1 Introduction

The content of these notes is also covered by chapter 4 section A of [1].

2 White noise

In applications infinitesimal Wiener increments are represented as

$$dw_t = \eta_t \, dt$$

The stochastic process η_t is referred to as *white noise*. Consistence of the definition requires η_t to be a Gaussian process with the following properties.

- *Zero average*:

$$\prec \eta_t \succ = 0$$

Namely we must have

$$0 = \prec w_{t+dt} - w_t \succ = \prec \eta_t \succ \; dt$$

- *δ-correlation*: at any instant of time

$$\prec \eta_t \eta_s \succ = \sigma^2 \, \delta^{(1)}(t-s) \, , \qquad \qquad \sigma^2 \; > \; 0$$

as it follows from the identification

$$\frac{d}{dt}\frac{d}{ds} \prec w_t w_s \succ = \prec \eta_t \eta_s \succ \tag{2.1}$$

Namely, writing the correlation of the Wiener process in terms of the Heaviside step function

$$\prec w_t w_s \succ = \sigma^2 [s \, H(t-s) + t \, H(s-t)] \tag{2.2}$$

and observing

$$\frac{d}{dt} H(t-s) = \delta^{(1)}(t-s)$$

we obtain, upon inserting (2.2) into (2.1)

$$\frac{d}{dt}\frac{d}{ds} \prec w_t w_s \succ = \sigma^2 \frac{d}{dt}[H(t-s) - s \, \delta^{(1)}(t-s)] + \sigma^2 \frac{d}{ds}[H(s-t) - t \, \delta^{(1)}(s-t)]$$

By construction the δ is even an even function of its argument: the right hand side can be couched into the form

$$\frac{d}{dt}\frac{d}{ds} \prec w_t w_s \succ = \sigma^2 \left[2 \, \delta^{(1)}(t-s) + (t-s) \, \frac{d}{dt} \delta^{(1)}(t-s) \right]$$

In order to interpret the meaning of the derivative, we integrate the right hand side over a smooth test function f

$$\begin{aligned} &\int_{s-\varepsilon}^{s+\varepsilon} dt \, f(t) \, (t-s) \, \frac{d}{dt} \delta^{(1)}(t-s) \\ &= -\int_{s-\varepsilon}^{\sigma+\varepsilon} dt \, \frac{df}{dt}(t) \, (t-s) \, \delta^{(1)}(t-s) - \int_{s-\varepsilon}^{\sigma+\varepsilon} dt \, f(t) \, \delta^{(1)}(t-s) \\ &= -\int_{s-\varepsilon}^{\sigma+\varepsilon} dt \, f(t) \, \delta^{(1)}(t-s) = -f(s) \end{aligned} \tag{2.3}$$

We conclude that

$$\frac{d}{dt}\frac{d}{ds} \prec w_t w_s \succ = \sigma^2 \delta^{(1)}(t-s)$$

the identity determining the value of the white noise correlation.

3 Paley-Wiener-Zygmund integral

Definition 3.1 (*Paley-Wiener-Zygmund integral*). *Let*

$$g : [0,T] \to \mathbb{R}$$

a continuously differentiable function such that

$$g(0) = g(T) = 0$$

The random variable

$$G_T = \int_0^T dw_t \, g(t)$$

is defined as

$$\int_0^T dw_t \, g(t) = -\int_0^T dt \, w_t \frac{dg}{dt}(t)$$

The Paley-Wiener-Zygmund integral can be tackled resorting to standard Lebesgue-Stieltjes integration theory. We can prove

Proposition 3.1. *i* $\prec G_T \succ = 0$

ii $\prec G_T^2 \succ = \int_0^T dt \, g^2(t)$

Proof. i it follows by exchanging the order between integral and average.

ii By definition we have

$$\prec G_T^2 \succ = \int_0^T dt \int_0^T ds \frac{dg}{dt}(t)\frac{dg}{ds}(s) \prec w_t w_s \succ = \int_0^T dt \int_0^T ds \, g(t)g(s)\frac{d^2}{dtds} \prec w_t w_s \succ$$

If we now use the calculation of section (2), we get into

$$\prec G_T^2 \succ = \int_0^T dt \int_0^T ds \, g(t)g(s) \, \delta^{(1)}(t-s) = \int_0^T dt \, g^2(t)$$

(for $\sigma^2 = 1$) which proves the claim.

