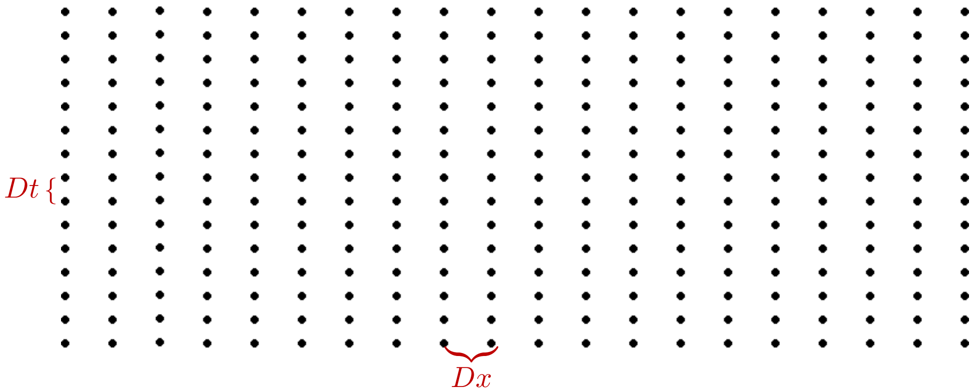
1 Introduction

The content of these notes is also covered by chapter 3 section A of [1].

2 Derivation of the Fokker-Planck equation

2.1 Lattice

A rectangular bi-dimensional lattice can be intuitively thought as a grid composed by rectangles of vertical mesh size Dt and horizontal mesh size Dx. Points on the lattice (grid) can be labeled by a pair integers of integers $(i\,,n)$ respectively specifying the horizontal and vertical coordinate. If the grid contains a countable number of points it can be identified with $\mathbb{Z}^2$:



2.2 Lattice dynamics

Probability on the lattice

$$P(\text{being at site } i \text{ at time } n) = \tilde{p}(i\,,n) \tag{2.1}$$

Transition probability

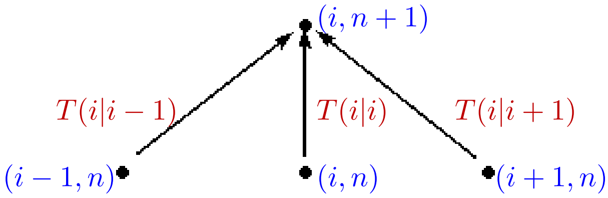
$$P(\text{jump from } j \text{ to } i \text{ in a unit of time }) = T(i|j) \tag{2.2}$$

At each time step the particle must either remain in the same place or migrate to the first nearest neighbors

$$T(j+1|j) + T(j|j) + T(j-1|j) = 1 \tag{2.3}$$

Balance equation

$$\tilde{p}(i\,,n+1) = T(i|i-1)\,\tilde{p}(i-1\,,n) + T(i|i)\,\tilde{p}(i\,,n) + T(i|i+1)\,\tilde{p}(i+1\,,n) \tag{2.4}$$



The variation of $\tilde{p}$ over one time unit is then

$$\tilde{p}(i\,,n+1) - \tilde{p}(i\,,n) = T(i|i-1)\,\tilde{p}(i-1\,,n) - [T(i+1|i) + T(i-1|i)]\,\tilde{p}(i\,,n) + T(i|i+1)\,\tilde{p}(i+1\,,n) \tag{2.5}$$

Define the lattice derivative operation

$$D_{n,\pm 1}\tilde{f}(n) = \tilde{f}(n \pm 1) - \tilde{f}(n) \tag{2.6}$$

The reason for introducing $D_{n,\pm 1}$ is that it satisfies a straightforward relation with Newton's difference quotient. Namely, if we set

$$\tilde{f}(n) = f(n\,Dx) \equiv f(x)$$

and

$$\tilde{f}(n+1) = f(n\,Dx + Dx) \equiv f(x + Dx)$$

then we get

$$\lim_{Dx\downarrow 0} \frac{D_{n,1}}{Dx}\tilde{f}(n) = \lim_{Dx\downarrow 0} \frac{f(x+Dx) - f(x)}{Dx} = \frac{\partial f}{\partial x}(x) \equiv \partial_x f \tag{2.7}$$

and

$$\lim_{Dx\downarrow 0} \frac{D_{n,1} + D_{n,-1}}{(Dx)^2}\tilde{f}(n) = \lim_{Dx\downarrow 0} \frac{f(x+Dx) - 2\,f(x) + f(x-Dx)}{(Dx)^2} = \frac{\partial^2 f}{\partial x^2}(x) \equiv \partial_x^2 f \tag{2.8}$$

as can be verified by Taylor-expanding the numerator around x. The identities (2.7) and (2.8) motivate the following terminology:

