1 Introduction

A possible reference for these notes are lectures-09 to 12 of [1].

2 Diffusion processes and Chapman-Kolmogorov equation

The material contained in this section is also discussed in [1] sections 09 and 11.

Definition 2.1 (*Diffusion process*)**.** *We call a diffusion process a Markov process* $\{\boldsymbol{\xi}_t \,|\, t \geq 0\}$ *with continuous paths with values in* $\mathbb{R}^d$.

Because of the Kolmogorov consistence conditions (see e.g. lecture 9 [2]) we characterize a diffusion process $\{\boldsymbol{\xi}_t \,|\, t \geq 0\}$ with a family of transition probabilities $p_\xi(\boldsymbol{x}, t \,|\boldsymbol{x}', t')$ (with respect to the Lebesgue measure) expressing the conditional probability density of the process at a future time t given the state at a previous time t' for all $t' \leq t \in \mathbb{R}_+$. The elements of the family must then satisfy

i Chapman-Kolmogorov equation

$$p_{\boldsymbol{\xi}}(\boldsymbol{x}\,, t\,|\boldsymbol{x}'\,, t') = \int_{\mathbb{R}^d} d^d z\, p_{\boldsymbol{\xi}}(\boldsymbol{x}\,, t\,|\boldsymbol{z}\,, s) p_{\boldsymbol{\xi}}(\boldsymbol{z}\,, s\,|\boldsymbol{x}'\,, t')$$

for any $t' \leq s \leq t$.

ii Kolmogorov-Čentsov theorem (see e.g. lecture [3]) which states that path-continuity holds almost surely if for some $\alpha\, \beta\,, K > 0$

$$\int_{\mathbb{R}^d} d^d z\, ||\boldsymbol{x} - \boldsymbol{x}'||^\alpha\, p_{\boldsymbol{\xi}}(\boldsymbol{x}\,, t\,|\boldsymbol{x}'\,, t') \leq K\, |t - t'|^{1+\beta}$$

If we somewhat strengthen the condition ii and require $\alpha > 2$ then the limits

$$\lim_{t \to t'} \frac{1}{t - t'} \int_{\mathbb{R}^d} d^d z\, (\boldsymbol{x} - \boldsymbol{x}')\, p_{\boldsymbol{\xi}}(\boldsymbol{x}\,, t\,|\boldsymbol{x}'\,, t') = \boldsymbol{b}(\boldsymbol{x}'\,, t')$$

$$\lim_{t \to t'} \frac{1}{t - t'} \int_{\mathbb{R}^d} d^d z\, (x - x')^i (x - x')^j\, p_{\boldsymbol{\xi}}(\boldsymbol{x}\,, t\,|\boldsymbol{x}'\,, t') = g^{ij}(\boldsymbol{x}'\,, t')$$

are well defined. They assert that increments of the Markov process are characterized by a *drift* (2.1a) and by a variance (2.1a). One way to generate a Gaussian random vector $\boldsymbol{\eta}$ with mean $\boldsymbol{b}\, dt$ and covariance matrix $g^{i\,j}\, dt$ is to start from a Gaussian vector $\boldsymbol{\zeta}$ with zero mean and covariance $1\, dt$ and proceed a linear transformation:

$$\boldsymbol{\eta} = \boldsymbol{b}\, dt + \boldsymbol{\sigma}[\boldsymbol{\zeta}]$$

where

$$\boldsymbol{g} = \boldsymbol{\sigma}\boldsymbol{\sigma}^\dagger$$

In this way one can choose $\boldsymbol{\sigma}$ to be the positive semi-definite symmetric square root of $\boldsymbol{g}$. Reasoning along these lines we conclude that

$$d\boldsymbol{\xi}_t = \boldsymbol{b}\,(\boldsymbol{\xi}_t\,, t)\; dt + \boldsymbol{\sigma}[d\boldsymbol{w}_t] \tag{2.2}$$

governs the increments of the diffusion process. Existence and uniqueness of (2.2) are guaranteed by Lipschitz continuity of the drift and diffusion fields. Since this latter (up to an orthogonal transformation) is the square root of $\boldsymbol{g}$, the Lipschitz condition can be imposed on $\boldsymbol{g}$ by requiring it either to be

- Lipschitz and *uniformly elliptic*:

$$c\sum_{i=1}^{d} x_i^2 \leq \sum_{ij} x_i g^{ij} x_j \leq C\sum_{i=1}^{d} x_i^2$$

  for some $0 < c \leq C < \infty$ and all $\boldsymbol{x} \in \mathbb{R}^d$.