□

4 Gaussian statistics and δ-correlation

Gaussian statistics and δ-correlation imply that η_t is independent of η_s for any $t \neq s$. The claim is verified by inspection of the characteristic function of the white-noise. The Paley-Wiener-Zygmund integral allows us to write for some smooth g chosen as in section 3

$$\prec e^{\imath\lambda \int_0^T dt \eta_t g(t)} \succ \equiv \prec e^{\imath\lambda \int_0^T dw_t g(t)} \succ = \prec e^{-\imath\lambda \int_0^T dt\, w_t g'(t)} \succ$$

The leftmost integral can be computed using e.g. the Karhunen-Loève representation of the Brownian motion

$$\prec e^{-\imath\lambda \int_0^T dt\, w_t g'(t)} \succ = \prec e^{-\imath\lambda \sum_n c_n \int_0^T dt\, \psi_n(t)\, g'(t)} \succ$$

here we used the shorthand notation

$$g' := \frac{dg}{dt}$$

Randomness is stored in the $\{c_n\}_{n=0}^{\infty}$ which form a sequence of independent Gaussian random variables with zero average and variance for c_n equal to the n-th eigenvalue of the operator defined by

$$R(t\,,s) = \prec w_t\, w_s \succ$$

Thus we obtain

$$\prec e^{-\imath\lambda \int_0^T dt\, w_t g'(t)} \succ = e^{-\frac{\lambda^2}{2} \sum_{n=0}^{\infty} \int_0^T dt \int_0^T ds\, r_n\, \psi_n(t)\psi_n(s)\, g'(t) g'(s)} = e^{-\frac{\lambda^2}{2} \int_0^T dt \int_0^T ds\, R(t\,,s)\, g'(t) g'(s)}$$

since by construction

$$R(t\,,s) = \sum_{n=0}^{\infty} r_n\, \psi_n(t)\psi_n(s)$$

The lemma in 3 then guarantees us

$$\prec e^{\imath\lambda \int_0^T dt \eta_t g(t)} \succ \equiv \prec e^{\imath\lambda \int_0^T dw_t g(t)} \succ = e^{-\frac{\lambda^2}{2} \int_0^T dt\, g^2(t)}$$

We can read the result in two ways.

- Characteristic function of

  $$G_T := \int_0^T dw_t\, g(t)$$

  We have just proved that

  $$\prec e^{\imath\,\lambda\, G_T} \succ = e^{-\frac{\lambda^2}{2} \prec G_T^2 \succ}$$

  i.e. that G_T has a Gaussian distribution.

- "Characteristic function" of the white noise. Let us set λ equal to unit and inspect

  $$\prec e^{\imath \int_0^T dt\, \eta_t\, g(t)} \succ = e^{-\frac{1}{2} \int_0^T dt\, g^2(t)} \tag{4.1}$$

Interpreting integrals as sums

$$\int_0^T dt\, \eta_t\, g(t) \sim \sum_k dt\, \eta_{t_k}\, g(t_k)$$

were the η_{t_k} a collection of random Gaussian variables with correlation

$$\prec \eta_{t_k} \eta_{t_l} \succ = C_{kl}$$

we would write

$$\prec e^{\imath \sum_k dt\, \eta_{t_k}\, g(t_k)} \succ = e^{-\frac{1}{2}\sum_k dt \sum_l dt\, g(t_k) C_{k\,l} g(t_l)}$$

Contrasting this latter result with (4.1) it is tempting to identify

$$\sum_k dt \sum_l dt\, g(t_k) C_{k\,l} g(t_l) \overset{dt\downarrow 0}{\rightarrow} \int_0^T dt \int_0^T ds\, g(t) C(t-s)\, g(s)$$

and

$$C(t-s) = \delta^{(1)}(t-s)$$

This is in agreement with the claim that white noise is "Gaussian" with zero average and δ-Dirac covariance.

