# Exercise set 1

## Exercise 1

Let $\xi : \Omega \to \mathbb{R}_+$ be a random variable with density

$$p_\xi(x) = \frac{e^{-\frac{x}{\bar{x}}}}{\bar{x}} \tag{0.1}$$

- Compute the *characteristic* and the *generating* functions of $\xi$. The generating function is defined as

  $$g_\xi(t) = \prec e^{-t\,\xi} \succ \qquad t > 0$$

- Explain what is the relation between the generating and the characteristic function. Given the generating function $g_\xi$, is it possible to reconstruct the density $p_\xi$?

- Consider a sequence $\{\xi_i\}_{i=1}^{\infty}$ of i.i.d. random variables with density specified by (0.1). What is the asymptotic distribution of

  $$S_n[\xi] = \frac{1}{n}\sum_{i=1}^{n} \xi_i$$

  for $n \uparrow \infty$?

## Exercise 2

Prove the following proposition:

**Definition 0.1** (*Chernoff inequality*)**.** *Let $\xi$ any random variable such that*

$$g_\xi(t) = \prec e^{t\,\xi} \succ < \infty$$

*then*

$$P(\xi \geq a) \leq \min_t e^{-a\,t} g_\xi(t)$$

Give the explicit expression of the bound for $\xi$ a Gaussian random variable with zero average and variance $\sigma^2$.

## Exercise 3

Let $\xi : \Omega \to \mathbb{N}$ a Poisson random variable:

$$P_{\xi}(i; \lambda\, t) = \frac{(\lambda\, t)^i}{\Gamma(i+1)} e^{-\lambda\, t}$$

It $P_{\xi}(i;\lambda)$ describes the probability of $i$ events of the same type occur simultaneously while being mutually independent and having the same probability per unit of time $\lambda$. Compute

- the characteristic function
- the mean value
- the variance of the Poisson distribution.

## Exercise 4

Consider a random variable $\xi : \Omega \to [0, 1]$ with uniform density

$$p_{\xi}(x) = 1$$

Find an invertible function $f$

$$f : [0, 1] \to \mathbb{R}_+$$

such that the random variable

$$\eta = f(\xi)$$

is exponentially distributed (i.e. according to (0.1))

## Exercise 5

Let $A_{i\,j}$ a *strictly positive definite* matrix in $d$-dimensions ($A \in \mathbb{R}^d \times \mathbb{R}^d$).

- Prove that for any $\boldsymbol{m} = (m_1 \, \dots \, m_d) \in \mathbb{R}^d$

$$p_{\boldsymbol{\xi}}(\boldsymbol{x}) = \frac{\sqrt{\det A}\, e^{-\frac{||A\,(\boldsymbol{x}-\boldsymbol{m})||^2}{2}}}{(2\,\pi)^{\frac{d}{2}}}$$
$$||A\,(\boldsymbol{x}-\boldsymbol{m})||^2 = \sum_{i\,j=1}^{d} (x_i - m_i)\, A_{i\,j}(x_j - m_j)$$

  is the probability density of the sequence $\boldsymbol{\xi} = (\xi_1 \, \dots \, \xi_d)$ Gaussian random variables.

- Compute the average of $\xi_i$
- Compute the *second order correlation* $\prec \xi_i \xi_j \succ$ ($i, j$ arbitrary)
- Compute the *fourth order correlation* $\prec \xi_i \xi_j \xi_l \xi_k \succ$ ($i, j, l, k$ arbitrary)

## Exercise 6

Let $(\Omega, \mathcal{F}, P)$ a finite dimensional probability space $\eta$ an arbitrary random variable on it and $\mathcal{P}_1 \subseteq \mathcal{P}_2 \subseteq \cdots \subseteq \mathcal{P}_n$ a growing sequence of partitions of $\Omega$. Prove that the sequence $\{\xi_i\}_{i=1}^n$ whose elements are defined by

$$\xi_k = \prec \eta | \mathcal{P}_k \succ \equiv E(\eta | \mathcal{P}_k)$$

is a martingale.