  OR

- if the g^{ij} is continuously twice differentiable with a bound on the second derivatives.

twice differentiable with

3 Fokker-Planck equation: forward Kolmogorov equation

Let us denote by $\boldsymbol{\phi}_t$ the fundamental solution of

$$d\boldsymbol{\xi}_t = \boldsymbol{b}\,(\boldsymbol{\xi}_t, t)\; dt + d\boldsymbol{\omega}_t(\boldsymbol{\xi}_t, t) \tag{3.1}$$

with

$$d\omega_t^i(\boldsymbol{\xi}_t, t) = \sigma^{ij}(\boldsymbol{\xi}_t, t) dw_t^j$$

By fundamental solution we mean that given an intial condition

$$\boldsymbol{\xi}_{t_o} = \boldsymbol{x}_o \tag{3.2}$$

the unique solution of (3.1), (3.2) can be expressed for any $t \geq t_o$ through the map

$$\boldsymbol{\xi}_t = \boldsymbol{\phi}_t(\boldsymbol{x}_o, t_o)$$

Henceforth, we assume that (3.1) not only satisfies the hypotheses of the theorem of existence ed uniqueness for any t but also that $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ are at least, respectively, differentiable and twice differentiable everywhere. For any given initial condition $\boldsymbol{x}_o$ at time t_o we have for $t \geq 0$

$$p_{\boldsymbol{\xi}}(\boldsymbol{x}, t, \,|\, \boldsymbol{x}_o, t_o) = \prec \delta^{(d)}(\boldsymbol{x} - \boldsymbol{\phi}_t(t_o, \boldsymbol{x}_o)) \succ$$

Differentiating both sides with respect to time applying Ito lemma and the martingale property of stochastic increments we get into

$$\partial_t p_{\boldsymbol{\xi}}(\boldsymbol{x}, t, \,|\, \boldsymbol{x}_o, t_o) = \partial_t \prec \delta^{(d)}(\boldsymbol{x} - \boldsymbol{\phi}_t(t_o, \boldsymbol{x}_o)) \succ$$
$$\prec \left[\boldsymbol{b}\,(\boldsymbol{\phi}_t, t) \cdot \partial_{\boldsymbol{\phi}_t} + \frac{(\sigma^{ik}\sigma^{jk})\,(\boldsymbol{\phi}_t, t)}{2} \partial_{\phi_t^i} \partial_{\phi_t^j}\right] \delta^{(d)}(\boldsymbol{x} - \boldsymbol{\phi}_t(t_o, \boldsymbol{x}_o)) \succ$$

Using the translational invariance of the δ-function we can write the right hand side as

$$\partial_t p_{\boldsymbol{\xi}}(\boldsymbol{x}, t, \,|\, \boldsymbol{x}_o, t_o) =$$
$$\prec \left[-\boldsymbol{b}\,(\boldsymbol{\phi}_t, t) \cdot \partial_{\boldsymbol{x}} + \frac{(\sigma^{ik}\sigma^{ik})\,(\boldsymbol{\phi}_t, t)}{2} \partial_{x^i} \partial_{x^j}\right] \delta^{(d)}(\boldsymbol{x} - \boldsymbol{\phi}_t(t_o, \boldsymbol{x}_o)) \succ$$

a fact which entitles us to carry the derivatives over the average sign

$$\partial_t p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) =$$
$$-\partial_{x^i} \prec b^i\,(\boldsymbol{\phi}_t,t)\,\delta^{(d)}(\boldsymbol{x}-\boldsymbol{\phi}_t(t_o,\boldsymbol{x}_o)) \succ +\partial_{x^i}\partial_{x^j} \prec \frac{(\sigma^{ik}\sigma^{jk})\,(\boldsymbol{\phi}_t,t)}{2}\delta^{(d)}(\boldsymbol{x}-\boldsymbol{\phi}_t(t_o,\boldsymbol{x}_o)) \succ$$

