1 Introduction

The topics of this lecture are covered by chapter 5 of [1].

2 Existence and uniqueness theorem

Theorem 2.1 (*Existence and uniqueness*)**.** *Suppose that for some* $T \in \mathbb{R}_{+}$

$$\boldsymbol{b} : \mathbb{R}^{d} \times [0, T] \to \mathbb{R}^{d}$$

and

$$\boldsymbol{\sigma} : \mathbb{R}^{d \times d} \times [0, T] \to \mathbb{R}^{d \times m}$$

are continuous and satisfy the following conditions in the Euclidean norm

$$||\boldsymbol{b}(\boldsymbol{x}, t) - \boldsymbol{b}(\boldsymbol{y}, t)|| < C\,||\boldsymbol{x} - \boldsymbol{y}|| \qquad \& \qquad ||\boldsymbol{\sigma}(\boldsymbol{x}, t) - \boldsymbol{\sigma}(\boldsymbol{y}, t)|| < C\,||\boldsymbol{x} - \boldsymbol{y}||$$

and

$$||\boldsymbol{b}(\boldsymbol{x}, t)|| < C\,(1 + ||\boldsymbol{x}||) \qquad \& \qquad ||\boldsymbol{\sigma}(\boldsymbol{x}, t)|| < C\,(1 + ||\boldsymbol{x}||)$$

for all $0 \leq t \leq T$ *and some positive constant* C*. Let also* $\boldsymbol{\xi}_o$

$$\boldsymbol{\xi}_o : \Omega \to \mathcal{R}^{d}$$

a random variable such that

$$\prec ||\boldsymbol{\xi}_o||^2 \succ < \infty$$

Furthermore $\boldsymbol{\xi}_o$ *is independent of the* σ*-algebra* $\mathcal{W}$ *generated by a given* m*-dimensional Wiener process for* $t \geq 0$*. Then, there exists a unique solution of*

$$d\boldsymbol{\xi}_t = \boldsymbol{b}(\boldsymbol{\xi}_t, t)dt + \boldsymbol{\sigma}(\boldsymbol{\xi}_t, t)[d\boldsymbol{w}_t] \tag{2.1a}$$

$$\boldsymbol{\xi}_0 = \boldsymbol{\xi}_o \tag{2.1b}$$

Uniqueness here means that any square integrable $\boldsymbol{\xi}_t$ *and* $\tilde{\boldsymbol{\xi}}_t$ *with continuous paths, satisfying (2.1a), (2.1b) then for all* $0 \leq t \leq T$

$$\boldsymbol{\xi}_t = \tilde{\boldsymbol{\xi}}_t \qquad a.s.$$

Proof. **Existence**

We start by constructing a Picard type sequence of approximations to the solution

$$\boldsymbol{\xi}_t^{(0)} := \boldsymbol{\xi}_o$$

$$\boldsymbol{\xi}_t^{(1)} := \boldsymbol{\xi}_o + \int_0^t ds\, \boldsymbol{b}\left(\boldsymbol{\xi}_s^{(0)}, s\right) + \int_0^t ds\, \boldsymbol{\sigma}\left(\boldsymbol{\xi}_s^{(0)}, s\right)[d\boldsymbol{w}_s] \tag{2.2a}$$

$\dots$

$$\boldsymbol{\xi}_t^{(n+1)} := \boldsymbol{\xi}_o + \int_0^t ds\, \boldsymbol{b}\left(\boldsymbol{\xi}_s^{(n)}, s\right) + \int_0^t ds\, \boldsymbol{\sigma}\left(\boldsymbol{\xi}_s^{(n)}, s\right)[d\boldsymbol{w}_s]$$

The aim is to prove that the sequence in mean square and almost surely converges in the sense of Cauchy. Namely if we set

$$d^{(0)}(t) := 1$$

and for $n \geq 0$

$$d^{(n+1)}(t) = \prec ||\boldsymbol{\xi}_t^{(n+1)} - \boldsymbol{\xi}_t^{(n)}||^2 \succ$$

then we have

$$d^{(n+1)}(t) \leq \frac{(M\,t)^{n+1}}{\Gamma(n+1)}$$

for some $M > 0$. The claim is proved by *induction*:

- First we inspect

$$\begin{aligned} d^{(1)}(t) &= \prec \left|\left|\int_0^t ds\, \boldsymbol{b}(\boldsymbol{\xi}_o, s) + \int_0^t ds\, \boldsymbol{\sigma}(\boldsymbol{\xi}_o, s)[d\boldsymbol{w}_s]\right|\right|^2 \succ \\ &\leq 2 \prec \left|\left|\int_0^t ds\, \boldsymbol{b}(\boldsymbol{\xi}_o, s)\right|\right|^2 \succ + 2\int_0^t ds \prec \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\sigma}^\dagger)(\boldsymbol{\xi}_o, s) \succ \end{aligned}$$

  having so bounded from above the cross product. In order to estimate the first term we can use the Cauchy-Schwartz inequality:

$$\prec \left|\left|\int_0^t ds\, \boldsymbol{b}(\boldsymbol{\xi}_o, s)\right|\right|^2 \succ \leq \int_0^t ds \prec ||\boldsymbol{b}(\boldsymbol{\xi}_o, s)||^2 \succ \int_0^t ds$$

  The Lipschitz condition yields upper bounds on the remaining expressions:

$$d^{(0)}(t) \leq 2\int_0^t ds\, L^2 T \prec (1 + ||\boldsymbol{\xi}_o||)^2 \succ + 2\int_0^t ds\, L^2 \prec (1 + ||\boldsymbol{\xi}_o||)^2 \succ \leq M\,t$$

  for

$$M \geq 4\,L^2(1+T) \prec 1 + ||\boldsymbol{\xi}_o||^2 \succ$$

  We can then proceed by *induction*.

- Then we *suppose* that

$$d^{(n)}(t) \leq \frac{(M\,t)^n}{\Gamma(n)}$$

  holds true.

- The last step is to observe that

$$d^{(n+1)}(t) =$$
$$\prec \left|\left|\int_0^t ds\,[\boldsymbol{b}\left(\boldsymbol{\xi}_s^{(n)}, s\right) \,-\boldsymbol{b}\left(\boldsymbol{\xi}_s^{(n-1)}, s\right)] + \int_0^t ds\,[\boldsymbol{\sigma}\left(\boldsymbol{\xi}_s^{(n)}, s\right) - \boldsymbol{\sigma}\left(\boldsymbol{\xi}_s^{(n-1)}, s\right)][d\boldsymbol{w}_s]\right|\right|^2 \succ$$

  satisfies the inequality

$$d^{(n+1)}(t) \leq 2\,L^2(1+T)\int_0^t ds\, \prec ||\boldsymbol{\xi}_s^{(n)} - \boldsymbol{\xi}_s^{(n-1)}||^2 \succ$$
$$\leq 2\,L^2(1+T)\int_0^t ds\,\frac{(M\,s)^n}{\Gamma(n)} = \frac{2\,L^2(1+T)\,M^n\,t^{n+1}}{\Gamma(n+1)}$$

  whence finally we are entitled to conclude

$$d^{(n+1)}(t) \leq \frac{M^{n+1}\,t^{n+1}}{\Gamma(n+1)}$$

The bound yields mean square convergence, but it is is not sufficient as such to prove the almost sure convergence of the Picard's iteration. Cauchy-Schwartz inequality, however, gives us

$$\max_{0\leq t\leq T}||\boldsymbol{\xi}_t^{(n+1)} - \boldsymbol{\xi}_t^{(n)}|| \leq$$
$$2\,T\,L^2\int_0^T dt \prec ||\boldsymbol{\xi}_t^{(n)} - \boldsymbol{\xi}_t^{(n-1)}||^2 \succ +2\max_{0\leq t\leq T}\left|\left|\int_0^T[\boldsymbol{\sigma}\left(\boldsymbol{\xi}_s^{(n)}, s\right) - \boldsymbol{\sigma}\left(\boldsymbol{\xi}_s^{(n-1)}, s\right)][dw_s]\right|\right|^2$$

We have proved however that for martingales

$$\prec \max_{0\leq\, t\leq\, T}||\xi_t||^p \succ \leq \max_{0\leq\, t\leq\, T}\frac{p}{p-1}\prec ||\xi_t||^p \succ$$

