N-1.
\end{aligned}$$

$$\beta = \min\{\rho/R_1, R_2/\rho\} < 1$$

$$\mu_{min} \geq C_6^{-1/3}\beta^{2N}$$

For $R_2 < r < R_1$, we have

$$\left|u_{\alpha,N}^{\delta}\left(re^{i\theta}\right) - u_N\left(re^{i\theta}\right)\right|$$
$$\leq \left(\frac{\sqrt{n}\delta}{\sqrt{2\alpha}} + 2\alpha C_6 \beta^{-6N}\sqrt{N}E\right) C_5\sqrt{N}$$
$$\leq \left(\frac{\delta}{\sqrt{2\alpha}} + 2\alpha C_6 \beta^{-6N} E\right) C_5\sqrt{3}N.$$

If take

$$N = N(\delta) = \frac{\log\sqrt{\delta}}{6\log\beta} \quad \text{and} \quad \alpha = c\delta$$

Then we have a convergence estimate

$$\left|u_{\alpha,N}^{\delta}\left(re^{i\theta}\right) - u_N\left(re^{i\theta}\right)\right|$$
$$\leq \sqrt{3}C_5\left(\frac{1}{\sqrt{2c}} + 2cC_6E\right)\frac{\sqrt{\delta}|\log\sqrt{\delta}|}{6|\log\beta|}$$
$$\leq K\sqrt{\delta}|\log\sqrt{\delta}|,$$

$$K = K(r, \rho, R_1, R_2, c) > 0$$

Theorem: Suppose that the Cauchy data f, g satisfy the conditions in Lemma and the measured data f^δ, g^δ satisfy the assumption, then there exist positive constants $C_1 = C_1(r,\rho)$, $C_2 = C_2(r,\rho,a)$, $C_3 = C_3(\rho,a)$, $C_4 = C_4(\rho)$, $K = K(r,\rho,R_1,R_2,c)$, if take $N = N(\delta) = \frac{\log\sqrt{\delta}}{6\log\beta}$ and choose the regularization parameter $\alpha = c\delta$ where $c > 0$ is a constant, then the following convergence result is satisfied for sufficiently small δ,

$$\left|u_{\alpha,N}^{\delta}\left(re^{i\theta}\right) - u\left(re^{i\theta}\right)\right| \tag{1}$$
$$\leq \begin{cases} K\sqrt{\delta}|\log\sqrt{\delta}| + \left(C_1 + C_2\left(\frac{\rho}{r}\right)^{N(\delta)}\right) Da^{N(\delta)}, & a\rho < r < \rho, \\ K\sqrt{\delta}|\log\sqrt{\delta}| + \left(C_3 + C_4 N(\delta)\right) Da^{N(\delta)}, & r = \rho, \\ K\sqrt{\delta}|\log\sqrt{\delta}| + \left(C_1 + C_2\left(\frac{r}{\rho}\right)^{N(\delta)}\right) Da^{N(\delta)}, & \rho < r < \rho/a, \end{cases}$$

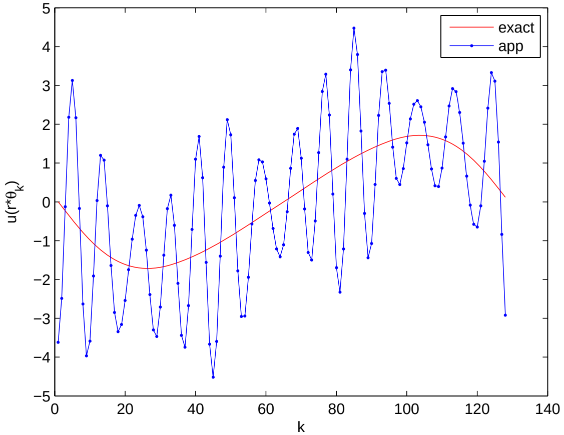
where

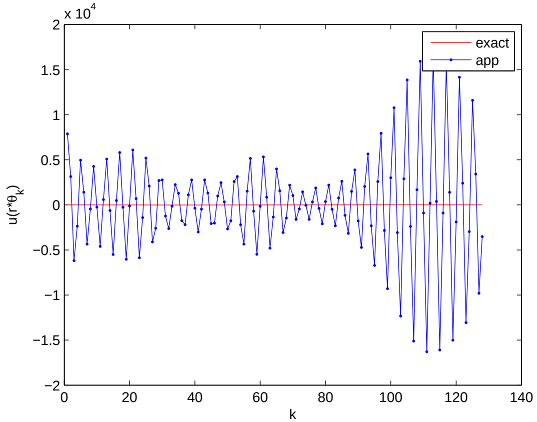
$$0 < a < 1$$

and $K = K(r,\rho,R_1,R_2,c) > 0$ is a constant independent of δ.

