Regularization Methods for Nonlinear Inverse Problems – Ideas for Challenging Joint Research Projects

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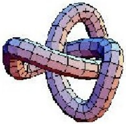

For initiation of joint research to be presented at Workshop of

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From German side research groups of the following colleagues are involved in project ideas of this talk:

THORSTEN HOHAGE (University of Göttingen)

THOMAS SCHUSTER (H. Schmidt Univ. Hamburg)

ULRICH TAUTENHAHN (UAS Zittau/Görlitz)

BERND HOFMANN (Chemnitz Univ. of Technology)

Outline

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1. Introduction
2. Variational inequalities and convergence rates
3. Error profiles in regularization for ill-posed problems with noisy operators and noisy data
4. Variational inequalities in combination with a posteriori parameter choice rules in Banach spaces
5. Parameter identification in partial differential equations
6. Inverse problems with Poisson data

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Nonlinear inverse problems

Let U, V denote infinite dimensional Hilbert or Banach spaces with norms $\|\cdot\|_U, \|\cdot\|_V$,

$F: D(F) \subseteq U \longrightarrow V$ **nonlinear forward operator**

with domain $D(F)$.

We consider the **ill-posed nonlinear** operator equation

$$F(u) = v \qquad (u \in D(F) \subseteq U,\ v \in V) \qquad (*)$$

with solution $u^* \in \mathcal{D}(F)$ and exact right-hand side $v^* = F(u^*)$.

For the stable approximate solution of $(*)$ two types of approaches are well-established:

- **Variational regularization methods**
  Regularized solutions minimize Tikhonov type functionals
  Under consideration in this talk!

- **Iterative regularization methods**
  Iterations with stopping rule as regularization parameter
  Incomplete selection of experts in Germany: Professors:
  MARTIN HANKE (Univ. of Mainz),
  PETER MAASS (Univ. of Bremen),
  ANDREAS RIEDER (University of Karlsruhe),
  THOMAS SCHUSTER (H. Schmidt Univ. Hamburg).

Challenging practical aspects of variational regularization:

- Exploit adapted variants of the methods under given a priori information concerning data noise model and expected solution
- Use appropriate (heuristic) rules for choosing the regularization parameter(s)

Challenging theoretical aspects:

- Find conditions for proving convergence and in particular convergence rates of the methods
- Impact of smoothness (of both solution and forward operator) on the behaviour of regularized solutions

variational regularization (Tikhonov type regularization)

with stabilizing functional $\Omega : \mathcal{D}(\Omega) \subseteq U :\to \mathbb{R}$

and for noisy data v^δ, e.g. assuming $\|v^* - v^\delta\|_V \le \delta$.

Regularized solutions u_α^δ minimize s.t. $u \in \mathcal{D}(F) \cap \mathcal{D}(\Omega)$:

$$T_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^p + \alpha\,\Omega(u) \to \min\,.$$

We exploit for Ω with subdifferential $\partial\Omega$ the **Bregman distance** $D_\xi(\cdot, u)$ of Ω at $u \in U$ and $\xi \in \partial\Omega(u) \subseteq U^*$ defined as

$$D_\xi(\tilde{u}, u) := \Omega(\tilde{u}) - \Omega(u) - \langle \xi, \tilde{u} - u \rangle_{U^*,U} \quad (u, \tilde{u} \in \mathcal{D}(\Omega) \subseteq U)\ .$$

The set

$$\mathcal{D}_B(\Omega) := \{u \in \mathcal{D}(\Omega) : \partial\Omega(u) \neq \emptyset\}$$

is called Bregman domain. An element $u^\dagger \in \mathcal{D}$ is called an Ω**-minimizing solution** if

$$\Omega(u^\dagger) = \min\{\Omega(u) : F(u) = v^*,\ u \in \mathcal{D}\} < \infty\ .$$

Such Ω-minimizing solutions exist under Assumption 1 if $(*)$ has a solution $u \in \mathcal{D}$.

For results on **existence, stability and convergence** see
$\triangleright$ H./KALTENBACHER/P./SCHERZER 2007, $\triangleright$ PÖSCHL 2008.