In consideration of such martingale inequality, we attain the bound

$$\prec \max_{0\leq t\leq T}||\boldsymbol{\xi}_t^{(n+1)} - \boldsymbol{\xi}_t^{(n)}|| \succ \leq C\int_0^T dt \prec ||\boldsymbol{\xi}_t^{(n)} - \boldsymbol{\xi}_t^{(n-1)}||^2 \succ \leq C\,\frac{M^n\,t^n}{\Gamma(n)}$$

which on its turn entitles us to use invoke Borel-Cantelli lemma. Namely if we pick any $0 < \varepsilon < 1$ and observe by Čebišev that

$$P\left(\max_{0\leq t\leq T}||\boldsymbol{\xi}_t^{(n+1)} - \boldsymbol{\xi}_t^{(n)}|| > \varepsilon^{n+1}\right) \leq \varepsilon^{-2\,(n+1)} \prec \max_{0\leq t\leq T}||\boldsymbol{\xi}_t^{(n+1)} - \boldsymbol{\xi}_t^{(n)}||^2 \succ \leq C\,\frac{(\varepsilon^{-2}\,M\,t)^{n+1}}{\Gamma(n+1)}$$

then

$$\sum_{n=0}^{\infty} P\left(\max_{0\leq t\leq T}||\boldsymbol{\xi}_t^{(n+1)} - \boldsymbol{\xi}_t^{(n)}|| > \varepsilon^{n+1}\right) < \infty$$

We conclude that

$$\boldsymbol{\xi}_t^{(n)} = \boldsymbol{\xi}_t^{(0)} + \sum_{n=0}^{n-1}\left(\boldsymbol{\xi}_t^{(n+1)} - \boldsymbol{\xi}_t^{(n)}\right) \stackrel{n\uparrow\infty}{\to} \boldsymbol{\xi}_t \qquad a.s.$$

with

$$\boldsymbol{\xi}_t = \boldsymbol{\xi}_o + \lim_{n\uparrow\infty} \left\{ \int_0^t ds\, \boldsymbol{b}\left(\boldsymbol{\xi}_s^{(n)}, s\right) + \int_0^t ds\, \boldsymbol{\sigma}\left(\boldsymbol{\xi}_s^{(n)}, s\right)[d\boldsymbol{w}_s] \right\}$$
$$= \boldsymbol{\xi}_o + \int_0^t ds\, \boldsymbol{b}\left(\boldsymbol{\xi}_s, s\right) + \int_0^t ds\, \boldsymbol{\sigma}\left(\boldsymbol{\xi}_s, s\right)[d\boldsymbol{w}_s]$$

having used the dominated convergence theorem. It remains now to show that the solution belongs to $\mathbb{L}^{(2)}(\Omega \times [0,T])$. There result follows from bounds similar to the above:

$$\prec \left|\left|\xi_t^{(n+1)}\right|\right|^2 \succ \leq 3 \prec ||\boldsymbol{\xi}_o|| \succ +3 \prec \left|\left|\int_0^t ds\, \boldsymbol{b}\left(\boldsymbol{\xi}_s^{(n)}, s\right)\right|\right|^2 \succ +3 \prec \left|\left|\int_0^t ds\, \boldsymbol{\sigma}\left(\boldsymbol{\xi}_s^{(n)}, s\right)[d\boldsymbol{w}_s]\right|\right|^2 \succ$$

whence

$$\prec \left|\left|\xi_t^{(n+1)}\right|\right|^2 \succ \leq 3 \prec ||\boldsymbol{\xi}_o|| \succ +6\, L^2(T+1) \int_0^t ds \prec 1 + \left|\left|\xi_s^{(n)}\right|\right|^2 \succ \leq \tilde{C} e^{\tilde{C} t}$$

by recursion for some $\tilde{C} > 0$. Passing to the limit yields the claim.

