① Funktion f keskiarvo joukossa A on

$$\frac{1}{a(A)} \iint_A f(x,y)\,dx\,dy,$$

missä $a(A) \in]0,\infty[$ on A:n pinta-ala. Nyt $A = [-2,2]\times[0,4]$, joten $a(A) = (2-(-2))\cdot(4-0) = 4\cdot 4 = 16$. ($A$:n pinta-alan voi laskea tietenkin myös Määritelmää 2.4 ja Lausetta 2.9 käyttäen

$$a(A) = \iint_A dx\,dy = \int_0^4 dy \int_{-2}^{2} dx = \int_0^4 dy \Big/_{-2}^{2} x = \int_0^4 dy\, 4 = \Big/_0^4 4y = 16.)$$

Lauseen 2.9 perusteella (katso myös Huomautus 2.10)

$$\iint_A f(x,y)\,dx\,dy = \int_{-2}^{2} dx \int_0^4 dy\, \frac{x+y}{y+1} \overset{(*)}{=} \int_{-2}^{2} dx \int_0^4 dy \left(\frac{x-1}{y+1} + 1\right)$$

$$= \int_{-2}^{2} dx \Big/_0^4 \big((x-1)\ln|y+1| + y\big) = \int_{-2}^{2} dx\,\big((x-1)\ln 5 + 4\big)$$

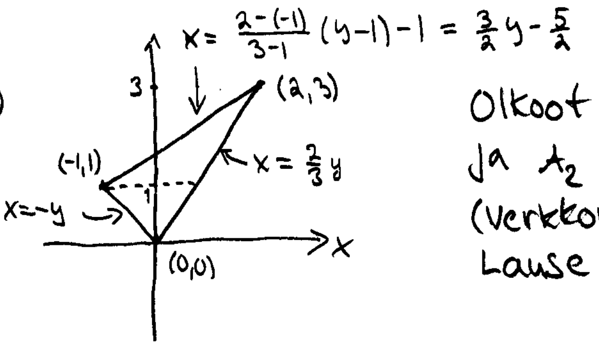
$$= \Big/_{-2}^{2} \left(\left(\frac{x^2}{2} - x\right)\ln 5 + 4x\right) = 8 - (4\ln 5 - 8) = 16 - 4\ln 5.$$

(*) $\dfrac{x+y}{y+1} = \dfrac{(x-1)+(y+1)}{y+1} = \dfrac{x-1}{y+1} + 1$, tai jakokulman avulla

$$\begin{array}{r|l} & 1 \\ y+1 & y+x \\ & \underline{y+1} \\ & x-1 \end{array} \qquad \text{siis} \quad \frac{y+x}{y+1} = 1 + \frac{x-1}{y+1}.$$

Siis f:n keskiarvo A:ssa on

$$\frac{1}{a(A)} \iint_A f(x,y)\,dx\,dy = \frac{1}{16}(16 - 4\ln 5) = 1 - \tfrac{1}{4}\ln 5.$$

② 

Olkoot A_1 se kolmion A osuus, jossa $0 \le y \le 1$, ja A_2 se osuus, jossa $1 \le y \le 3$. Nyt (verkkomonisteen sivulla 44 oleva kohta (iv) ja Lause 2.15):

$$\iint_A y^2\,dx\,dy = \iint_{A_1} y^2\,dx\,dy + \iint_{A_2} y^2\,dx\,dy$$

$$= \int_0^1 dy \int_{-y}^{\frac{2}{3}y} dx\, y^2 + \int_1^3 dy \int_{\frac{3}{2}y-\frac{5}{2}}^{\frac{2}{3}y} dx\, y^2$$

$$= \int_0^1 dy \Big/_{-y}^{\frac{2}{3}y} xy^2 + \int_1^3 dy \Big/_{\frac{3}{2}y-\frac{5}{2}}^{\frac{2}{3}y} xy^2$$

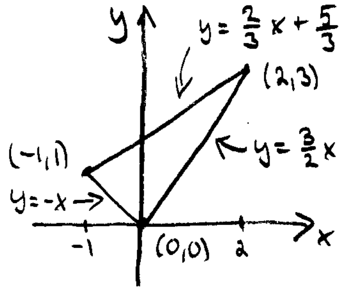
$$= \int_0^1 dy\, \frac{5}{3}y^3 + \int_1^3 dy \left(-\frac{5}{6}y^3 + \frac{5}{2}y^2\right)$$

$$= \Big/_0^1 \frac{5}{12}y^4 + \Big/_1^3 \left(-\frac{5}{24}y^4 + \frac{5}{6}y^3\right)$$

$$= \frac{5}{12} - \frac{405}{24} + \frac{135}{6} + \frac{5}{24} - \frac{5}{6} = \frac{5}{12} - \frac{400}{24} + \frac{130}{6}$$

$$= \frac{5}{12} - \frac{200}{12} + \frac{260}{12} = \frac{65}{12} = 5\frac{5}{12}.$$