From the properties of the δ-function we finally conclude

$$\partial_t p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) = \partial_{\boldsymbol{x}}\cdot\boldsymbol{J}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) \tag{3.3a}$$

$$J^i(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) = -b^i\,(\boldsymbol{x},t)\,p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) + \partial_{x^j}\left[\frac{(\sigma^{ik}\sigma^{jk})\,(\boldsymbol{x},t)}{2}p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o)\right] \tag{3.3b}$$

It is readily seen that inserting the (conditional) probability *current* (3.3b) into (3.3a) we recover a Fokker-Planck equation:

$$\partial_t p_{\boldsymbol{\xi}} = -\partial_{x^i}(b^i p_{\boldsymbol{\xi}}) + \frac{1}{2}\partial_{x^i}\partial_{x^j}(g^{ij}\,p_{\boldsymbol{\xi}}) \tag{3.4}$$

with g being the covariance

$$g^{ij} = \sigma^{ik}\sigma^{jk}$$

of the diffusion process specified by (3.1). In the probabilistic literature (3.3a) or equivalently (3.4) are referred to as *forward Kolmogorov* equation. The describe the forward in time t evolution of a transition probability density satisfying under our hypothesis the *initial condition*

$$\lim_{t\to t_o} p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) = \delta^{(d)}(\boldsymbol{x}-\boldsymbol{x}_o) \tag{3.5}$$

3.1 Boundary conditions in $\mathbb{R}^d$

To simplify the discussion we suppose that $\boldsymbol{g}$ is *uniformly elliptic*. Technically, (3.4) is a parabolic differential equation. Under our working hypotheses it admits a unique solution once we specify an initial condition in time such as (3.5) and boundary conditions in space. In establishing the correspondence between (3.1) and (3.4) we implied that the normalization condition

$$\int_{\mathbb{R}^d} d^d x\, p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) = 1$$

to hold true. The corresponding spatial boundary conditions for (3.4) are

$$\lim_{||\boldsymbol{x}||\uparrow\infty} ||\boldsymbol{x}||^{d+\varepsilon}\,p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) = 0$$

for some $\varepsilon > 0$.

3.2 Boundary conditions in $\mathbb{A}\subset\mathbb{R}^d$

WE may think of (3.4) as a providing the solution of (3.1) in probability and use it to construct diffusion processes in a finite subset $\mathbb{A}_d$ of $\mathbb{R}^d$. In order to accomplish such goal we can exploit the divergence form of the right hand side of (3.3a) and invoke the dominated convergence theorem to derive the consequences of *probability conservation*

$$0 = \partial_t \int_{\mathcal{A}} d^d x\, p_{\boldsymbol{\xi}}\,(\boldsymbol{x},t\,|\,\boldsymbol{x}_o\,t_o) = \boldsymbol{n}\cdot\boldsymbol{J}|_{\partial\mathbb{A}_d}$$

where $\boldsymbol{n}$ is the unit outward pointing vector normal orthogonal to the boundary of $\mathbb{A}_d$. We see that probability conservation is naturally enforced by the requirement of vanishing probability current on the boundary of $\mathcal{A}_d$:

$$0 = n_i \left\{ -b^i\,(\boldsymbol{x},t)\, p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) + \frac{1}{2}\partial_{x^j}\left[g^{ij}\,(\boldsymbol{x},t)\, p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o)\right]\right\}_{\mathbb{A}_d} \tag{3.6}$$

If we formally write the current as the sum

$$\boldsymbol{J} = \boldsymbol{J}_{\text{outwards}} + \boldsymbol{J}_{\text{inwards}} \qquad \text{such that} \qquad \begin{aligned} \boldsymbol{n}\cdot\boldsymbol{J}_{\text{outwards}}|_{\mathbb{A}_d} &\geq 0 \\ \boldsymbol{n}\cdot\boldsymbol{J}_{\text{inwards}}|_{\mathbb{A}_d} &< 0 \end{aligned}$$

we can interpret (3.6) as a *reflecting boundary* condition: all incoming trajectories from the interior of $\mathbb{A}_d$ to the boundary $\partial\mathbb{A}_d$ are subsequently reflected to the interior of $\mathbb{A}_d$.

