Qualitative Results in Inverse Scattering and Impedance Tomography

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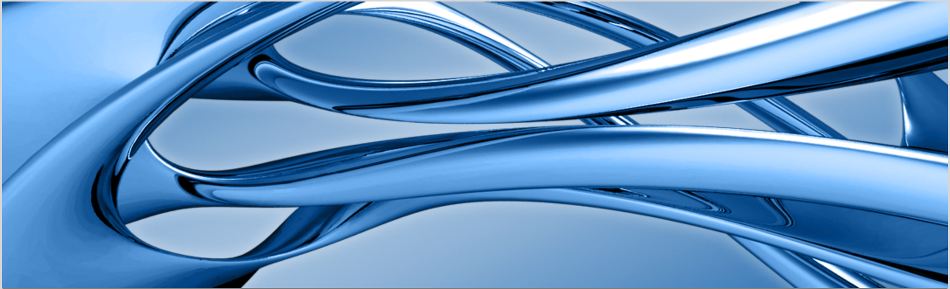

Overview

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- Example of a problem in impedance tomography
- Example of an inverse scattering problem
- Questions of uniqueness and stability
- Characterizations of shape
- Transmission eigenvalues
- Asymptotics
- Stochastic models

Example (A)

Impedance Tomography

$B \subset \mathbb{R}^2$ bounded domain with smooth ∂B and $\sigma \in L^\infty(B)$ of the form

$$\sigma = \begin{cases} \sigma_0 & \text{in } B \setminus D, \\ \sigma_0 + q & \text{in } D, \end{cases} \qquad \text{where } \overline{D} \subset B\,.$$

Direct Problem: Given $\sigma \in L^\infty(B)$ and $f \in H_\diamond^{-1/2}(\partial B)$, determine $u \in H^1(B)$ with

$$\operatorname{div}(\sigma \nabla u) = 0 \text{ in } B\,, \quad \sigma \frac{\partial u}{\partial \nu} = f \text{ on } \partial B\,.$$

Inverse Problem: Given Neumann-Dirichlet map $\Lambda : H_\diamond^{-1/2}(\partial B) \to H_\diamond^{+1/2}(\partial B)$, $f \mapsto u|_{\partial B}$, determine σ or at least $D = \operatorname{supp}(q)$!

Example (B)

Scattering Problem for time-harmonic acoustic waves:

Contrast $q \in L^\infty(\mathbb{R}^d)$ (where $d = 2, 3$) with $q = 0$ outside of bounded domain $D \subset \mathbb{R}^d$,
incident field $u^i(x) = \exp(ik\, x \cdot \hat{\theta})$ with wave number $k = \omega/c > 0$ and direction $\hat{\theta} \in S^{d-1}$ (unit sphere).

Direct Problem: Given q and u^i, determine $u \in H^1_{loc}(\mathbb{R}^d)$ with

$$\Delta u + k^2(1 + q)u = 0 \text{ in } \mathbb{R}^d\,, \quad u = u^i + u^s\,,$$

and u^s satisfies a radiation condition.

rad. cond. implies: $$u^s(x) = \frac{\exp(ik|x|)}{|x|^{(d-1)/2}} \left[u^\infty(\hat{x}, \hat{\theta}, k) + \frac{1}{|x|} \right]$$

Inverse Problem: Given $u^\infty(\hat{x}, \hat{\theta}, k)$ for many (all?) $\hat{x}, \hat{\theta} \in S^{d-1}$, $k > 0$, determine q or at least $D = \mathrm{supp}(q)$!

Remarks:

Variations wrt

- boundary conditions on ∂D (impenetrable inclusions/scatterers): $u = 0$ or $\partial u / \partial \nu = 0$ or $\partial u / \partial \nu + \lambda u = 0$ on ∂D
- models, e.g.: diffuse tomography, optical tomography, elastic wave propagation (Navier's equations), electromagnetic wave propagation (Maxwell's equations), coupled equations
- geometries: layered media, gratings, wave guides, cracks