Jos integrointijärjestys valitaan toisin päin, tulee laskuista mutkikkaampia. Olkoot tällöin A_1 se kolmion osuus, jossa $-1 \le x \le 0$, ja A_2 se osuus, jossa $0 \le x \le 2$. Nyt



$$\iint_A y^2\,dx\,dy = \iint_{A_1} y^2\,dx\,dy + \iint_{A_2} y^2\,dx\,dy$$

$$= \int_{-1}^0 dx \int_{-x}^{\frac{2}{3}x+\frac{5}{3}} dy\, y^2 + \int_0^2 dx \int_{\frac{3}{2}x}^{\frac{2}{3}x+\frac{5}{3}} dy\, y^2 = \ldots = 5\frac{5}{12}.$$

(3) A:n pinta-ala (katso integrointirajat edellisen tehtävän jälkimmäisestä kuvasta):

$$a(A) = \iint_A dx\,dy = \int_{-1}^0 dx \int_{-x}^{\frac{2}{3}x+\frac{5}{3}} dy + \int_0^2 dx \int_{\frac{3}{2}x}^{\frac{2}{3}x+\frac{5}{3}} dy$$

$$= \int_{-1}^0 dx \Big/_{-x}^{\frac{2}{3}x+\frac{5}{3}} y + \int_0^2 dx \Big/_{\frac{3}{2}x}^{\frac{2}{3}x+\frac{5}{3}} y = \int_{-1}^0 dx\left(\frac{5}{3}x+\frac{5}{3}\right) + \int_0^2 dx\left(-\frac{5}{6}x+\frac{5}{3}\right)$$

$$= \Big/_{-1}^{0} \left(\frac{5}{6}x^2 + \frac{5}{3}x\right) + \Big/_{0}^{2} \left(-\frac{5}{12}x^2 + \frac{5}{3}x\right) = -\frac{5}{6} + \frac{5}{3} - \frac{5}{3} + \frac{10}{3}$$

$$= -\frac{5}{6} + \frac{20}{6} = \frac{15}{6} = \frac{5}{2}.$$

Myös seuraavassa integraalissa nähdään integroimisrajat tehtävän 2 jälkimmäisestä kuvasta.

$$\iint_A x\,dx\,dy = \int_{-1}^{0} dx \int_{-x}^{\frac{2}{3}x+\frac{5}{3}} dy\, x + \int_{0}^{2} dx \int_{\frac{3}{2}x}^{\frac{2}{3}x+\frac{5}{3}} dy\, x$$

$$= \int_{-1}^{0} dx \Big/_{-x}^{\frac{2}{3}x+\frac{5}{3}} xy + \int_{0}^{2} dx \Big/_{\frac{3}{2}x}^{\frac{2}{3}x+\frac{5}{3}} xy$$

$$= \int_{-1}^{0} dx \left(\frac{5}{3}x^2 + \frac{5}{3}x\right) + \int_{0}^{2} dx \left(-\frac{5}{6}x^2 + \frac{5}{3}x\right)$$

$$= \Big/_{-1}^{0} \left(\frac{5}{9}x^3 + \frac{5}{6}x^2\right) + \Big/_{0}^{2} \left(-\frac{5}{18}x^3 + \frac{5}{6}x^2\right) = \frac{5}{9} - \frac{5}{6} - \frac{40}{18} + \frac{20}{6}$$

$$= \frac{10 - 15 - 40 + 60}{18} = \frac{15}{18} = \frac{5}{6}.$$

Seuraavassa integraalissa nähdään integroimisrajat tehtävän 2 ensimmäisestä kuvasta.

$$\iint_A y\,dx\,dy = \int_{0}^{1} dy \int_{-y}^{\frac{2}{3}y} dx\, y + \int_{1}^{3} dy \int_{\frac{3}{2}y-\frac{5}{2}}^{\frac{2}{3}y} dx\, y$$

$$= \int_{0}^{1} dy \Big/_{-y}^{\frac{2}{3}y} xy + \int_{1}^{3} dy \Big/_{\frac{3}{2}y-\frac{5}{2}}^{\frac{2}{3}y} xy$$

$$= \int_{0}^{1} dy\, \frac{5}{3}y^2 + \int_{1}^{3} dy \left(-\frac{5}{6}y^2 + \frac{5}{2}y\right)$$