3.3 Probability conservation and Stratonovich calculus

The probability (mass) transport by the fundamental solution $\boldsymbol{\phi}_t$ of an ordinary differential equation

$$\dot{\boldsymbol{\xi}}_t = \boldsymbol{v}\,(\boldsymbol{\xi}_t,t)$$

can be shown using ordinary calculus and proceeding as in section 3 to satisfy

$$\partial_t p_{\boldsymbol{\xi}}(\boldsymbol{x},t) = -\partial_{\boldsymbol{x}}[\boldsymbol{v}\,(\boldsymbol{x},t)\, p_{\boldsymbol{\xi}}(\boldsymbol{x},t)] \tag{3.7a}$$

$$p_{\boldsymbol{\xi}}(\boldsymbol{x},t_o) = p_{\boldsymbol{\xi}_o}(\boldsymbol{x}) \tag{3.7b}$$

Ordinary calculus can be applied also to the Stratonovich version of (3.1)

$$d\boldsymbol{\xi}_t = \left[\boldsymbol{b}\,(\boldsymbol{\xi}_t,t) - \frac{\boldsymbol{I}}{2}(\boldsymbol{\xi}_t,t)\right]dt + d\boldsymbol{\omega}_t(\boldsymbol{\xi}_t,t)$$

where $\boldsymbol{I}$ denotes the Ito drift

$$I^i = \sigma^{jk}\partial_{x^j}\sigma^{ik}$$

We get into

$$\begin{aligned} &\partial_t p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o) = \\ &\quad -\partial_{x^i}\left[\left(b^i(\boldsymbol{x},t) - \frac{I^i(\boldsymbol{x},t)}{2}\right)p_{\boldsymbol{\xi}}(\boldsymbol{x},t,\,|\,\boldsymbol{x}_o,t_o)\right] + \partial_{x^i} \prec \sigma^{ij}(\boldsymbol{x},t)\,\delta^{(d)}(\boldsymbol{x}-\boldsymbol{\phi}_t(t_o,\boldsymbol{x}_o))dw_t^j \succ \end{aligned} \tag{3.8}$$

where now the last term is non vanishing as the Stratonovich stochastic increment does *not* enjoy the martingale property. The overall result cannot, however, depend upon our choice to represent diffusion increments in Ito or Stratonovich form. We can therefore determine the average by equating the right hand sides of (3.10) and (3.4). We get into

$$\begin{aligned} &\partial_{x^i}[\sigma^{ij}(\boldsymbol{x},t) \prec \delta^{(d)}(\boldsymbol{x}-\boldsymbol{\phi}_t(t_o,\boldsymbol{x}_o))dw_t^j \succ] \\ &\quad = \frac{1}{2}\partial_{x^i}\left\{\left[\partial_{x^j}(\sigma^{jk}\sigma^{ik})(\boldsymbol{x},t) - \left(\sigma^{jk}\partial_{x^j}\sigma^{ik}\right)(\boldsymbol{x},t)\right]\, p_{\boldsymbol{\xi}}(\boldsymbol{x},t\,|\,\boldsymbol{x}_o,t_o)\right\} \\ &\quad = \frac{1}{2}\partial_{x^i}\left\{\sigma^{ik}(\boldsymbol{x},t)\partial_{x^j}\left[\sigma^{jk}(\boldsymbol{x},t)p_{\boldsymbol{\xi}}(\boldsymbol{x},t\,|\,\boldsymbol{x}_o,t_o)\right]\right\} \end{aligned} \tag{3.9}$$

We have in this way somewhat indirectly derived the general expression for averages over Stratonovich increments:

$$\prec \delta^{(d)}(\boldsymbol{x} - \boldsymbol{\phi}_t(t_o, \boldsymbol{x}_o)) dw_t^i \succ = \frac{1}{2} \partial_{x^j} \left[\sigma^{ji}(\boldsymbol{x}, t) p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{x}_o, t_o) \right]$$