Uniqueness

Suppose there is an $\tilde{\boldsymbol{\xi}}_t$ also satisfying the stochastic differential equation. Then

$$\prec \left|\left|\boldsymbol{\xi}_t - \tilde{\boldsymbol{\xi}}_t\right|\right|^2 \succ = \prec \left|\left|\int_0^t ds\, [\boldsymbol{b}\left(\boldsymbol{\xi}_s, s\right) - \boldsymbol{b}\left(\tilde{\boldsymbol{\xi}}_s, s\right)] + \int_0^t ds\, [\boldsymbol{\sigma}\left(\boldsymbol{\xi}_s, s\right) - \boldsymbol{\sigma}\left(\tilde{\boldsymbol{\xi}}_s, s\right)][d\boldsymbol{w}_s]\right|\right|^2 \succ$$

By the same inequalities as above there is a positive constant $K > 0$ such that

$$\prec \left|\left|\boldsymbol{\xi}_t - \tilde{\boldsymbol{\xi}}_t\right|\right|^2 \succ \leq K \int_0^t ds \prec \left|\left|\boldsymbol{\xi}_t - \tilde{\boldsymbol{\xi}}_t\right|\right|^2 \succ$$

Gronwall lemma (see appendix A) for a function vanishing at the lower boundary allows us to conclude that

$$\tilde{\boldsymbol{\xi}}_t = \boldsymbol{\xi}_t$$

in mean square. The martingale inequality ensures in such a case that the same equality holds almost surely. □

2.1 Example: absence of Lipschitz continuity

Consider the ordinary differential equation:

$$\dot{\xi} = C \xi^{1/3}$$

The field

$$f = C\, x^{1/3}$$

is *not* differentiable in zero therefore not Lipschitz continuous there. As a consequence the equation has multiple solutions

$$\xi_t = \begin{cases} 0 & t < t_o \\ \tilde{C}\, t^{3/2} & t \geq t_o \end{cases}$$

for arbitrary t_o.

3 Solution by iteration

If $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ are smooth

$$\begin{aligned}\boldsymbol{\xi}_t &= \boldsymbol{\xi}_o + \int_0^t ds\, \boldsymbol{b}(\boldsymbol{\xi}_s, s) + \int_0^t \boldsymbol{\sigma}(\boldsymbol{\xi}_s, s)[d\boldsymbol{w}_s] \\ &= \boldsymbol{\xi}_o + \boldsymbol{b}(\boldsymbol{\xi}_o, 0)\, t + \boldsymbol{\sigma}(\boldsymbol{\xi}_o, 0)[d\boldsymbol{w}_t] + \int_0^t ds \int_0^s d\boldsymbol{b}(\boldsymbol{\xi}_u, u) + \int_0^t \int_0^s d\boldsymbol{\sigma}(\boldsymbol{x}_u, u)[d\boldsymbol{w}_s] \\ &= \boldsymbol{\xi}_o + \boldsymbol{b}(\boldsymbol{\xi}_o, 0)\, t + \boldsymbol{\sigma}(\boldsymbol{\xi}_o, 0)[d\boldsymbol{w}_t] + \int_0^t d\boldsymbol{b}(\boldsymbol{\xi}_s, s)\,(t-s) + \int_0^t d\boldsymbol{\sigma}(\boldsymbol{x}_s, s)[\boldsymbol{w}_t - \boldsymbol{w}_s] \end{aligned} \tag{3.1}$$

We then apply Ito lemma to $\boldsymbol{b}$ and $\boldsymbol{\sigma}$ and iterate. In such a way the solution is constructed as a power series in t and w_t.

Example 3.1 (*$1d$-linear case*)**.** Consider the Ito SDE

$$d\xi_t = \frac{\xi_t}{\tau} dt + \sigma\, \xi_t dw_t \tag{3.2}$$

we can remove the drift by setting

$$\xi_t = \tilde{\xi}_t e^{\frac{t}{\tau}}$$

The new process $\tilde{\xi}_t$ is related to the original by a function independent of the Wiener process. Hence, Ito calculus lemma

$$d(\tilde{\xi}_t \eta_t) = (d\tilde{\xi}_t)\eta_t + \tilde{\xi}_t d\eta_t + < d\tilde{\xi}_t\,, d\eta_t >$$

($< \bullet\,, \bullet >$ is the quadratic co-variation) reduces for

$$\eta_t = e^{\frac{t}{\tau}}$$

to the standard Leibniz rule. We find

$$d(\tilde{\xi}_t e^{\frac{t}{\tau}}) = (d\tilde{\xi}_t) e^{\frac{t}{\tau}} + \tilde{\xi}_t \frac{e^{\frac{t}{\tau}}}{\tau}$$