Uniqueness, Stability

- Uniqueness for 3D inverse scattering problem: Nachman, Novikov, Ramm 1988, 2D - case just settled.
- Inverse scattering by Dirichlet or Neumann boundary conditions: Uniqueness for one wave number and one (or dimension d, resp.) incident wave for polygonal regions: Elschner, Yamamoto, H. Liu et al since 2003, general case open.
- Uniqueness for 2D impedance tomography answered by Astala, Päivärinta et al since 2006, 3D - case is open.
- Backscattering (next slide)
- Stability EIT: Alessandrini, di Cristo et al since 1988
- Stability inverse scattering Problem: Isakov since 1992, Hohage, Potthast since 2000

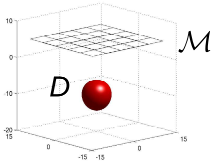
Backscattering

Example electrostatic backscattering: Given an insulating $D \subset \mathbb{R}^3$, determine $u_x = \Phi(\cdot, x) + U_x$ (where Φ fundamental sol., U_x smooth, $x \in \mathcal{M}$) s.t.

$$\Delta u_x = \delta_x \text{ in } \mathbb{R}^3 \setminus \overline{D}\,, \quad \frac{\partial u_x}{\partial \nu} = 0 \text{ on } \partial D\,.$$

Note: u_x is the Neumann function for the exterior of D

Then define “backscattered data” as $b = b(x) = U_x(x)$ for $x \in \mathcal{M}$



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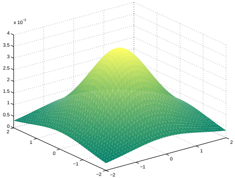


backscattered data

Open problem: Can one reconst. D from backscattered data? Preliminary results exist for 2D data within bounded domains.

Characterizations

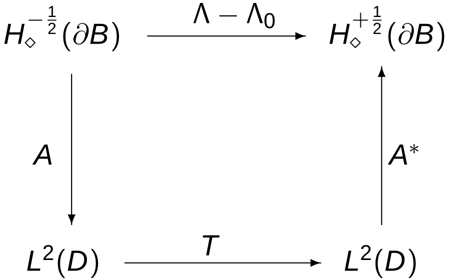
- Sampling methods, e.g.:
  Linear Sampling Method: Colton, Kirsch et al since 1996,
  Singular Sources Method: Potthast et al since 1996,
  Probe Method: Ikehata since 1998,
  Factorization Method: Kirsch, Hanke, Hyvönen et al since 1998
- Source supports:
  Scattering support: Kusiak, Sylvester since 2003,
  Electrostatics: Hanke, Hyvönen et al 2008
- Transmission eigenvalues (appear with far field operator):
  Colton, Monk, Kirsch, Päivärinta et al since 1988

Example: Factorization Method

Data for impedance tomography problem: $\Lambda - \Lambda_0$ where

$$\begin{aligned} \Lambda &= \text{D-N-operator with defect,} \\ \Lambda_0 &= \text{D-N-operator without defect} \end{aligned}$$

$$\sigma = \sigma_0 + \chi_D \, q$$



Factorization:

$$\Lambda - \Lambda_0 = A^* T A$$

where $A : H_\diamond^{-1/2}(\partial B) \to L^2(D)$ compact and $T : L^2(D) \to L^2(D)$ isomorphism. Then with $\phi_z(x) = d \cdot (x - z)/|x - z|^2$, $x \in \partial B$, (if B disc, $\sigma_0 = 1$, d fixed vector, and further assumptions):

- $\phi_z \in \mathcal{R}(A^*) \iff z \in D$
- $\mathcal{R}(A^*) = \mathcal{R}(|\Lambda - \Lambda_0|^{1/2})$

$\Big\}\quad z \in D \iff \phi_z \in \mathcal{R}(|\Lambda - \Lambda_0|^{1/2})$

Example: Scattering Support

Recall: Let $f \in L^2(\mathbb{R}^3)$ (or distribution) of compact support and u radiating solution of $\Delta u + k^2 u = f$ in $\mathbb{R}^3$.

