1 Introduction

The content of these notes is also covered by chapter 3 section C of [1].An alternative reference for Brownian motion and it properties is provided by Varadhan's lecture notes [2].

2 Kolmogorov–Čentsov theorem

Theorem 2.1 (*Kolmogorov-Čentsov*)**.** *If $\xi_t(\omega)$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ satisfying*

$$\prec |\xi_t - \xi_s|^{\beta} \succ \leq C\,|t-s|^{1+\alpha}$$

for some positive constants α, β and C, then if necessary , $\xi_t(\omega)$ can be modified for each t on a set of measure zero, to obtain an equivalent version $\tilde{\xi}_t(\omega)$ that is almost surely continuous with exponent γ for every $\gamma \in [0, \alpha/\beta]$ and some $\delta > 0$:

$$P\left(\omega\,;\; \sup_{\substack{0\,<\,t-s<h(\omega)\\ t,s\in[0\,,T]}} \frac{|\tilde{\xi}_t(\omega)-\tilde{\xi}_s(\omega)|}{|t-s|^{\gamma}} \leq \delta\right) = 1$$

Proof. Let

$$\mathcal{T}_n = \left\{\frac{j\,T}{2^n}\right\}_{j=0}^{2^n}$$

The countable union $\mathcal{T} := \cup_{n=0}^{\infty}\mathcal{T}_n$ is a *countable dense* subset of $[0\,,T]$. By linear interpolation we can construct a sequence of approximations to the original process $\xi(t)$ coinciding with it on dyadic rationals e.g.

- the $\mathcal{T}_n$-based interpolation is

$$\xi_n(t) = \xi\left(\frac{j}{2^n}T\right) + \left(t - \frac{j\,T}{2^n}\right)\frac{\xi\left(\frac{j+1}{2^n}T\right) - \xi\left(\frac{j}{2^n}\right)}{\frac{T}{2^n}} \qquad \& \qquad t \in [\frac{j}{2^n}T\,, \frac{j+1}{2^n}T] \equiv \mathcal{I}_j^n$$

- The $\mathcal{T}_{n+1}$-based interpolation is

$$\xi_{n+1}(t) = \xi\left(\frac{j}{2^n}T\right) + \left(t - \frac{j\,T}{2^n}\right)\frac{\xi\left(\frac{2\,j+1}{2^{n+1}}T\right) - \xi\left(\frac{j\,T}{2^n}\right)}{\frac{T}{2^{n+1}}} \qquad \& \qquad t \in [\frac{j\,T}{2^n}\,, \frac{(2\,j+1)\,T}{2^{n+1}}] \equiv \mathcal{I}_{2\,j}^{n+1}$$

so that $t \in \mathcal{T}_n$ we have

$$\xi(t) = \xi_n(t) = \xi_{n+1}(t) \qquad\qquad t \in \mathcal{T}_n$$

and for $t \in \mathcal{I}_j^n \cap \mathcal{I}_{2\,j}^{n+1}$

$$|\xi_{n+1}(t) - \xi_n(t)| = \left(t - \frac{j\,T}{2^n}\right)\left|\frac{\xi\left(\frac{2\,j+1}{2^{n+1}}T\right) - \xi\left(\frac{j\,T}{2^n}\right)}{\frac{T}{2^{n+1}}} - \frac{\xi\left(\frac{j+1}{2^n}T\right) - \xi\left(\frac{j\,T}{2^n}\right)}{\frac{T}{2^n}}\right|$$

which can be rewritten as

$$|\xi_{n+1}(t) - \xi_n(t)| = \frac{2^{n+1}}{T}\left(t - \frac{j\,T}{2^n}\right)\left|\xi\left(\frac{2\,j+1}{2^{n+1}}T\right) - \frac{\xi\left(\frac{j\,T}{2^n}\right) + \xi\left(\frac{j+1}{2^n}T\right)}{2}\right|$$

so that by the triangular inequality:

