1 Introduction

The presentation of Girsanov's theorem follows [1] where from further details can be found.

2 Girsanov formula

This section follows [1].

Proposition 2.1 (*Exponential martingale*)**.** *Let* $\boldsymbol{w}_t$ *is a* d*-dimensional Wiener process (Brownian motion) and assume that*

$$\boldsymbol{b}\colon \Omega \times \mathbb{R} \to \mathbb{R}^d$$

is non-anticipative, and such that

$$\prec e^{c\int_0^T dt ||\boldsymbol{b}||^2} \succ < \infty \tag{2.1}$$

for some $c > 0$ *then*

$$M_t = e^{\int_0^t d\boldsymbol{w}_s \cdot \boldsymbol{b}_s - \int_0^t ds \, \frac{||\boldsymbol{b}_s||^2}{2}} \qquad\qquad 0 \leq t \leq T$$

is a martingale

Proof. By direct application of Ito lemma

$$dM_t = d\boldsymbol{w}_t \cdot \boldsymbol{b}_t \, M_t$$

thus

$$\prec dM_t \succ = 0$$

which shows that M_t is a *local* martingale. The condition (2.1) is a technical condition ensuring that

$$\prec |M_t| \succ < \infty$$

whence for all $t \leq T$ we have

$$\prec |M_t| \succ = \prec M_t \succ = 1$$

□

Theorem 2.1 (*Girsanov*)**.** *Consider a probability measure* P *on the space of paths* $\{\boldsymbol{w}_t \,|\, 0 \leq t \leq T\}$ *such that* $\boldsymbol{w}_t$ *is a* d*-dimensional Wiener process (Brownian motion) and assume that for*

$$\boldsymbol{b}\colon \Omega \times \mathbb{R} \to \mathbb{R}^d$$

non-anticipative

$$M_t := e^{\int_0^t d\boldsymbol{w}_s \cdot \boldsymbol{b}_s - \int_0^t ds \, \frac{||\boldsymbol{b}_s||^2}{2}} \qquad\qquad 0 \leq t \leq T$$

is a martingale. In such a case, we can define a new measure Q *on path space* $\{\boldsymbol{w}_t \,|\, 0 \leq t \leq T\}$ *such that its Radon-Nikodym derivative with respect to* P *is*

$$\frac{dQ_t}{dP_t} = M_t$$

meaning that for any functional F of $\boldsymbol{w}_t$ the identity

$$\int dQ\, F := \int dP\, F\, M_t \qquad \textit{(alternative notation: } \prec F \succ_Q = \prec F\, M_t \succ \textit{)}$$

holds true. Then, the stochastic process

$$\boldsymbol{\zeta}_t = \boldsymbol{w}_t - \int_0^t ds\, \boldsymbol{b}_s$$

is a Wiener process with respect to Q .

Proof. We need to show that increments of $\boldsymbol{\zeta}_t$ are independent each with Gaussian generating (characteristic) function.

- Gaussian for of the Generating function: we need to prove

  $$\prec e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_t} \succ_Q = e^{\frac{||\boldsymbol{\lambda}||^2\, t}{2}}$$

  Namely

  $$\prec e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_t} \succ_Q = \prec e^{\boldsymbol{\lambda}\cdot\boldsymbol{w_t} - \int_0^t ds\, \boldsymbol{\lambda}\cdot\boldsymbol{b}_s} e^{\int_0^t d\boldsymbol{w}_s\cdot\boldsymbol{b}_s - \int_0^t ds\, \frac{||\boldsymbol{b}_s||^2}{2}} \succ$$

  can be written as

  $$\prec e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_t} \succ_Q = e^{\frac{||\boldsymbol{\lambda}||^2 t}{2}} \prec e^{\int_0^t d\boldsymbol{w}_s\cdot(\boldsymbol{\lambda}+\boldsymbol{b}_s) - \int_0^t ds\, \frac{||\boldsymbol{b}_s+\boldsymbol{\lambda}||^2}{2}} \succ = e^{\frac{||\boldsymbol{\lambda}||^2 t}{2}}$$

  since

  $$M_t^{(\lambda)} = e^{\int_0^t d\boldsymbol{w}_s\cdot(\boldsymbol{\lambda}+\boldsymbol{b}_s) - \int_0^t ds\, \frac{||\boldsymbol{b}_s+\boldsymbol{\lambda}||^2}{2}}$$

  is a martingale, if M_t is.

- Independence of increments: by construction

  $$\boldsymbol{\zeta}_{t+t_o} - \boldsymbol{\zeta}_{t_o} = \boldsymbol{w}_{t+t_o} - \boldsymbol{w}_{t_o} + \int_{t_o}^t ds\, \boldsymbol{b}_s$$

  is for any $t\,, t_o$ independent of $\boldsymbol{z}_{t_o}$. Then

  $$\prec e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t+t_o}} \succ_Q = \prec e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t+t_o} - \boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t_o}} e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t_o}} \succ_Q = \prec \left[e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t+t_o} - \boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t_o}} \frac{M_{t+t_o}}{M_{t_o}} \right] e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t_o}} M_{t_o} \succ$$