An immediate consequence of this equation is that Fokker-Planck equations are sometime cast in the form

$$\begin{aligned} &\partial_t p_{\boldsymbol{\xi}}(\boldsymbol{x}, t, \,|\, \boldsymbol{x}_o, t_o) = \\ &\quad -\partial_{x^i} \left[\tilde{b}^i(\boldsymbol{x}, t) p_{\boldsymbol{\xi}}(\boldsymbol{x}, t, \,|\, \boldsymbol{x}_o, t_o) \right] + \frac{1}{2} \partial_{x^i} \left[(\sigma^{jk} \partial_{x^j} \sigma^{ik})(\boldsymbol{x}, t) p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{x}_o, t_o) \right] \end{aligned}$$

with

$$\tilde{\boldsymbol{b}} := \boldsymbol{b} - \frac{\boldsymbol{I}}{2}$$

4 Backward Kolmogorov equation

Suppose

$$f : \mathbb{R}^d \to \mathbb{R}^d$$

is $p_{\boldsymbol{\xi}}$-integrable i.e. the conditional average

$$\prec \; f(\boldsymbol{\xi}_t) \succ_{(\boldsymbol{y}, s)} = \int_{\mathbb{R}^d} d^d x \, f(\boldsymbol{x}) \, p_{\boldsymbol{\xi}}(\boldsymbol{x}, t | \boldsymbol{y}, s)$$

is well defined. Then

$$F(\boldsymbol{y}, s) = \prec \; f(\boldsymbol{\xi}_t) \succ_{(\boldsymbol{y}, s)}$$

satisfies the *backward Kolmogorov equation*:

$$\left\{ \partial_s + b^i(\boldsymbol{y}, s) \partial_{y^i} + \frac{g^{i\,j}(\boldsymbol{y}, s)}{2} \partial_{y^i} \partial_{y^j} \right\} F(\boldsymbol{y}, s) = 0 \tag{4.1}$$

with

$$g^{i\,j}(\boldsymbol{y}, s) = (\sigma^{ik} \sigma^{jk})(\boldsymbol{y}, s)$$

Proof. It is instructive to derive the proof in two equivalent ways

- First way. Using the Chapman-Kolmogorov equation

$$\begin{aligned} ds \, \partial_s F(\boldsymbol{y}, s) &= \int d^d x f(\boldsymbol{x}) \left\{ p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, t | \boldsymbol{y}, s + ds \right) - p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, t | \boldsymbol{y}, s \right) \right\} \\ &= \int d^d x d^d z \, f(\boldsymbol{x}) \left\{ p_{\boldsymbol{\xi}_t} \left(\boldsymbol{x}, t | \boldsymbol{y}, s + ds \right) - p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, t | \boldsymbol{z}, s + ds \right) \right\} p_{\boldsymbol{\xi}} \left(\boldsymbol{z}, s + ds | \boldsymbol{y}, s \right) \end{aligned}$$

  Using the Fokker-Planck (forward Kolmogorov) equation

$$\begin{aligned} ds \, \partial_s F(\boldsymbol{y}, s) = &\int d^d x d^d z \, f(\boldsymbol{x}) \left\{ p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, t | \boldsymbol{y}, s + ds \right) - p_{\boldsymbol{\xi}} \left(\boldsymbol{x}, t | \boldsymbol{z}, s + ds \right) \right\} \\ &\times \left\{ -\partial_{z^i} b^i \left(\boldsymbol{z}, s + ds \right) + \partial_{z^i} \partial_{z^j} \frac{\sigma^{ik}(\boldsymbol{z}, s + ds) \sigma^{jk}(\boldsymbol{z}, s + ds)}{2} \right\} \left\{ ds \, p_{\boldsymbol{\xi}} \left(\boldsymbol{z}, s | \boldsymbol{y}, s \right) + O(ds^2) \right\} \end{aligned}$$

Since

$$p_{\boldsymbol{\xi}}\left(\boldsymbol{z}, s|\boldsymbol{y}, s\right) = \delta^{(d)}(\boldsymbol{z} - \boldsymbol{y})$$

we have after integration by parts

$$ds\,\partial_s F(\boldsymbol{y}, s) = -\,ds \int d^d x d^d z\, f(\boldsymbol{x})\delta^{(d)}(\boldsymbol{z} - \boldsymbol{y}) \\ \times \left\{ b^i\left(\boldsymbol{z}, s + ds\right)\partial_{z^i} + \frac{\sigma^{ik}(\boldsymbol{z}, s + ds)\sigma^{jk}(\boldsymbol{z}, s + ds)}{2}\partial_{z^i}\partial_{z^j} \right\} p_{\boldsymbol{\xi}}\left(\boldsymbol{x}, t|\boldsymbol{z}, s + ds\right)$$

whence the claim.