$$|\xi_{n+1}(t)-\xi_n(t)| \le \frac{\left|\xi\left(\frac{2\,j+1}{2^{n+1}}T\right)-\xi\left(\frac{j\,T}{2^n}\right)\right|+\left|\xi\left(\frac{2\,j+1}{2^{n+1}}T\right)-\xi\left(\frac{j+1}{2^n}T\right)\right|}{2}$$

$$\le \left|\xi\left(\frac{2\,j+1}{2^{n+1}}T\right)-\xi\left(\frac{(j+1)\,T}{2^n}\right)\right| \vee \left|\xi\left(\frac{2\,j+1}{2^{n+1}}T\right)-\xi\left(\frac{j\,T}{2^n}\right)\right|$$

The inequality entails that

$$\sup_{0\le t\le T}|\xi_{n+1}(t)-\xi_n(t)| \le \sup_{0\le j<2^n-1}\left|\xi\left(\frac{2\,j+1}{2^{n+1}}T\right)-\xi\left(\frac{(j+1)\,T}{2^n}\right)\right| \vee \left|\xi\left(\frac{2\,j+1}{2^{n+1}}T\right)-\xi\left(\frac{j\,T}{2^n}\right)\right|$$

The following inequalities hold true:

- if the occurrence of the event B implies the occurrence of the event A we must have $A \subseteq B$ and therefore $P(A) \le P(B)$ which translates for us into

$$P\left(\sup_{0\le t\le 1}|\xi_{n+1}(t)-\xi_n(t)| \ge 2^{-n\,\gamma}\right)$$
$$\le P\left(\sup_{0\le j\le 2^{n+1}-1}\left|\xi\left(\frac{(j+1)\,T}{2^{n+1}}\right)-\xi_n\left(\frac{j\,T}{2^{n+1}}\right)\right| \ge 2^{-n\,\gamma}\right)$$

- as $P(\cup_k A_k) \le \sum_k P(A_k)$:

$$P\left(\sup_{0\le j\le 2^{n+1}-1}\left|\xi\left(\frac{(j+1)\,T}{2^{n+1}}\right)-\xi_n\left(\frac{j\,T}{2^{n+1}}\right)\right| \ge 2^{-n\,\gamma}\right)$$
$$\le 2^{n+1}\sup_{0\le j\le 2^{n+1}-1} P\left(\left|\xi\left(\frac{(j+1)\,T}{2^{n+1}}\right)-\xi_n\left(\frac{j\,T}{2^{n+1}}\right)\right| \ge 2^{-n\,\gamma}\right)$$

- By Čebyšev inequality and using the theorem's hypothesis

$$P\left(\left|\xi\left(\frac{(j+1)\,T}{2^{n+1}}\right)-\xi_n\left(\frac{j\,T}{2^{n+1}}\right)\right| \ge 2^{-n\,\gamma}\right)$$
$$\le \frac{\prec |\xi\left(\frac{(j+1)\,T}{2^{n+1}}\right)-\xi_n\left(\frac{j\,T}{2^{n+1}}\right)|^{\beta} \succ}{2^{-n\,\beta\,\gamma}} \le 2^{n\beta\,\gamma}\,C\,\left(\frac{T}{2^{n+1}}\right)^{\alpha+1}$$

Gleaning the above information together we obtain

$$P\left(\sup_{0\le t\le T}|\xi_n\,(t)-\xi_{n+1}\,(t)\,|\right) \le C\,T^{\alpha+1}\,\left(\frac{1}{2}\right)^{n\,(\alpha-\gamma\,\beta)}$$

If

$$\gamma > \frac{\alpha}{\beta}$$

the inequality entitle us to apply Borel-Cantelli lemma and conclude

$$\sup_{0\le t\le T}|\xi_n\,(t)-\xi_{n+1}\,(t)\,| \to 0 \qquad a.s.$$

and consequently

$$\lim_{n\uparrow\infty}\xi_n\,(t)=\tilde{\xi}\,(t)$$

The limit $\xi\tilde{(t)}$ will be continuous on $[0\,,T]$ and will coincide with $\xi(t)$ on $\mathcal{T}$ thereby establishing our result. □