Then:
$$u(x) = \frac{\exp(ik|x|)}{|x|}\, u_f^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right),\ |x| \to \infty\,.$$

Definition: Let $\alpha = u_f^\infty$ for some $f \in L^2(D)$ with compact support.
c-supp$(\alpha) = \bigcap_{M \in \mathcal{M}_\alpha} M$ is called convex scattering support of α where

$$\mathcal{M}_\alpha = \left\{ M \subset \mathbb{R}^3 : \begin{array}{l} M \text{ convex and compact with} \\ u_g^\infty = \alpha \text{ for some } g \in L^2(\mathbb{R}^3) \text{ with} \\ \text{support in } M \end{array} \right\}$$

Goal: Determine $\mathcal{M}_\alpha$ from α!

Example: Transmission Eigenvalue Probl.

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Definition: $k > 0$ is called interior transmission eigenvalue if there exist nontrivial $u, w \in L^2(D)$ with $u - w \in H^2(D)$ such that

$$
\begin{aligned}
\Delta u + k^2(1+q)\,u &= 0 \quad \text{in } D\,,\\
\Delta w + k^2\,w &= 0 \quad \text{in } D\,,\\
u = w \text{ on } \partial D\,, \qquad & \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} \text{ on } \partial D\,.
\end{aligned}
$$

Discreteness: Colton, Kirsch, Päivärinta since 1989

Existence: Päivärinta, Sylvester 2008, Cakoni (infinite number of eigenvalues and general q 2009)

Open problem: Existence of complex eigenvalues!

Ultimate goal: Determine properties of q from eigenvalues/eigenfunctions

Asymptotics

- Small inclusions (Ammari/Kang 2004, Hanke, Griesmaier, Vogelius, et al)

$$(\Lambda_\varepsilon - \Lambda_0)f \;=\; -\varepsilon^2 \nabla_y N(\cdot, z) \cdot M \nabla u_0(z) \;+\; \mathcal{O}(\varepsilon^3)$$

- Small wave numbers (Justification of Factorization Method for point source incidence): Hanke, Gebauer 2008
- Topological derivative (Sokolowski, Masmoudi et al):

$$\mathrm{div}(\sigma \nabla u_\varepsilon) = 0 \text{ in } B \setminus K_\varepsilon(z)\,, \quad \sigma \frac{\partial u_\varepsilon}{\partial \nu} = f \text{ on } \partial B\,,$$

$$\sigma \frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ on } \partial K_\varepsilon(z)\,. \quad \text{Set } J_\varepsilon(z) := \int_{\partial B} |u_\varepsilon - g|^2 ds\,.$$

Then compute: $\quad T(z) \;:=\; \lim_{\varepsilon \to 0} \big[J_\varepsilon(z) - J_0(z) \big] / (2\pi\varepsilon^2)$

and plot the region $\{z \in B : T(z) < 0\}$!

Stochastic Models

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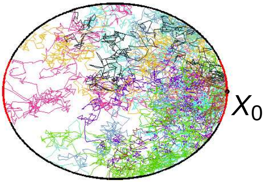
Example EIT-Direct Problem. Given $\sigma \in L^\infty(B)$, $f \in L^2(\partial B)$, determine $u \in H^1(B)$ with

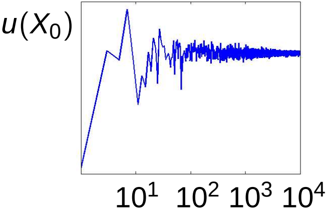
$$\operatorname{div}(\sigma \nabla u) = 0 \text{ in } B\,, \quad \sigma \frac{\partial u}{\partial \nu} = f \text{ on } \partial B\,.$$

Probabilistic approach: If σ is of class $\mathcal{C}^1$, $X_0 \in \overline{B}$, and if

$$X_t = X_0 + \int_0^t \nabla\sigma(X_s)\mathrm{d}s + \int_0^t \sqrt{2\sigma(X_s)}\mathrm{d}W_s - \int_0^t (\sigma\nu)(X_s)\mathrm{d}L_s$$

then one has $u(X_0) = \lim_{T\to\infty} \mathbb{E}\left[\int_0^T f(X_t)\mathrm{d}L_t\right]$ (Feynman/Kac)





Stochastic EIT-Problem (cont.)

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Open Problem: What about general conductivities?

In general, X is not a semi-martingale. However, the differential operator acts locally, so that in regions, where σ is smooth, X behaves like a “good”diffusion. What happens at the surface of discontinuity?