- Second way. Using the expression of the average in terms of the fundamental solution of the stochastic differential equation

$$\prec f(\boldsymbol{\xi}_t) \succ_{(\boldsymbol{y}, s+ds)} - \prec f(\boldsymbol{\xi}_t) \succ_{(\boldsymbol{y}, s)} = \prec f(\boldsymbol{\phi}\left(t; s + ds, \boldsymbol{y}\right)) \succ - \prec f(\boldsymbol{\phi}\left(t; s, \boldsymbol{y}\right)) \succ \tag{4.2}$$

Observing that

$$\prec f(\boldsymbol{\phi}(t; s, \boldsymbol{y})) \succ = \prec f(\boldsymbol{\phi}(t; s + ds, \boldsymbol{\phi}(s + ds; s, \boldsymbol{y})) \succ \tag{4.3}$$

(4.2) becomes

$$\prec f(\boldsymbol{\xi}_t) \succ_{(\boldsymbol{y}, s+ds)} - \prec f(\boldsymbol{\xi}_t) \succ_{(\boldsymbol{y}, s)} = \\ \prec f(\boldsymbol{\phi}(t; s + ds, \boldsymbol{y}) \succ - \prec f(\boldsymbol{\phi}(t; s + ds, \boldsymbol{\phi}(s + ds; s, \boldsymbol{y})) \succ$$

The increment is acts on the initial position, whilst keeping fixed the initial and final time. It is expedient to define

$$\tilde{f}(\boldsymbol{\phi}(u; s, \boldsymbol{y})) = f(\boldsymbol{\phi}(t; s + ds, \boldsymbol{\phi}(u; s, \boldsymbol{y})) \tag{4.4}$$

and to couch the increment into the form

$$\prec f(\boldsymbol{\xi}_t) \succ_{(\boldsymbol{y}, s+ds)} - \prec f(\boldsymbol{\xi}_t) \succ_{(\boldsymbol{y}, s)} = \prec \tilde{f}(\boldsymbol{y}) - \tilde{f}(\boldsymbol{\phi}\left(s + ds; s, \boldsymbol{y}\right)) \succ \tag{4.5}$$

We can then apply Ito Lemma to the function g to get into

$$d_s \prec f(\boldsymbol{\xi}_t) \succ_{(\boldsymbol{y}, s+ds)} \\ = -ds \prec \left\{ b^i(\boldsymbol{\phi}, s + ds)\partial_{y^i} + \frac{\sigma^{ik}(\boldsymbol{\phi}, s + ds)\sigma^{jk}(\boldsymbol{\phi}, s + ds)}{2}\partial_{y^i}\partial_{y^j} \right\} \tilde{f}(\boldsymbol{\phi}\left(s + ds; s, \boldsymbol{y}\right)) \succ \\ = -ds \left\{ b^i(\boldsymbol{y}, s)\partial_{y^i} + \frac{\sigma^{ik}(\boldsymbol{y}, s)\sigma^{jk}(\boldsymbol{y}, s)}{2}\partial_{y^j}\partial_{y^j} \right\} \tilde{f}(\boldsymbol{y}) \tag{4.6}$$

□

4.1 Boundary conditions for the backward Kolmogorov equation

We can re-write the backward Kolmogorov equation into the form

$$\partial_s p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{y}, s) = -\mathfrak{L}_{\boldsymbol{y}} p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{y}, s) \tag{4.7}$$

with $\mathfrak{L}_{\boldsymbol{y}}$ the generator of the diffusion process:

$$\mathfrak{L}_{\boldsymbol{y}} := b^i(\boldsymbol{y}, s)\partial_{y^i} + \frac{g^{ij}(\boldsymbol{y}, s)}{2}\partial_{y^i}\partial_{y^j}$$

The boundary condition in time is clearly

$$\lim_{s\uparrow t} p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{y}, s) = \delta^{(d)}(\boldsymbol{x} - \boldsymbol{y})$$