Numerical example

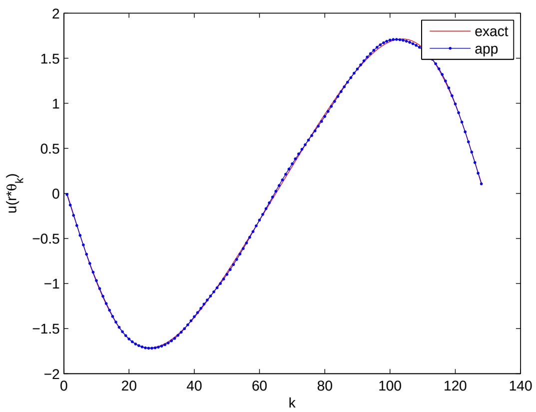
- Parameters $\rho = 1, \quad R_1 = 2, \quad R_2 = 0.5$
- Noisy data
$$\begin{aligned} f^\delta(x_k) &= f(x_k) + \delta \mathsf{rand}(k), \qquad \delta = 0.01 \\ g^\delta(x_k) &= g(x_k) + \delta \mathsf{rand}(k), \end{aligned}$$
- The root mean square error
$$\varepsilon(r) = \left(\frac{1}{m} \sum_{j=1}^{m} (u^\delta_{\alpha,N}(\bar{x}_j) - u(\bar{x}_j))^2 \right)^{1/2}, \qquad m = 128$$
- The exact solution $u(x,y) = -\frac{y}{(x-0.1)^2 + y^2},$
- Convergence region is $0.5 < r < 2$