Example: Standard situation in Hilbert spaces

U, V **Hilbert spaces**,

$\Omega(u) := \|u - \bar{u}\|_U^2, \qquad u^*$ is called $\bar{u}$-minimum norm solution

$$T_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^2 + \alpha\, \|u - \bar{u}\|_U^2$$

$\mathcal{D}(\Omega) = \mathcal{D}_B(\Omega) = U,$ since $\partial\Omega(u)$ is singleton

$$\xi := \Omega'(u^*) = 2(u^* - \bar{u})$$

$$D_\xi(\tilde{u}, u) = \|\tilde{u} - u\|_U^2$$

Example: Regularization with differential operators

U, V **Hilbert spaces**,

$\Omega(u) := \|Bu\|_U^2$ with unbounded s.a. operator $B : \mathcal{D}(B) \subset U \to U$

$$T_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^2 + \alpha \, \|Bu\|_U^2$$

$\mathcal{D}(\Omega) = \widetilde{U}$ Hilbert space with stronger norm $\|u\|_{\widetilde{U}} := \|Bu\|_U$

$$\xi := \Omega'(u^*) = 2B^2 u^*$$

$D_\xi(\tilde{u}, u) = \|B(\tilde{u} - u)\|_U^2$ with $\mathcal{D}_B(\Omega) = \mathcal{D}(B^2)$

Example: Power type penalties in Banach spaces

$$U, V \ \textbf{Banach spaces}, \quad \Omega(u) := \frac{\|u\|_U^q}{q} \ (q > 1),$$

$$T_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^p + \alpha \left(\frac{1}{q} \|u\|_U^q \right) \qquad (p, q > 1)$$

$\mathcal{D}(\Omega) = \mathcal{D}_B(\Omega) = U$, since $\Omega(u)$ is differentiable with

$\xi := \Omega'(u^*) = J_q(u^*)$ with $J_q : U \to U^*$ duality mapping

$$D_\xi(\tilde{u}, u) = \frac{1}{q}\|\tilde{u}\|_U^q - \frac{1}{q}\|u\|_U^q - \langle J_q(u), \tilde{u} - u \rangle_{U^*,U}$$

In recent publications the distinguished role of

variational inequalities

$$\langle \xi, u^\dagger - u \rangle_{U^*,U} \le \beta_1 D_\xi(u, u^\dagger) + \beta_2 \|F(u) - F(u^\dagger)\|_V^\kappa \quad (**)$$

for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$ with some $\xi \in \partial\Omega(u^\dagger)$,
two multipliers $0 \le \beta_1 < 1, \quad \beta_2 \ge 0$,
and an exponent $\kappa > 0$ was elaborated.

Classical theory of convergence rates in Tikhonov regularization for nonlinear ill-posed equations in Hilbert spaces due to

- ▷ ENGL/KUNISCH/NEUBAUER *Inverse Problems* 1989

for the standard minimization problem

$$T_\alpha^\delta(u) := \|F(u) - v^\delta\|_V^2 + \alpha\,\|u - \bar{u}\|_U^2 \to \min$$

separates the following both components

1. Smoothing properties and nonlinearity of the forward operator

$$\|F(u) - F(u^\dagger) - F'(u^\dagger)(u - u^\dagger)\|_V \le \frac{L}{2}\,\|u - u^\dagger\|_U^2\,.$$

2. Solution smoothness

$$u^\dagger - \bar{u} \,=\, F'(u^\dagger)^*\,w, \qquad L\|w\|_V < 1\,.$$

Both ingredients are **united in variational inequalities.**

This allows handling of **non-smooth** situations for $u^\dagger$ and F!

Theorem – convergence rates & variational inequalities

Under the standing assumptions and assuming the existence of an Ω-minimizing solution from the Bregman domain $u^\dagger \in \mathcal{D}_B(\Omega)$ let there exist an element $\xi \in \partial\Omega(u^\dagger)$ and constants $0 \le \beta_1 < 1,\ \beta_2 \ge 0$, and $0 < \kappa \le 1$ such that the variational inequality $(**)$ holds for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$.