- $D_{n,1}$ is called the lattice forward derivative with respect to n.
- $D_{n,-1}$ is called the lattice backward derivative with respect to n.
- $D_{n,1} + D_{n,-1}$ is called the lattice Laplacian with respect to n

Note that the lattice Laplacian is symmetric with respect to increments. A symmetric definition of the derivative is obtained by considering

$$\frac{D_{n,1} - D_{n,-1}}{2}\tilde{f}(n) \equiv \frac{\tilde{f}(n+1) - \tilde{f}(n-1)}{2} \tag{2.9}$$

where the factor 2 is introduced to take into account that the function $\tilde{f}$ is sampled at lattice sites separated by two times the unit mesh size.
In the following two subsection, under hypotheses of increasing generality the balance equation (2.4) will be re-written in terms of the lattice derivatives with respect to the discrete "time" n and "position" i coordinates. The scope is to ease the study of the continuum limit done in section 2.3 which is used to define the evolution of the probability density as the mesh sizes Dx and Dt are sent to zero in an appropriate way.

2.2.1 Constant drift and diffusion

Suppose that $T(i|j)$ distinguishes only whether the particle is going to the right or to the left

$$T(i|j) = \begin{cases} T_+ & i - j = 1 \\ T_- & i - j = -1 \end{cases}$$

In such a case

$$D_{n,1}\tilde{p}(i\,,n) = \frac{T_+ + T_-}{2}\left(D_{i,1} + D_{i,-1}\right)\tilde{p}(i\,,n) - \frac{T_+ - T_-}{2}\left(D_{i,1} - D_{i,-1}\right)\tilde{p}(i\,,n) \tag{2.10}$$

Setting

$$\bar{T} := \frac{T_+ + T_-}{2} \qquad \text{diffusion}$$
$$\tilde{T} := T_+ - T_- \qquad \text{drift} \tag{2.11}$$

one finds

$$D_{n,1}\tilde{p}(i\,,n) = \bar{T}\,(D_{i,1} + D_{i,-1})\,\tilde{p}(i\,,n) - \tilde{T}\frac{D_{i,1} - D_{i,-1}}{2}\tilde{p}(i\,,n) \tag{2.12}$$

2.2.2 More general local case

$T(i|j)$ distinguishes whether the particle is going to the right or to the left in a point dependent way

$$T(i|j) = \begin{cases} T_+(j) & i - j = 1 \\ T_-(j) & i - j = -1 \end{cases}$$

In such a case

$$D_n\tilde{p}(i\,,n) = T_+(i-1)\,\tilde{p}(i-1\,,n) - [T_+(i) + T_-(i)]\,\tilde{p}(i\,,n) + T_-(i+1)\,\tilde{p}(i+1\,,n) \tag{2.13}$$

Adding and subtracting $T_+(i-1)\,\tilde{p}(i\,,n)$ and $T_-(i+1)\,\tilde{p}(i\,,n)$ yields

$$D_{n,1}\tilde{p}(i\,,n) =$$
$$T_+(i-1)\,D_{i,-1}\tilde{p}(i\,,n) + \tilde{p}(i\,,n)\,[D_{i,-1}T_+(i) + D_{i,1}T_-(i)] + T_-(i+1)\,D_{i,1}\tilde{p}(i\,,n)$$