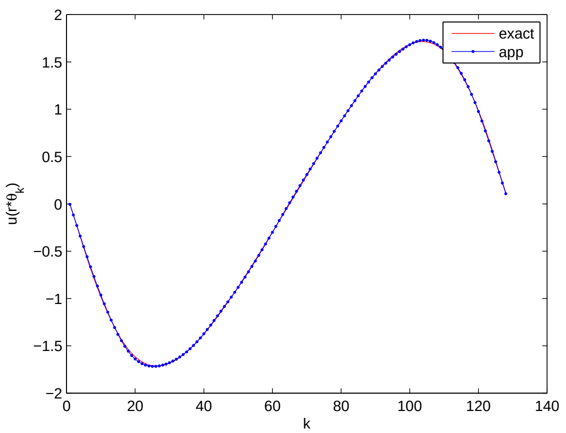




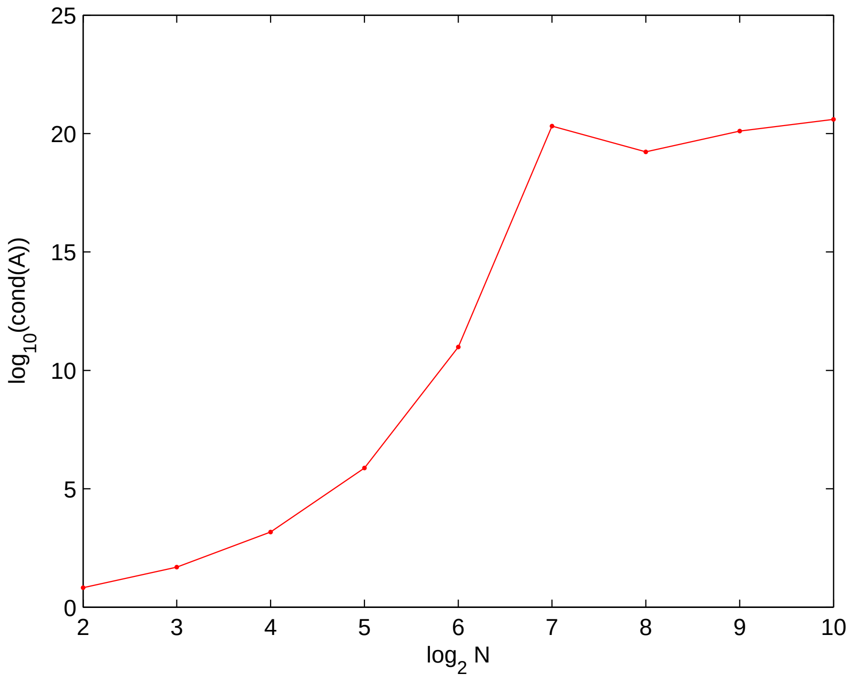
N=32, r=0.6, the MFS solution

N=64, r=0.6, the MFS solution

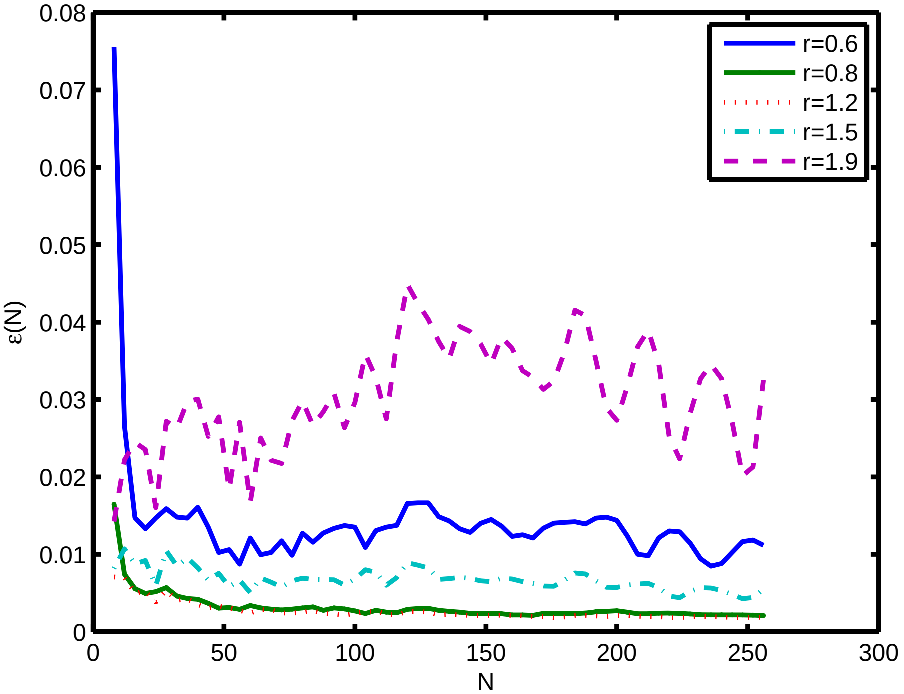




$\alpha = \delta = 0.01$ the regularized MFS solution



Condition numbers of *A* versus *N*



Root mean square errors versus N for r=0.6, 0.8, 1.2, 1.5, 1.9

Numerical results are not sensitive to N.

This result is better than our theorem.

Open Problems

- How about the convergence result when the Cauchy data are not analytic?
- How about convergence analysis for a general multi-connected domain? (In working)
- Convergence for a general simply connected domain?
- The regularized MFS can be used to solve other inverse problem?

Thank you!

The method of fundamental solution for solving two inverse problems

- The Cauchy problem of elliptic equation in a simply connected domain

$$\begin{cases} Lu = 0 \quad in\ \Omega \\ u = f \quad on\ \Gamma \\ \dfrac{\partial u}{\partial n} = g \quad on\ \Gamma \end{cases}$$

$$L = \triangle, \triangle \pm k^2 I$$

- Inverse heat conduction problem

$$u_t - \Delta u = 0, \ in \ D$$

$$u|_{t=0} = \varphi$$

$$u|_{\Gamma} = f$$

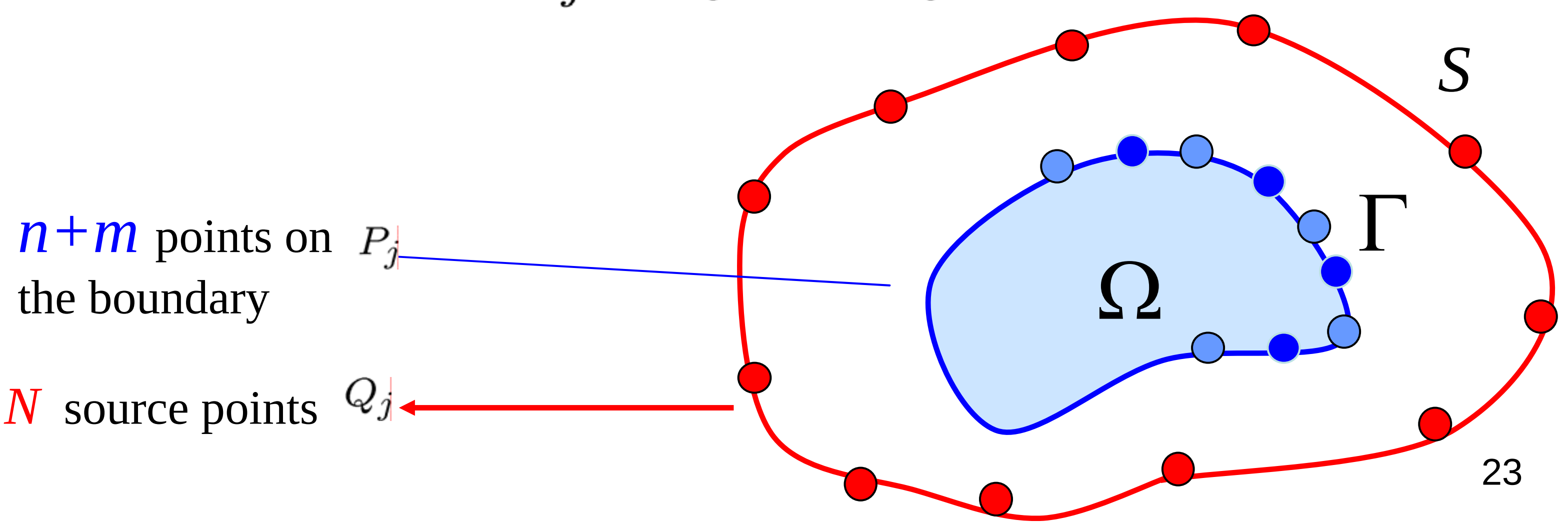
$$\frac{\partial u}{\partial n}|_{\Gamma} = g$$

$$D = \Omega \times (0, t_{\max})$$

MFS for elliptic equations in a simply connected domain

- Let *G(P,Q)* be a fundamental solution for an elliptic operator *L*.
- Choose a surface *S* containing Ω in its interior and N=*n*+*m* points $\{Q_j\}$ on *S*.
- Approximate *u* by

$$u_N = \sum_{j=1}^{N} \lambda_j G(P, Q_j)$$



Collocation method

- Choose collocation points $\{P_i\}_{i=1}^{n+m}$ on Γ and set

$$\sum_{j=1}^{N} \lambda_j G(P_i, Q_j) = f^\delta(P_i), i = 1, ..., n$$
$$\sum_{j=1}^{N} \lambda_j \frac{\partial G(P_i, Q_j)}{\partial n} = g^\delta(P_i), i = n+1, ..., n+m$$