Then for $p > 1$ we have the convergence rate

$$D_\xi(u^\delta_{\alpha(\delta)}, u^\dagger) \,=\, \mathcal{O}\left(\delta^\kappa\right) \quad \text{as} \quad \delta \to 0$$

for an a priori parameter choice $\alpha(\delta) \asymp \delta^{p-\kappa}$.

Comparison of Hölder convergence rates

$D_\xi(u^\delta_{\alpha(\delta)}, u^\dagger) = \mathcal{O}(\delta^\nu)$ for variational regularization with $\psi(t) = t^p$:

Low rate world $0 < \nu \le 1$: Proof ansatz $T^\delta_\alpha(u^\delta_\alpha) \le T^\delta_\alpha(u^\dagger)$ under low order source conditions

$0 < \nu = \kappa \le 1$ obtained for arbitrary reflexive Banach spaces U and V, $p > 1$, and diversified properties expressed by κ

with a priori choice $\alpha(\delta) \asymp \delta^{p-\nu}$

Enhanced rate world $\nu > 1$: Proof ansatz $T^\delta_\alpha(u^\delta_\alpha) \le T^\delta_\alpha(u^\dagger - z)$ under high order source conditions

$1 < \nu \le \frac{2s}{s+1}$ obtained for s-smooth Banach space V ($s > 1$)

and a priori choice $\alpha(\delta) \asymp \delta^{(p-1)\frac{s}{s+1}}$

Upper rate limit: $\nu = \frac{4}{3}$ in Hilbert space V ($s = 2$)

Optimal rate independent of p $\ge$ 1!

($\triangleright$ sc Neubauer/Hein/H./Kindermann/Tautenhahn 2009/10)

Structural conditions of F locally in $u^\dagger$ can be expressed by:

Definition (degree of nonlinearity)

Let $0 \le c_1, c_2 \le 1$ and $c_1 + c_2 > 0$. We define F to be **nonlinear of degree** (c_1, c_2) for the Bregman distance D_ξ of Ω at $u^\dagger$ and at $\xi \in \partial\Omega(u^*)$ if there is a constant $K > 0$ such that

$$\left\|F(u)-F(u^\dagger)-F'(u^\dagger)(u-u^\dagger)\right\|_V \le K \left\|F(u)-F(u^\dagger)\right\|_V^{c_1} D_\xi(u,u^\dagger)^{c_2}$$

for all $u \in \mathcal{M}_{\bar{\alpha}}(\rho)$.

Extensions to **non-metric misfit functionals** S

(e.g., Kullback-Leibler divergence) are of interest in theory and practice:

$$T_\alpha^\delta(u) := S(F(u), v^\delta) + \alpha\,\Omega(u) \to \min\,.$$

Use of new variats of the **penalty functional** Ω.

Extensions to **non-standard error measures**

(more than Bregman distance) are of interest.

How to handle regularization in **non-reflexive** and **non-separable** Banach spaces?

Error profiles in regularization for ill-posed problems with noisy operators and noisy data

Joint research ideas of the Chemnitz Research group with

Dr. SHUAI LU (RICAM Linz and Fudan University Shanghai)

and

Professor ULRICH TAUTENHAHN (Zittau)

To be included:

Yuanyuan Shao (PhD student, Chemnitz/Zittau).

and possibly

Research group of Professor CHU-LI FU (Lanzhou University).

Ingredients for the project:

1. ▷ B. HOFMANN, P. MATHÉ:
Analysis of profile functions for general linear regularization methods. *SIAM Journal Numerical Analysis* 45 (2007), 1122-1141.

Impact of smoothness on regularization + nonlinearity

2. **Method of approximate source conditions** using distance functions for measuring the violation of benchmark source conditions to be extended to the **case of noisy operators**

3. Former research proposal by PEREVERZEV, LU, TAUTENHAHN entitled

"**Multi-parameter regularization** for ill-posed problems with noisy data".

