1 Introduction

The content of these notes is also covered by chapter 3 section A and B of [1].

2 Counter-example: process with jumps

In the derivation of the Fokker-Planck equation we used

$$Dx^2 = \sigma^2 \, Dt \qquad\qquad (x = i\, Dx \quad \& \quad t = n\, Dt) \tag{2.1}$$

with Dx, Dt respectively the "space" and "time" mesh (spacing) sizes of the lattice $(i, n) \in \mathbb{Z}^2$ and σ^2 some finite function $\sigma^2 \equiv \sigma^2(x, t)$. By doing so, we are ruling out the possibility that the process performs *finite jumps* in space in an infinitesimal amount of time. The hypothesis (2.1) is necessary as far we are looking for an evolution law for (conditional) probability densities governed by a parabolic partial differential equation (i.e. the Fokker-Planck equation itself). It is, however, not necessary as well defined master equations governing the evolution of (conditional) probability densities are also obtained in the presence of finite jumps. In order to see this, let us consider

$$\tilde{p}_{Dt}(i, n+1) = \sum_{k=-\infty}^{\infty} \tilde{W}_{Dt}(i, n+1|k, n)\, \tilde{p}_{Dt}(k, n)$$

or equivalently

$$\tilde{p}_{Dt}(i, n+1) - \tilde{p}_{Dt}(i, n) = \sum_{k=-\infty}^{\infty} \tilde{W}_{Dt}(i, n+1|k, n)\, \tilde{p}_{Dt}(k, n) - \sum_{k=-\infty}^{\infty} \tilde{W}_{Dt}(k, n+1|i, n)\, \tilde{p}_{Dt}(i, n)$$

using the normalization condition

$$\sum_{k=-\infty}^{\infty} \tilde{W}_{Dt}(k, n+1|i, n) = 1 \tag{2.2}$$

for the transition probabilities $\tilde{W}_{Dt}(i, n+1|j, n) \;\geq\; 0, \forall\,, i, j, n$. We now suppose that

$$\tilde{W}_{Dt}(k, n+1|i, n) = \bar{W}\, \delta_{k,i} + T(k, t|i)\, Dt + O(Dt^2)$$

and pass to the limit $Dt \downarrow 0$ whilst keeping $t = n\, Dt$ finite and "quantized" (i.e. lattice valued) the spatial variable. We obtain

$$\partial_t p(n, t) = \sum_{n=-\infty}^{\infty} T(n, t|n')\, p(n', t) - \sum_{n=-\infty}^{\infty} T(n', t|n)\, p(n, t) \tag{2.3}$$

i.e. a master equation for the evolution in a continuous time of a lattice valued stochastic process.

2.1 Poisson process

Let us now further assume for $\gamma \in \mathbb{R}_+$

$$T(n, t|n') = \gamma\, \delta_{n\,, n'+1}$$

The master equation reduces to

$$\partial_t p(k\,, t) = \gamma\, p(k-1\,, t) - \gamma\, p(k\,, t)$$

This is the evolution for a process that can make (or not make) jumps only towards the right of its current position. If we assume that the initial distribution

$$p(i,0) = p_o(i)$$

has support on $\mathbb{N}$ then the process will stay there for any further time. The equation can be solved exactly by computing the characteristic function

$$\check{p}(u,t) := \sum_{k=0}^{\infty} e^{\imath\, k\, u}\, p(k,t)$$

Namely, it is straightforward to see that $\check{p}(u,t)$ satisfies:

$$\partial_t \check{p}(u,t) = \gamma\,(e^{\imath\, u} - 1)\,\check{p}(u,t)$$

The solution for the initial condition $\check{p}(u,0) = \check{p}_o(u)$

$$\check{p}(u,t) = e^{\gamma\, t\,(e^{\imath\, u}-1)}\check{p}_o(u)$$

If we specialize for an initial condition

$$p_o(i) = \delta_{i\,0}$$

(i.e. we assume that the process starts from the origin) we obtain

$$\check{p}(u,t) = e^{\gamma\, t\,(e^{\imath\, u}-1)}$$

In order to infer the probability distribution associated to the characteristic function we can write

$$\check{p}(u,t) = e^{\gamma\, t\, e^{\imath\, u}} e^{-\gamma\, t} = e^{-\gamma\, t} \sum_{j=0}^{\infty} \frac{(\gamma\, t)^j}{\Gamma\,(j+1)} e^{\imath\, u\, j}$$

which implies that $\check{p}(u,t)$ is the characteristic function of the Poisson process, with probability distribution:

$$p(j\,,t) = \frac{(\gamma\, t)^j}{\Gamma\,(j+1)} e^{-\gamma\, t}$$

3 An alternative derivation of the Fokker-Planck equation

In order to further emphasize the assumptions underlying the derivation of the continuum limit of the lattice dynamics leading to the Fokker-Planck equation, we can try to recover the result by considering the following *master equation*

$$\partial_t p(x,t) = \int_{\mathbb{R}} dy\, W(x\,,t|y) p(y,t) - \int_{\mathbb{R}} dy\, W(y\,,t|x) p(x,t) \tag{3.1}$$

with *continuous* transition rates $W(x\,,t|y)$. It is readily seen that (2.3) and (3.1) have the same structure. We make now the following fundamental assumption:

$$W(x\,,t|y) = f(x-y, y, t)$$

with

$$f(x,y,t) \begin{cases} \text{sharply peaked around the origin of the first argument} & (x) \\ \text{smoothly varying in its second argument} & (y) \end{cases}$$

Note that the first argument controls the size of the displacement from y to x. Furthermore we assume that

$$\left| \int_{\mathbb{R}} dx\, x^n\, f(x, y, t) \right| < \infty$$

for any $n \in \mathbb{N}$. We can then write

$$\begin{aligned} \partial_t p(x,t) &= \int_{\mathbb{R}} dy\, f(x-y, y, t)\, p(y,t) - \int_{\mathbb{R}} dy\, f(y-x, x, t)\, p(x,t) \\ &= \int_{\mathbb{R}} dy\, f(y, -y+x, t)\, p(-y+x, t) - \int_{\mathbb{R}} dy\, f(y, x, t)\, p(x,t) \end{aligned}$$

Taylor expanding yields

$$\partial_t p(x,t) = -\int_{\mathbb{R}} dy\, y \partial_x \left[f(x,y,t) p(x,t) \right] + \frac{1}{2} \int_{\mathbb{R}} dy\, y^2 \partial_{x^2} \left[f(x,y,t) p(x,t) \right] + \dots \tag{3.2}$$

Upon setting

$$b(x,t) := \int_{\mathbb{R}} dy\, y\, f(x,y,t)$$

$$\sigma^2(x,t) := \int_{\mathbb{R}} dy\, y^2\, f(x,y,t)$$

and neglecting higher order terms, we recover the Fokker-Planck equation:

$$\partial_t p\,(x,t) + \partial_x \left[b\,(x,t)\, p\,(x,t) \right] = \frac{1}{2} \partial_x^2 \left[\sigma^2\,(x,t)\, p\,(x,t) \right] \tag{3.3}$$

3.1 Comments on the derivation of the Fokker-Planck equation

Some comments and observations are in order.

- Both when we allowed for jumps (section 2) and in the derivation of the Fokker-Planck equation, we worked under the assumption that the probability at discrete time $n+1$ depends only on the state of the system at discrete time n. This means that the master or Fokker-Planck equations brought about by the continuum limit, always describe so called Markov processes. A loose definition of Markov process is that of a stochastic process for which the *future state depends only upon the current state* (and not on past states) of the process.

- The Taylor expansion in retains (and requires the existence) of the first two moments of f. The observation establishes a point of contact with the central limit theorem on which we will further elaborate.

- Suppose now that σ^2 vanishes, so that we truncate the Taylor expansion at first order. Then (3.3) reduces to

  $$\partial_t p(x,t) + \partial_x [b(x,t) p(x,t)] = 0 \tag{3.4}$$

  This equation describes the evolution of a probability density or alternatively the mass transport by the flow solution of

  $$\frac{dx_t}{dt} = b\,(x_t\,, t) \tag{3.5}$$

  Namely if we take the initial condition

  $$p\,(x\,, t_o) = p_o(x) \qquad \& \qquad x_{t_0} = x_o$$

then we see that

$$p(x_t\,,t) = p_o(x_o)e^{-\int_{t_o}^{t} ds\,(\partial_{x_s} b)(x_s,s)} \tag{3.6}$$

yields an *implicit* solution of (3.4) provided

$$x_t = \varphi(t; t_o, x_o)$$

satisfies (3.5). Namely, taking the total derivative with respect to time of both sides

$$\partial_t p(x_t, t) + \frac{dx_t}{dt} \partial_{x_t} p(x_t, t) = -p(x_t, t)\, \partial_{x_t} b(x_t, t)$$

Furthermore if $p_o(x)$ is chosen positive, then (3.6) remains positive at any subsequent time. Note finally that in order to extract from (3.6) the explicit representation of the solution we need to invert the relation

$$x = x_t(x_o)$$

for any fixed x.

References

[1] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, `http://math.berkeley.edu/~evans/` 1