=> $A\lambda = b^\delta$

$$A = \begin{pmatrix} G(P_i, Q_q) \\ \frac{\partial G}{\partial n}(P_j, Q_q) \end{pmatrix}, \quad b^\delta = \begin{pmatrix} f^\delta(P_i) \\ g^\delta(P_j) \end{pmatrix},$$

Note: For ill-posed problems, the condition number of A and $A^T A$ is very large

Tikhonov regularization

- A regularized solution λ_α is given by solving

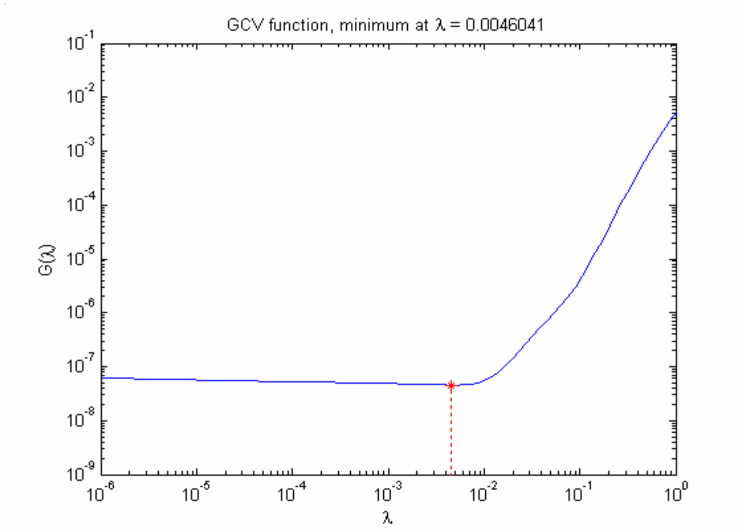
$$\min_{\lambda} \left\{ \|A\lambda - b^{\delta}\|^2 + \alpha^2 \|\lambda\|^2 \right\}$$

- The choice of a regularization parameter
    - Generalized Cross Validation method
    - L-curve method

Generalized cross validation: This method is to choose a regularization parameter which minimizes the GCV function

$$G(\alpha) = \frac{\|A\lambda_\alpha - b^\delta\|^2}{(trace(I_{n+m} - AA^I))^2}$$

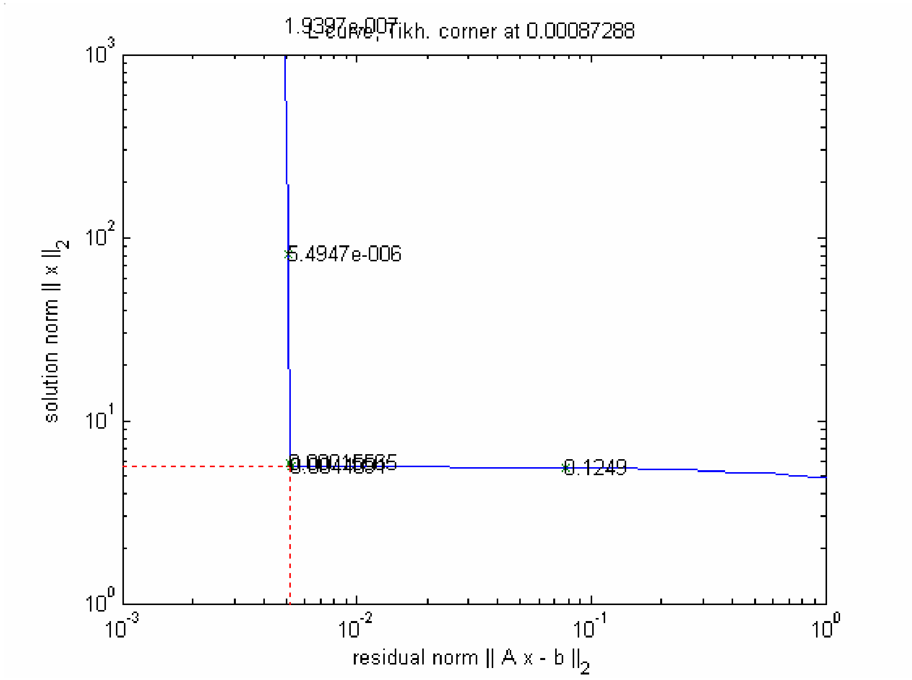
where A^I is a matrix which produces the regularized solution when multiplied with b^δ, i.e., $\lambda_\alpha = A^I b^\delta$.