Variational inequalities in combination with a posteriori parameter choice rules in Banach spaces

Research idea for joint work with German colleagues by

Dr. Shuai Lu in cooperation with Prof. Sergei V. Pereverzev

Variational inequalities on level sets form a new powerful tool for convergence rates of regularization in Banach spcaces.

Only a few rules for the choice of the regularization parameter were adapted to nonlinear regularization and to a Banach space setting:

- discrepancy method
- quasi-optimality method

We may explore the results for other rules

- modified L-curve method
- balancing method

Parameter identification in partial differential equations

- Stable approximate solution of PI problems frequently requires regularization approaches
- Wide ranges of applications in natural sciences and engineering
- Examples in chapter for Handbook of Imaging 2010 (Ed. by O. SCHERZER) with

  Professor JIN CHENG (Fudan University Shanghai)

  ▷ J. CHENG AND B. HOFMANN:
  Regularization Methods for Nonlinear Ill-Posed Problems.

Research group of Professor THOMAS SCHUSTER (Hamburg):

Structural Health Monitoring (SHM) of carbon fiber reinforced plastics

Goals:

- early detection of delaminations and cracks
- enhanced operating safety
- optimization of maintanance intervals
- cost efficieny due to optimized operation

SHM as parameter identification

Inverse Problem:

Detection of the material tensor $C = C_{ij}(x)$ from out-of-plane displacement measurements $\mathbf{u}_z^\delta$ of emitted Lamb waves

Optimal control including appropriate FEM solvers:

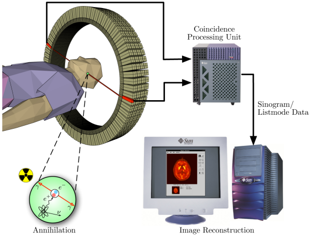
$$\min_{(u,C)\in U'\times V} \mathcal{J}(u,C) \qquad \text{subject to} \qquad \mathcal{F}(C)u = 0$$

where $\mathcal{F}(C)u = 0$ is the anisotropic wave equation and the object function is e.g. given as

$$\mathcal{J}_\alpha(u,C) = \frac{1}{2}\|u_z - u_z^\delta\|_U^2 + \frac{\alpha}{2}\|C - \bar{C}\|_V^2$$

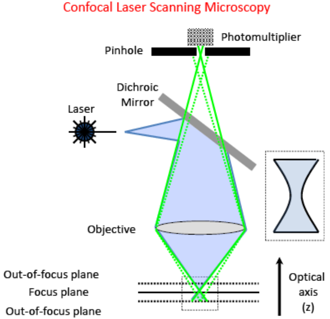
Research group of Professor THORSTEN HOHAGE (Göttingen):

positron emission tomography



source: Wikipedia

confocal microscopy



coherent x-ray imaging

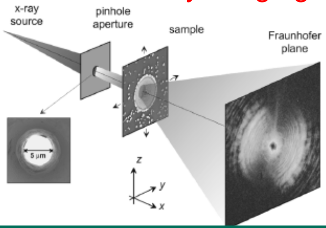
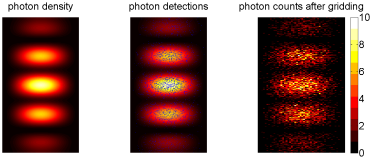


illustration of photon count data



Proposed continuous framework

- Let U, V Banach spaces, $V \subset L^1(D)$. Formally we want to approximately solve an operator equation $F(u) = v$ with an injective operator $F : D(F) \subset U \to V$ satisfying

$$F(u) \geq 0 \qquad \text{for all } u \in D(F).$$

- data: $v = \sum_{i=1}^{n} \delta_{x_i}$ drawn from a Poisson process with mean $g^\dagger = F(u)$, i.e. for all measurable $D' \subset D$ the number $\#\{i : x_i \in D'\}$ is a Poisson distributed random variable with expected value $\int_{D'} g(x)\, dx$.
- natural variational (Tikhonov-type) regularization:

$$\mathrm{KL}(F(u); v) + \alpha\Omega(u) = \min!$$

KL = Kullback-Leibler divergence; Ω penalty term

possible Finnish collaborators: M. LASSAS, S. SILTANEN