  Introducing the conditional expectation with respect to the σ-algebra $\mathcal{W}_{t_o}$ induced by the Wiener-process up to time t_o we can also write

  $$\eta_{t,t_o} = \prec e^{\boldsymbol{\lambda}\cdot(\boldsymbol{\zeta}_{t+t_o} - \boldsymbol{\zeta}_{t_o})} \frac{M_{t+t_o}}{M_{t_o}} | \mathcal{W}_{t_o} \succ$$

  By the coarsening property of conditional expectation (see e.g. section H of chapter 2 of [2]) we have

  $$\prec e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t+t_o}} \succ_Q = \prec \eta_{t,t_o}\, e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t_o}} M_{t_o} \succ = \prec \eta_{t,t_o}\, e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t_o}} \succ_Q$$

  But η_{t,t_o} is by construction independent of the history of $\boldsymbol{\zeta}_{t'}$ for $t' \leq t_o$:

  $$\prec e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t+t_o}} \succ_Q = \prec \eta_{t,t_o} \succ_Q \prec e^{\boldsymbol{\lambda}\cdot\boldsymbol{\zeta}_{t_o}} \succ_Q = e^{\frac{||\boldsymbol{\lambda}||^2 t}{2}} e^{\frac{||\boldsymbol{\lambda}||^2 t_o}{2}}$$

  i.e. independent increments have Gaussian distribution.

□

Girsanov theorem is very useful for the following reason. Consider the Ito stochastic differential equations

$$d\boldsymbol{\xi}_t = \boldsymbol{b}_t dt + d\boldsymbol{\omega}_t$$

and

$$d\boldsymbol{\zeta}_t = \tilde{\boldsymbol{b}}_t dt + d\boldsymbol{\omega}_t$$

such that

$$d\omega_t^i := \sigma_t^{ij} dw_t^j \qquad \text{(alternative notation: } d\boldsymbol{\omega}_t = \boldsymbol{\sigma}[d\boldsymbol{w}_t])$$

both satisfying the hypotheses of the existence and uniqueness theorem for $t \in [0, T]$. Then, if $P_{\boldsymbol{\xi}_t}$ and $P_{\boldsymbol{\zeta}_t}$ denote the probability measures over the realizations respectively of $\boldsymbol{\xi}_t$ and $\boldsymbol{\zeta}_t$ they satisfy

$$\frac{dP_{\boldsymbol{\zeta}_t}}{dP_{\boldsymbol{\xi}_t}} = e^{\int_0^t d\boldsymbol{w}_s \cdot \boldsymbol{\phi}_t - \int_0^t ds \frac{||\boldsymbol{\phi}_t||^2}{2}}$$

for

$$\tilde{\boldsymbol{b}}_t = \boldsymbol{b}_t + \boldsymbol{\sigma}[\boldsymbol{\phi}]_t$$

Example 2.1 (*Wiener process with constant drift*)**.** Consider the process

$$d\xi_t = v\,dt + \sigma\,dw_t$$
$$\xi_0 = x_o$$

for $v\,,\sigma\ \in\ \mathbb{R}_+$ and

$$d\zeta_t = \sigma\,dw_t$$
$$\zeta_0 = x_o \tag{2.2}$$

By Girsanov theorem

$$\frac{dP_{\xi_t}}{dP_{\zeta_t}} = e^{\int_0^t dw_s \frac{v}{\sigma} - \int_0^t ds\, \frac{v^2}{2\,\sigma^2}} = e^{w_t \frac{v}{\sigma} - \frac{v^2\,t}{2\,\sigma^2}}$$

so that

$$\prec e^{\lambda \xi_t} \succ_Q = e^{\lambda\,x_o} \prec e^{\lambda \zeta_t} e^{w_t \frac{v}{\sigma} - \frac{v^2\,t}{2\,\sigma^2}} \succ_P$$

where Q denotes the measure on the paths of ξ_t and P the measure on the paths of ζ_t. This latter measure we can relate to that of the Wiener process through

$$\zeta_t = \sigma\,w_t \tag{2.3}$$

so that

$$\prec e^{\lambda \xi_t} \succ_Q = e^{\lambda\,x_o} \prec e^{\lambda \sigma\,w_t} e^{w_t \frac{v}{\sigma} - \frac{v^2\,t}{2\,\sigma^2}} \succ \tag{2.4}$$

the average on the right hand side being with respect to the paths of the Wiener-process w_t

$$\prec e^{\lambda \xi_t} \succ_Q = e^{\lambda\,x_o - \frac{v^2\,t}{2\,\sigma^2}} \prec e^{\left(\lambda\sigma + \frac{v}{\sigma}\right) w_t} \succ = e^{\lambda\,x_o - \frac{v^2\,t}{2\,\sigma^2}} e^{\left(\lambda\sigma + \frac{v}{\sigma}\right)^2 \frac{t}{2}} = e^{\lambda\,(x_o + v\,t) + \frac{\lambda^2\,\sigma^2\,t}{2}} \tag{2.5}$$

References

[1] M. Avellaneda, *Ito processes, continuous-time martingales and Girsanov's Theorem*, lecture notes, `http://www.math.nyu.edu/faculty/avellane/rm3.ps` 1

[2] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, `http://math.berkeley.edu/~evans/` 2