The L-curve criterion : define the curve

$$L = \{ \quad (log(\|A\lambda_\alpha - b^\delta\|^2), log(\|\lambda_\alpha\|^2)), \quad \alpha > 0\}.$$

The curve is known as L-shape and a suitable regularization parameter α^* is one that near the "corner" of the L-curve

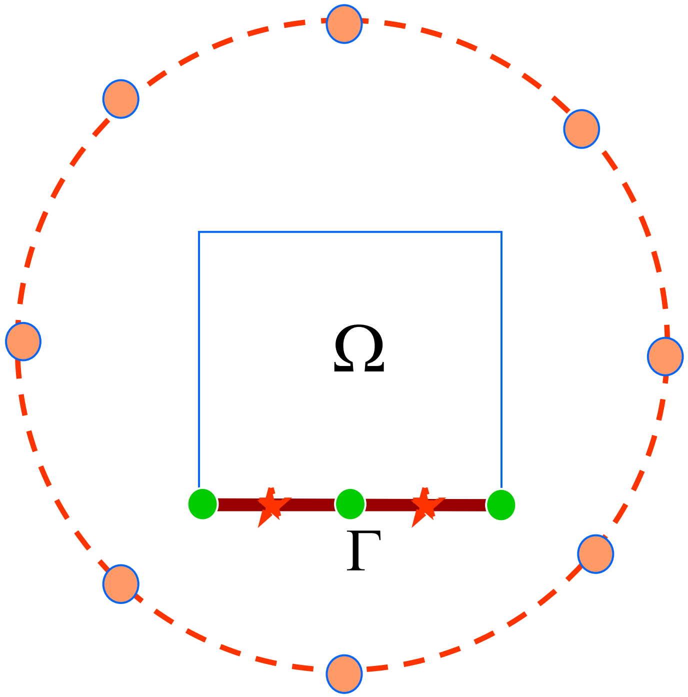


Example 1. $L=\Delta$ in 2D

Take:

$$\Delta u(x,y) = 0, \qquad (x,y) \in \Omega,$$

$$u(x,y) = x^3 - 3xy^2 + e^{2y}\sin(2x) - e^x\cos(y).$$



★ - - - Collocation points for Neumann condition

● - - - Collocation points for Dirichlet condition

○ - - - Source points

Example 2. $L = \Delta + 4I$ **in 2D**

$$\Delta u + 4u = 0$$

$$u(x,y) = 1/4 \sin(2\sqrt{2}x)e^{2y}$$

Example 3. $L = \Delta - 4I$ **in 2D**

$$\Delta u - 4u = 0$$

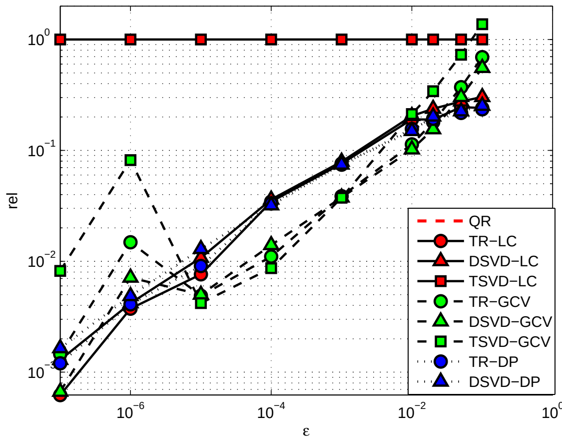
$$u(x,y) = 1/4 \sin(2x)e^{2\sqrt{2}y}$$

Numerical results for examples 1, 2 , 3

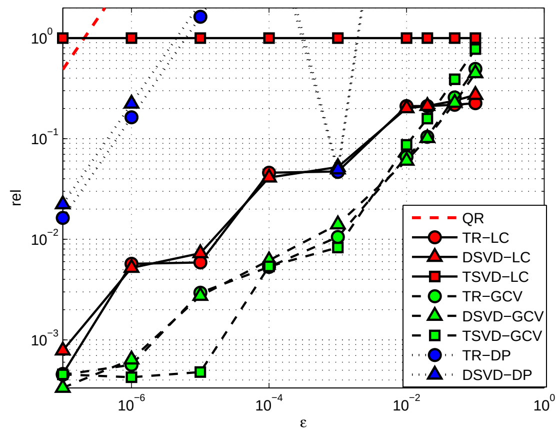
Numerical results are computed by using
Tikhonov regularization and GCV

rel(u)	Exact input data	Noisy input data
Example 1	2.6631e-5	0.0386
Example 2	2.0235e-5	0.0106
Example 3	2.5541e-6	0.0374