The expression can be also rewritten as

$$D_{n,1}\tilde{p}(i\,,n) = [T_+(i) + (D_{i,-1}T_+)(i)]\,D_{i,-1}\tilde{p}(i\,,n) +$$
$$+\tilde{p}(i\,,n)\,[D_{i,-1}T_+(i) + D_{i,1}T_-(i)] + [T_-(i) \;+\; (D_{i,1}T_-)(i)]\,D_{i,1}\tilde{p}(i\,,n)$$

which is reminiscent of the Leibniz rule in the continuum. The analogy is complete by setting as in the previous case

$$\bar{T}(i) = \frac{T_+(i) + T_-(i)}{2} \quad \text{diffusion} \qquad\qquad T_+(i) = \bar{T}(i) + \frac{\tilde{T}(i)}{2}$$
$$\Rightarrow$$
$$\tilde{T}(i) = T_+(i) - T_-(i) \quad \text{drift} \qquad\qquad T_-(i) = \bar{T}(i) - \frac{\tilde{T}(i)}{2}$$

In such a case one gets into

$$\begin{aligned} D_{n,1}\tilde{p}(i\,,n) = {} & \bar{T}(i)\,(D_{i,-1} + D_{i,1})\tilde{p}(i\,,n) + \tilde{p}(i\,,n)(D_{i,-1} + D_{i,1})\bar{T}(i) \\ & + \left[(D_{i,-1}\bar{T})(i)\,D_{i,-1}\tilde{p}(i\,,n) + (D_{i,1}\bar{T})(i)\,D_{i,1}\tilde{p}(i\,,n)\right] \\ & + \frac{(D_{i,-1}\tilde{T})(i)\,D_{i,-1}\tilde{p}(i\,,n) - (D_{i,1}\tilde{T})(i)\,D_{i,1}\tilde{p}(i\,,n)}{2} \\ & - \tilde{T}(i)\,\frac{D_{i,1} - D_{i,-1}}{2}\,\tilde{p}(i\,,n) - \tilde{p}(i\,,n)\frac{D_{i,1} - D_{i,-1}}{2}\tilde{T}(i) \end{aligned} \tag{2.14}$$

which can be more compactly couched into the form

$$D_{n,1}\tilde{p}(i\,,n) = \mathfrak{L}_i\,\tilde{p}(i\,,n)$$

where the operator $\mathfrak{L}_i$ acts linearly on $\tilde{p}(i\,,n)$ in the way specified by the right hand side of (2.14).

2.3 Continuum limit

Define the continuum time coordinate as

$$t \,=\, n\, Dt \tag{2.15}$$

and the continuum space coordinate as

$$x \,=\, i\, Dx \tag{2.16}$$

We will seek a continuum limit for the equations (2.12), (2.14) assuming the scaling hypothesis

$$(Dx)^2 = \sigma^2(x)\,(Dt) \tag{2.17}$$

with σ a a finite strictly-positive function with dimensions

$$[\sigma^2] = \left[\frac{length^2}{time}\right]$$

Furthermore we assume that

$$\begin{aligned}
\bar{T}(i) &= \sigma^2(x)\,\frac{T_+(i)+T_-(i)}{2} \overset{Dx\downarrow 0}{\to} \frac{\sigma^2(x)}{2} \,>\, 0 \qquad &\text{diffusion:} \qquad [\sigma^2] = \left[\tfrac{length^2}{time}\right] \\
\frac{\tilde{T}(i)}{Dx} &= \frac{T_+(i)-T_-(i)}{Dx} \overset{Dx\downarrow 0}{\to} b(x) \qquad &\text{drift:} \qquad [b] = \left[\tfrac{length}{time}\right]
\end{aligned} \tag{2.18}$$

and denote

$$\tilde{p}(i\,,n) \overset{Dx\,,Dt\,\downarrow\,0}{\to} p(x\,,t) \tag{2.19}$$

Using (2.7), (2.8) it is straightforward to verify that these assumptions yield for

$$\lim_{\substack{Dx\downarrow 0\\ Dt\downarrow 0\\ (Dx)^2=\sigma^2\,Dt}} \frac{D_{n,1}\tilde{p}(i,n)}{Dt} = \lim_{\substack{Dx\downarrow 0\\ Dt\downarrow 0\\ (Dx)^2=\sigma^2\,Dt}} \frac{\mathfrak{L}_i\,\tilde{p}(i,n)}{Dt}$$

the continuum limit

Fokker-Planck eq.: $$\partial_t p(x,t) + \partial_x\left\{b(x)\,p(x,t)\right\} = \frac{1}{2}\partial_x^2\left\{\sigma^2(x)\,p(x,t)\right\} \tag{2.20}$$

