Exercise set 4

Exercise 1

A d-dimensional Brownian motion is the stochastic process

$$\boldsymbol{w}_t : \Omega \times \mathcal{R}_+ \to \mathbb{R}^d$$

such that

$$\boldsymbol{w}_t = \begin{bmatrix} w_t^{(1)} \\ w_t^{(2)} \\ \vdots \\ w_t^{(d)} \end{bmatrix} \tag{0.1}$$

with $\left\{ w_t^{(i)} \right\}_{i=1}^{d}$ independent Brownian motions. Prove that

- $\boldsymbol{w}_{t+s} - \boldsymbol{w}_s$ is a Brownian motion for all $t\,,s\,,\in\,\mathbb{R}_+$
- $c\,\boldsymbol{w}_{\frac{t}{c^2}}$ is a Brownian motion for all $c\,,\in\,\mathbb{R}_+$. Note: this is called the Brownian scaling property

Exercise 2

Let w_t a one-dimensional Brownian motion. Prove that

$$\lim_{m\uparrow\infty} \frac{w_m}{m} = 0 \qquad a.s.$$

Hint.: use the Brownian scaling property and Borel-Cantelli lemma.

Exercise 3

Let w_t a one-dimensional Brownian motion ($\prec w_t^2 \succ = t$). Define the stochastic process ξ_t as

$$\xi_t = e^{-t} w_{e^{2t}}$$

show that for any $t\,,s\in\,\mathbb{R}$

$$\prec \xi_t \xi_s \succ = e^{-|t-s|}$$