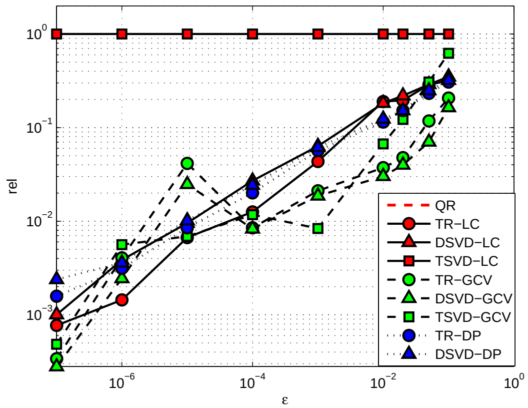
Relative error level is 0.001



Laplace Equation

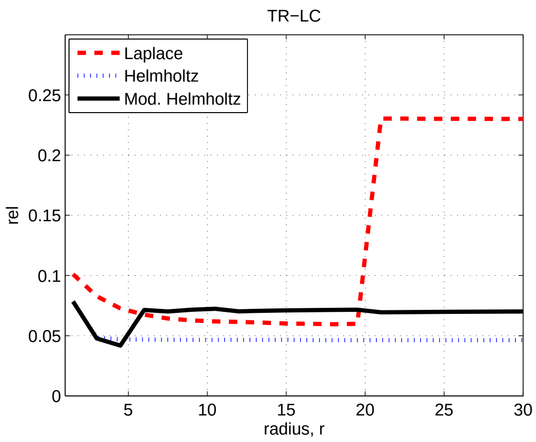


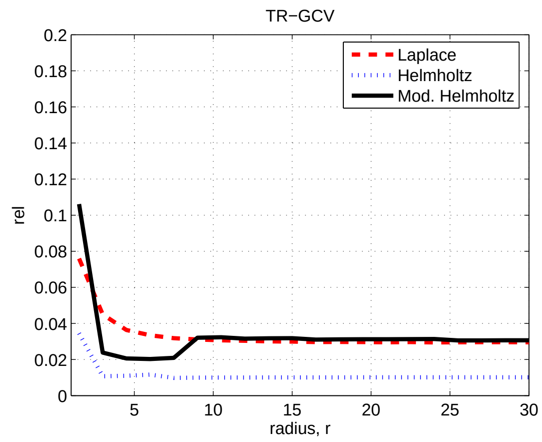
Modified Helmholtz Equation



Helmholtz Equation

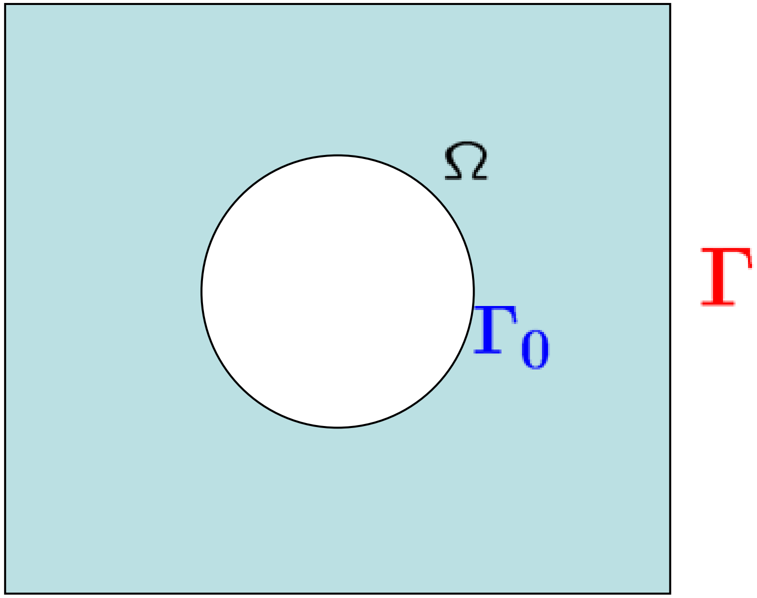
Relative root mean square errors versus the noise levels





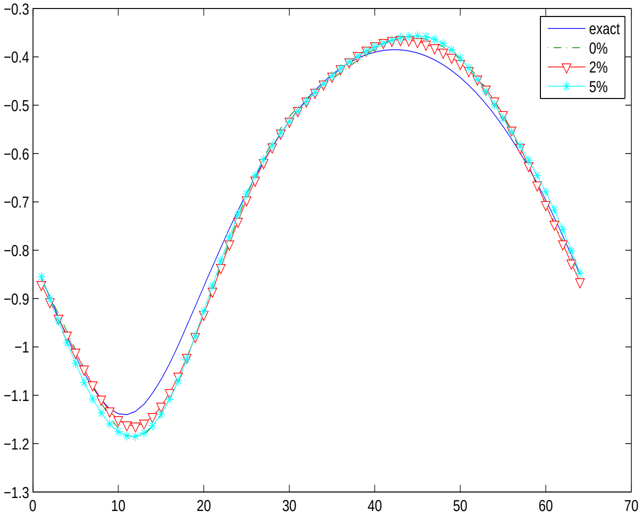
The relative error profiles versus different radii of source points $1.5 \leq r \leq 30$ with a fixed relative noise level $\varepsilon=0.001$.

Example 4. Multi-connected domain



r=0.5

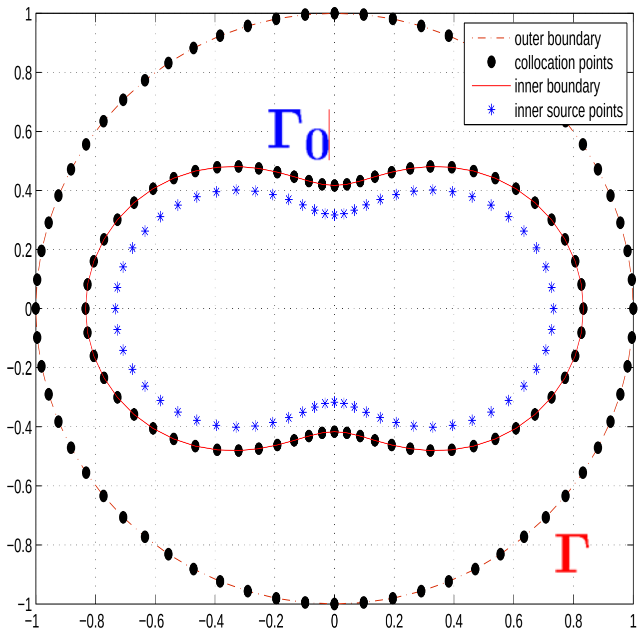
$$u = \log\left((x-0.1)^2 + (y-0.15)^2\right).$$