From (4.1) we can also read all the moments of the white noise. In order to do so we need to take functional derivatives with respect to $g(t)$ (see appendix A, in practice treat the function argument as an index and replace δ-Kroenecker with δ-functions). For any $0 < s < T$

$$\begin{aligned}\frac{\delta}{\delta g(s)} \prec e^{\imath \int_0^T dt\, g(t) \eta_t} \succ &= \frac{\delta}{\delta g(s)} e^{-\frac{1}{2}\int_0^T dt\, g^2(t)} \\ &= -e^{-\frac{1}{2}\int_0^T dt\, g^2(t)} \int_0^T dt\, g(t)\, \delta^{(1)}(t-s) = -g(s) e^{-\frac{1}{2}\int_0^T dt\, g^2(t)}\end{aligned}$$

implies

$$\prec \eta_s \succ = 0$$

Furthermore for $0 < s\, u < T$

$$\begin{aligned}\frac{\delta^2}{\delta g(s) \delta g(u)} \prec e^{\imath \int_0^T dt\, g(t) \eta_t} \succ &= -\frac{\delta}{\delta g(u)} g(s)\, e^{-\frac{1}{2}\int_0^T dt\, g^2(t)} \\ &= -\delta^{(1)}(u-s)\, e^{-\frac{1}{2}\int_0^T dt\, g^2(t)} + g(s)\, g(u)\, e^{-\frac{1}{2}\int_0^T dt\, g^2(t)}\end{aligned}$$

implies

$$\prec \eta_s \eta_u \succ = \delta^{(1)}(u-s)$$

which recovers for $\sigma^2 = 1$ the result obtained in section 2

A Functional derivative (for practical purposes)

Consider a functional space of continuous/smooth functions ϕ (eventually also satisfying certain boundary conditions) and a functional $F[\phi]$. We define the functional derivative of F, denoted $\delta F/\delta\phi(\boldsymbol{x})$, as the distribution $\delta F[\phi]$ such that for all test functions f:

$$\int_{\mathbb{R}^d} d^d x \, \frac{\delta F[\phi]}{\delta \phi(\boldsymbol{x})} \, f(\boldsymbol{x}) := \left. \frac{d}{d\varepsilon} F[\phi + \varepsilon\, f] \right|_{\epsilon=0}$$

Thus if

$$F[\phi] = \int_{\mathbb{R}^d} d^d x \, \phi^2(\boldsymbol{x}) \tag{A.1}$$

the definition yields

$$\int_{\mathbb{R}^d} d^d x \, \frac{\delta F[\phi]}{\delta \phi(\boldsymbol{x})} \, f(\boldsymbol{x}) = 2 \int_{\mathbb{R}^d} d^d x \, \phi(\boldsymbol{x}) f(\boldsymbol{x})$$

Alternatively, we can define

$$\frac{\delta F[\phi]}{\delta \phi(\boldsymbol{x})} = \lim_{\varepsilon \to 0} \frac{F[\varphi_{\boldsymbol{x}}] - F[\phi]}{\varepsilon}$$

with $\varphi_{\boldsymbol{x}}$ specified by

$$\varphi_{\boldsymbol{x}}(\boldsymbol{y}) = \phi(\boldsymbol{y}) + \varepsilon\, \delta^{(d)}(\boldsymbol{x} - \boldsymbol{y})$$

For the the example (A.1) this means

$$\frac{\delta F[\phi]}{\delta \phi(\boldsymbol{x})} = 2\, \phi(\boldsymbol{x})$$

References

[1] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, `http://math.berkeley.edu/~evans/`