3 Summary of notions of convergence

There are free notion of convergence

- *Convergence in probability*: $\{\xi_i\}_{i=1}^{\infty}$ converges to ξ in probability if for every positive ϵ

$$\lim_{n\uparrow\infty} P(|\xi_n - \xi| < \epsilon) = 0$$

- *Mean square convergence*:$\{\xi_i\}_{i=1}^{\infty}$ converges to ξ in mean square if for every positive ϵ

$$\lim_{n\uparrow\infty} \prec (\xi_n - \xi)^2 \succ = 0 \qquad \xi_n \overset{n\uparrow\infty}{\to} \xi \quad m.s.$$

- *Almost sure convergence*: $\{\xi_i\}_{i=1}^{\infty}$ converges to ξ almost surely (i.e. $P = 1$) if the event

$$\left\{\xi_n \overset{n\uparrow\infty}{\to} \xi\right\} \equiv \left\{\omega \in \Omega \mid \xi_n(\omega) \overset{n\uparrow\infty}{\to} \xi\right\}$$

  has probability one.

Convergence in probability is the weakest notion

$$\xi_n \overset{m.s.}{\to} \xi \quad \Rightarrow \xi_n \quad \overset{P}{\to} \xi \qquad \text{by Čebyšev}$$
$$\xi_n \overset{a.s.}{\to} \xi \quad \Rightarrow \quad \xi_n \overset{P}{\to} \xi$$

Almost sure convergence *does not* imply mean square convergence:

$$\xi_n \overset{a.s.}{\to} \xi \quad \not\Rightarrow \quad \xi_n \overset{m.s.}{\to} \xi$$

For example let $\{\xi_n\}_{n=0}^{\infty}$ independent uniformly distributed such that

$$\xi_n = \begin{cases} n & \omega \in [0\,, 1/n] \\ 0 & \end{cases}$$

then

$$\xi_n \overset{a.s.}{\to} 0$$

but

$$\prec \xi_n^2 \succ = n$$

Conversely, mean square convergence *does not* almost sure convergence:

$$\xi_n \overset{m.s.}{\to} \xi \quad \not\Rightarrow \quad \xi_n \overset{a.s.}{\to} \xi$$

4 Non differentiability of the Brownian motion

Even if we adopt the weakest notion of convergence, i.e. convergence in probability, Brownian motion turns out to be non-differentiable.

$$p_{\frac{w_{t+h}-w_t}{h}}(x) = \int_{\mathbb{R}} dy\, \delta^{(1)}\left(x - \frac{y}{h}\right) \frac{e^{-\frac{y^2}{2\,\sigma^2\,h}}}{\sqrt{2\,\sigma^2\,h}} = \frac{\sqrt{|h|}e^{-\frac{h\,x^2}{2\,\sigma^2}}}{\sqrt{2\,\pi\,\sigma^2}} \overset{h\downarrow 0}{\to} 0$$

so that for any $c \in \mathbb{R}_+$

$$P\left(-c \leq \frac{w_{t+h} - w_t}{h} \leq c\right) \leq C\sqrt{\frac{|h|}{2\,\pi}} \overset{h\downarrow 0}{\to} 0$$

the measure concentrates for infinite values of the differential ratio.

References

[1] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, http://math.berkeley.edu/~evans/

[2] S.R.S. Varadhan, *Stochastic Calculus, lecture notes*, http://www.math.nyu.edu/faculty/varadhan/stochastic.fall08.html

[3] T. Sapsis, and P. Lermusiaux, *Dynamically Orthogonal field equations for continuous stochastic dynamical systems*, Physica D, **238** 2347 (2009)and http://web.mit.edu/sapsis/www/papers/DO_fld_eqs.pdf