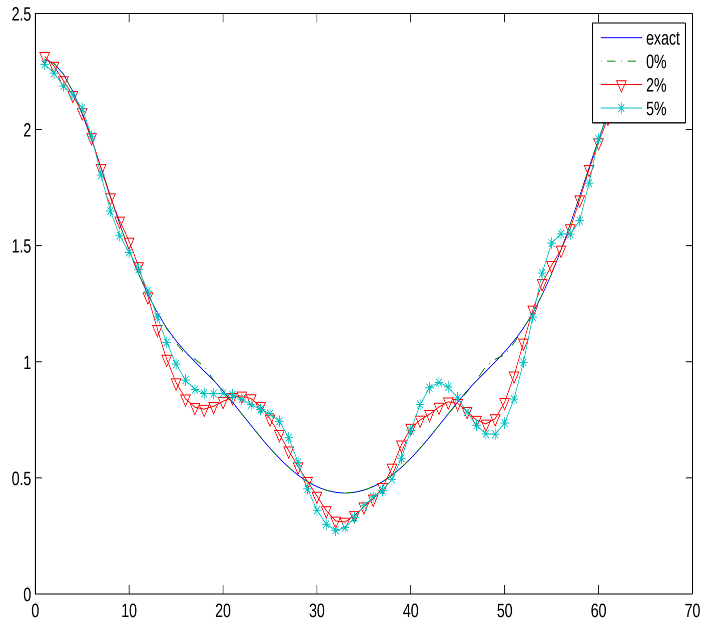


N=64

$$\begin{aligned} \triangle u &= 0 \quad in\ D, \\ u|_{\Gamma} &= 2 + x, \\ u|_{\Gamma_0} &= \exp(x). \end{aligned}$$

The Neumann data on Γ were obtained by solving a direct problem.





Reconstruction of a moving boundary from Cauchy data in one dimensional heat equation

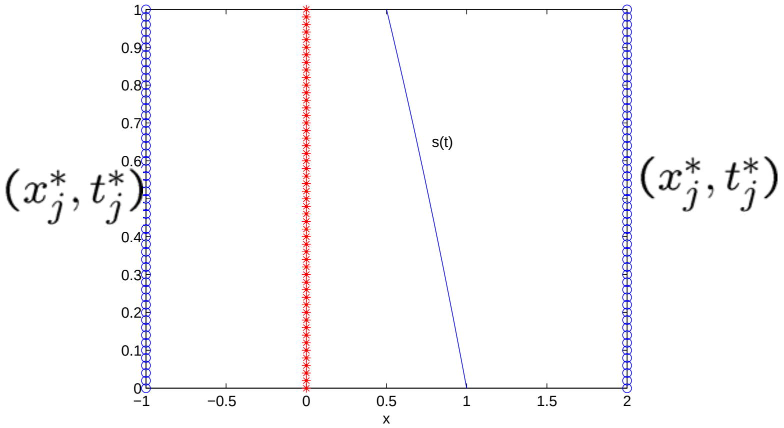
$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 < x < s(t), \quad 0 < t < T.$$

$$u(0,t) = u_0(t), \quad 0 < t < T,$$

$$\frac{\partial u(0,t)}{\partial x} = q_0(t), \quad 0 < t < T,$$

$$u(s(t),t) = u_s(t).$$

$$\Longrightarrow x = s(t)$$



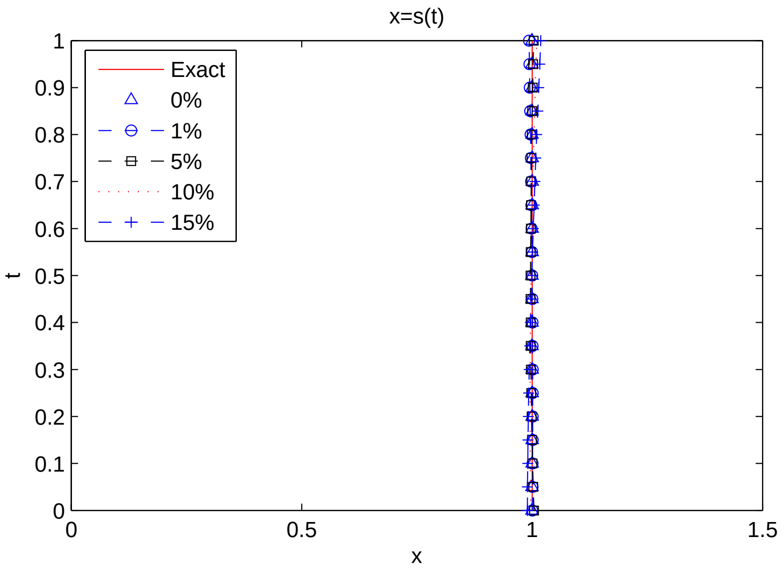
The MFS solution is

$$u_n(x,t) = \sum_{j=1}^{n} \lambda_j G(x - x_j^*, t - t_j^* + \tau), \quad \tau > T$$