The new Ito stochastic differential equation is

$$d\tilde{\xi}_t = \sigma\, \tilde{\xi}_t dw_t$$

If we apply the recursion equations (3.1) we get into

$$\begin{aligned}\tilde{\xi}_t &= \tilde{\xi}_o + \sigma\, \tilde{\xi}_o\, w_t + \sigma \int_0^t dw_s \int_0^s d\tilde{\xi}_{s_1} \\ &= \tilde{\xi}_o + \sigma\, \tilde{\xi}_o\, w_t + \sigma^2 \tilde{\xi}_o \int_0^t dw_s \int_0^s dw_{s_1} + \sigma^2 \int_0^t dw_s \int_0^s dw_{s_1} \int_0^{s_2} d\tilde{\xi}_{s_2}\end{aligned}$$

Repeating for an arbitrary number of steps

$$\tilde{\xi}_t = \tilde{\xi}_o + \tilde{\xi}_o \sum_{i=1}^{\infty} \sigma^i \int_0^t dw_{s_1} \prod_{j=1}^{i-1} \int_0^{s_j} dw_{s_j} \tag{3.3}$$

We have proved in a previous lecture that

$$\int_0^t dw_{s_1} \prod_{j=1}^{i-1} \int_0^{s_j} dw_{s_j} = h_i(w_t, t)$$

with h_i the Hermite polynomial

$$h_i(x,t) = \frac{t^n}{\Gamma(i+1)} \left.\frac{d^n}{dz^n}\right|_{z=0} e^{\frac{z\,x}{t} - \frac{z^2}{2\,t}} = \frac{1}{\Gamma(i+1)} \left.\frac{d^n}{d\lambda^n}\right|_{z=0} e^{\lambda\,x - \frac{\lambda^2\,t}{2}}$$

Upon inserting in (3.3), we get into

$$\tilde{\xi}_t = \tilde{\xi}_o \left\{ 1 + \sum_{i=1}^{\infty} \frac{\sigma^i h_i(w_t, t)}{\Gamma(i+1)} \right\} = \xi_o\, e^{\sigma w_t - \frac{\sigma^2\,t}{2}}$$

and consequently

$$\xi_t = \xi_o\, e^{\frac{t}{\tau} + \sigma w_t - \frac{\sigma^2\,t}{2}}$$

The same result is straightforwardly obtained by converting (3.2) to *Stratonovich form*

$$d\xi_t = \left(1 - \frac{\sigma^2\,\tau}{2}\right) \xi_t \frac{dt}{\tau} + \sigma\, \xi_t\, dw_t$$

and by integrating it according to the usual rules of calculus

$$\xi_t = \xi_o\, e^{\left(1 - \frac{\sigma^2\,\tau}{2}\right)\frac{t}{\tau} + \sigma\, w_t}$$

Appendix

A Gronwall lemma

Lemma A.1 (*Gronwall*). *Let*

$$\phi : [0,T] \to \mathbb{R}_+ \qquad \& \qquad f : [0,T] \to \mathbb{R}_+$$

and let $C_0 \geq 0$ *a real constant. If for all* $0 \leq t \leq T$

$$\phi_t \leq C_0 + \int_0^t ds\, f_s\, \phi_s$$

then

$$\phi_t \leq C_0 e^{\int_0^t ds\, f_s}$$

Proof. First observe

$$\frac{d\phi_t}{dt} \leq f_t\, \phi_t$$

ϕ_t may vanish, so we cannot divide both side by ϕ_t in order to couch the left hand side in the form of a logarithmic derivative. Instead we set

$$\Phi_t = C_0 + \int_0^t ds\, f_s\, \phi_s$$

and obtain

$$\frac{d\Phi_t}{dt} = f_t\, \phi_t \leq \; f_t\, \Phi_t \qquad\qquad \phi_t \leq \Phi_t$$

Then

$$\frac{d}{dt}\left[e^{-\int_0^t ds\, f_s}\, \Phi_t\right] \;\leq\; 0$$

implying that

$$e^{-\int_0^t ds\, f_s}\, \Phi_t \leq \Phi_0 \leq \; C_0$$

and therefore (the exponential is strictly positive under current hypotheses)

$$\Phi_t \leq C_0\, e^{\int_0^t ds\, f_s}$$

□

References

[1] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, `http://math.berkeley.edu/~evans/`