3 Probability conservation

The Fokker-Planck equation provides a self-consistent description for the time evolution of a probability density. Namely (hints of a proof)

- it ensures probability conservation:

$$\partial_t \int_{\mathbb{R}} dx\, p(x,t) = \int_{\mathbb{R}} dx\,\partial_x \left\{\partial_x \frac{\sigma^2(x)\,p(x,t)}{2} - b(x)\,p(x,t)\right\} = 0$$

  for any $p(x,t)$ decaying sufficiently fast at infinity.

- If the initial density is positive definite, the density remains positive definite at any time:

$$p(x,t) = e^{V(x,t)}$$

yields

$$\partial_t V = \partial_x \left\{ \partial_x \frac{\sigma^2}{2} + b \right\} + \frac{\sigma^2}{2} \left\{ \partial_x^2 V + (\partial_x V)^2 \right\} + \left\{ \partial_x \sigma^2 + b \right\} (\partial_x V)$$

which for

$$\sigma^2, b : \mathbb{R} \to \mathbb{R} \tag{3.1}$$

specifies a evolution for a real quantity V.

4 Extensions to the multi-dimensional case

Inspection of the derivation of (2.20) evinces that the steps involved do not require the diffusion and drift to be time independent. More laborious but conceptually identical is to repeat the calculation of section 2 in the multi-dimensional case. The result in the most general case is

Fokker-Planck eq.: $$\partial_t p(\boldsymbol{x},t) + \frac{\partial}{\partial x^i} \left\{ b^i(\boldsymbol{x},t)\, p(\boldsymbol{x},t) \right\} = \frac{1}{2} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \left\{ \sigma_{i\,l}(\boldsymbol{x},t)\, \sigma_{i\,l}(\boldsymbol{x},t)\, p(\boldsymbol{x},t) \right\} \tag{4.1}$$

where repeated indices imply summations over i, j, l ranging from 1 to the number of dimensions d (Einstein convention) and

$$\sigma_{i\,l}\sigma_{j\,l} : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{d \times d} \qquad \text{tensor field}$$

$$b^i : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d \qquad \text{vector field}$$

Note that

$$\sigma_{i\,l}\sigma_{j\,l} = \left\{ \sigma\, \sigma^\dagger \right\}_{i\,j}$$

where tr denotes matrix transposition. Thus the tensor field is positive definite with respect to the standard scalar product in $\mathbb{R}^d$.

5 Diffusion equation

A special case of the Fokker-Planck equation (2.20) is obtained for

$$b = 0$$
$$\sigma^2 = \text{constant}$$

In such a case we get into the diffusion equation

Diffusion eq.: $$\partial_t p(x,t) = \frac{\sigma^2}{2}\, \partial_x^2\, p(x,t) \tag{5.1}$$

The solution of the diffusion equation in the space of probability densities defined over the entire real axis

$$p : \mathbb{R} \to \mathbb{R}_+$$

for a given initial condition $p_o(x) = p(x, 0)$ can be explicitly obtained by considering the Fourier transform of the diffusion equation

$$\partial_t \check{p}(q,t) = -\frac{\sigma^2 q^2}{2}\, \check{p}(q,t)$$

which yields the solution

$$\check{p}(q,t) = \check{p}_o(q)\, e^{-\frac{\sigma^2 q^2 t}{2}}$$

Note that the choice

$$\check{p}_o(q) = e^{\imath\, q\, y}$$

yields

$$p(x,t) = \frac{e^{-\frac{(x-y)^2}{2\,\sigma^2\, t}}}{(2\,\pi\,\sigma^2\, t)} \equiv g_{y\,\sigma\sqrt{t}}(x)$$

References

[1] L.C. Evans, *An Introduction to Stochastic Differential Equations*, lecture notes, http://math.berkeley.edu/~evans/