By $u_n(x,t)=0$, we can get an approximate boundary $x=s^*(t)$

Example 1: $u(x,t) = 1 - x.$

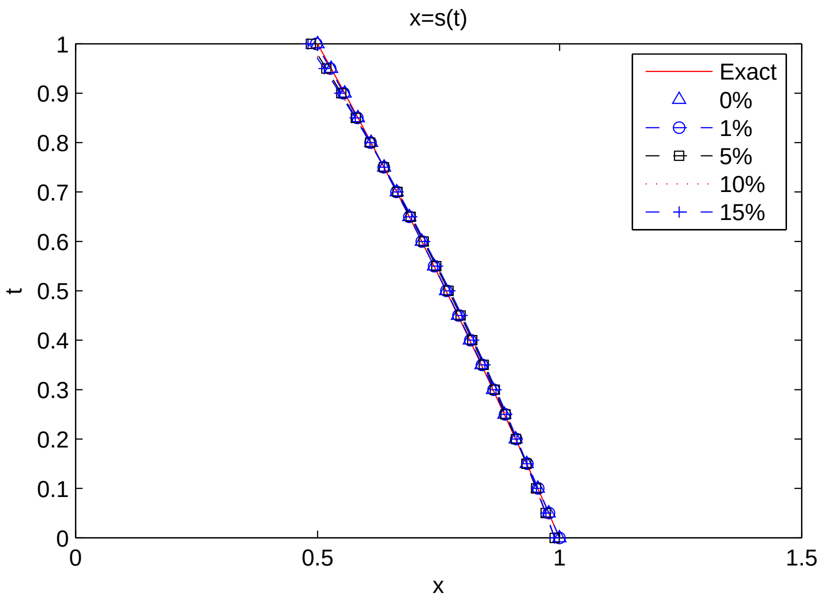
$$s(t) \equiv 1, u_s(t) = 0.$$



Example 2: $u(x,t) = \left(x + \frac{5}{4}\right)^2 + 2\left(t - \frac{81}{32}\right)$

$$u_s(t) = 0$$

$$s(t) = \left(\frac{81}{16} - 2t\right)^{1/2} - \frac{5}{4}.$$



Example 3:

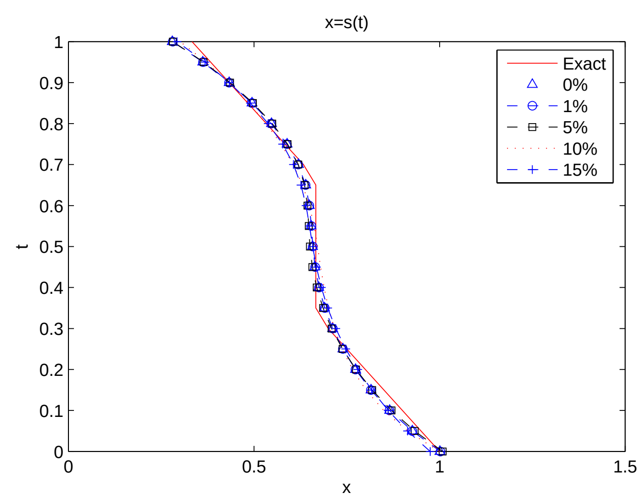
$$s(t) = \begin{cases} 1 - t, & \text{for} \quad 0 < t \leq \frac{1}{3}, \\ \frac{2}{3}, & \text{for} \quad \frac{1}{3} < t \leq \frac{2}{3}, \\ \frac{4}{3} - t, & \text{for} \quad \frac{2}{3} < t \leq 1, \end{cases}$$

$$u_s(t) \equiv 0$$

$$q_0(t) = 1$$

$$u(x, 0) = x - 1$$

u(0,t) is obtained by solving a direct problem using the finite difference method.



Details can be found in the following papers

[1] Y. C. Hon and T. Wei, The method of fundamental solution for solving multidimensional inverse heat conduction problems, *Computer Modeling in Engineering & Sciences*, 7(2005), No. 2, 119-132.

[2] Y. C. Hon and T. Wei, A fundamental solution method for inverse heat conduction problem, *Engineering Analysis with Boundary Elements*, 28(2004), 489-495.

[3] T. Wei, Y. C. Hon and L. Ling, Method of fundamental solutions with regularization techniques for Cauchy problems of elliptic operators, *Engineering Analysis with Boundary Elements*, 31（2007), 373-385.

[4] D. Y. Zhou and T. Wei, The method of fundamental solution for solving a Cauchy problem of Laplace's equation in a multi-connected domain, *Inverse Problems in Science and Engineering*, 16(3) 389-411, 2008

[5] T. Wei, M. Yamamoto, Reconstruction of a moving boundary from Cauchy data in one dimensional heat equation, *Inverse Problems in Science and Engineering*, 17(4), 2009, 551-567.

[6] T. Wei and Y. S. Li, An inverse boundary problem for one-dimensional heat equation with a multilayer domain, *Engineering Analysis with Boundary Elements*, 33（2009), 225-232

[7] T. Wei and D.Y. Zhou, Convergence analysis for the Cauchy problem of Laplace's equation by a regularized method of fundamental solutions, AiCM, in press.

Thank you!