In order to derive spatial boundary conditions compatible with the interpretation of $p_{\boldsymbol{\xi}}$ as conditional probability density, we may differentiate the Chapman-Kolmogorov equation in the form

$$\begin{aligned} 0 = \partial_u p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{y}, s) = & \\ \int_{\mathbb{A}_d} d^d z \, [\partial_u p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{z}, u)] p_{\boldsymbol{\xi}}(\boldsymbol{z}, u \,|\, \boldsymbol{y}, s) &+ \int_{\mathbb{A}_d} d^d z \, p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{z}, u)\partial_u p_{\boldsymbol{\xi}}(\boldsymbol{z}, u \,|\, \boldsymbol{y}, s) \end{aligned} \tag{4.8}$$

for any $s \le u \le t$. Upon defining

$$\mathfrak{L}^{\dagger}_{\boldsymbol{x}} f(\boldsymbol{x}) := -\partial_{x^i}[b^i(\boldsymbol{x}, s) f(\boldsymbol{x})] + \frac{1}{2}\partial_{x^j}\partial_{x^j}[g^{ij}(\boldsymbol{x}, s)\, f\,(\boldsymbol{x})]$$

we can couch (4.8) in to the form

$$\begin{aligned} 0 &= \partial_u p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{y}, s) = \\ &= \int_{\mathbb{A}_d} d^d z \, [\mathfrak{L}_{\boldsymbol{z}} p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{z}, u)]\, p_{\boldsymbol{\xi}}(\boldsymbol{z}, u \,|\, \boldsymbol{y}, s) + \int_{\mathbb{A}_d} d^d z \, p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{z}, u)\mathfrak{L}_{\boldsymbol{z}} p_{\boldsymbol{\xi}}(\boldsymbol{z}, u \,|\, \boldsymbol{y}, s) \end{aligned}$$

whence we infer that for arbitrary $(\boldsymbol{x}\,, t)$ and $(\boldsymbol{y}\,, s)$ the equality

$$\begin{aligned} 0 = n_i &\left[p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{z}, u) \left(b^i(\boldsymbol{z}, s) + \frac{g^{ij}(\boldsymbol{z}, s)}{2}\partial_{z^j} \right) p_{\boldsymbol{\xi}}(\boldsymbol{z}, u \,|\, \boldsymbol{y}, s) \right]_{\boldsymbol{z} \in \mathbb{A}_d} \\ &- n_i \left[(\partial_{z^j} p_{\boldsymbol{\xi}})(\boldsymbol{x}, t \,|\, \boldsymbol{z}, u)\, g^{ij}(\boldsymbol{z}, s)\, p_{\boldsymbol{\xi}}(\boldsymbol{z}, u \,|\, \boldsymbol{y}, s) \right]_{\boldsymbol{z} \in \mathbb{A}_d} \end{aligned} \tag{4.9}$$

must hold true. for $\boldsymbol{n}$ as above denoting the unit outward pointing vector normal orthogonal to the boundary of $\mathbb{A}_d$. The equality is satisfied if $p_{\boldsymbol{\xi}}$ satisfies reflecting boundary conditions as a probability density (i.e. in $(\boldsymbol{z}\,, u)$) and

$$n_i\, g^{ij}(\boldsymbol{z}, s)\, (\partial_{z^j} p)_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{z}, u)|_{\boldsymbol{z} \in \mathbb{A}_d} = 0$$

In such a case we can interpret, as the notation suggests, $\mathfrak{L}$ and $\mathfrak{L}^{\dagger}$ as mutually adjoint operators acting on the space of transition probability density associated to the diffusion process with drift $\boldsymbol{b}$ and covariance $\boldsymbol{g}$. Interestingly, there are other boundary conditions under which $\mathfrak{L}$, $\mathfrak{L}^{\dagger}$ are adjoint operators. Of particular relevance in applications are *absorbing* boundary conditions

$$p_{\boldsymbol{\xi}}(\boldsymbol{x}, t \,|\, \boldsymbol{z}, u)|_{\boldsymbol{z} \in \mathbb{A}_d} = p_{\boldsymbol{\xi}}(\boldsymbol{z}, u \,|\, \boldsymbol{y}, s)|_{\boldsymbol{z} \in \mathbb{A}_d} = 0$$

for arbitrary $(\boldsymbol{x}\,, t)$ and $(\boldsymbol{y}\,, s)$.

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[4] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, http://math.berkeley.edu/~evans/