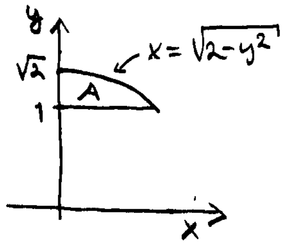
$$= \Big/_{0}^{1} \frac{5}{9}y^3 + \Big/_{1}^{3} \left(-\frac{5}{18}y^3 + \frac{5}{4}y^2\right) = \frac{5}{9} - \frac{135}{18} + \frac{45}{4} + \frac{5}{18} - \frac{5}{4}$$

$$= \frac{10 - 135 + 5}{18} + \frac{40}{4} = -\frac{120}{18} + 10 = -\frac{20}{3} + 10 = \frac{10}{3}.$$

Siis joukon A keskiö on

$$\frac{1}{a(A)}\left(\iint_A x\,dx\,dy,\ \iint_A y\,dx\,dy\right) = \frac{2}{5}\left(\frac{5}{6}, \frac{10}{3}\right) = \left(\frac{1}{3}, \frac{4}{3}\right).$$

(4)



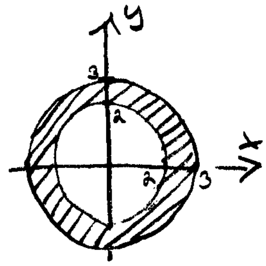
$$\iint_A \frac{x}{y}\,dx\,dy = \int_1^{\sqrt{2}} dy \int_0^{\sqrt{2-y^2}} dx\, \frac{x}{y}$$

$$= \int_1^{\sqrt{2}} dy \Big/_0^{\sqrt{2-y^2}} \frac{x^2}{2y} = \int_1^{\sqrt{2}} dy\, \frac{2-y^2}{2y}$$

$$= \int_1^{\sqrt{2}} dy \left(\frac{1}{y} - \frac{y}{2}\right) = \Big/_1^{\sqrt{2}} \left(\ln y - \frac{y^2}{4}\right)$$

$$= \ln 2^{\frac{1}{2}} - \frac{2}{4} - \ln 1 + \frac{1}{4} = \frac{1}{2}\ln 2 - \frac{1}{4}.$$

(5) $A = \{(x,y) \mid 2^2 \leq x^2+y^2 \leq 3^2\}$



Siirrytään napakoordinaatteihin (katso verkkomoniste)

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$ Jacobin determinantti $J_g(r,\varphi) = r.$

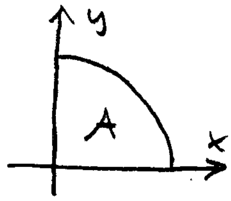
Merkitään $B = [2,3] \times [0, 2\pi]$. Nyt saadaan verkkomonisteessa esitetyn teorian perusteella

$$\iint_A (x^2+y^2)\,dx\,dy = \iint_B \underbrace{(r^2\cos^2\varphi + r^2\sin^2\varphi)}_{=r^2(\cos^2\varphi+\sin^2\varphi)=r^2} |J_g(r,\varphi)|\,dr\,d\varphi$$

$$= \int_2^3 dr \int_0^{2\pi} d\varphi\, r^3 = \int_2^3 dr \Big/_0^{2\pi} \varphi r^3 = 2\pi \int_2^3 dr\, r^3 = 2\pi \Big/_2^3 \frac{r^4}{4}$$

$$= 2\pi\left(\frac{81}{4} - \frac{16}{4}\right) = \frac{65\pi}{2}.$$

6. $A = \{(x,y) \mid x \geq 0,\ y \geq 0 \text{ ja } x^2+y^2 \leq 1\}$



Siirrytään napakoordinaatteihin

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \text{Jacobin determinantti } J_g(r,\varphi) = r.$$

Merkitään $B = [0,1] \times [0, \pi/2]$. Joukon A pinta-ala on

$$a(A) = \iint_A dx\,dy = \iint_B |J_g(r,\varphi)|\,dr\,d\varphi = \int_0^1 dr \int_0^{\pi/2} d\varphi\, r$$

$$= \int_0^1 dr \Big/_0^{\pi/2} \varphi r = \frac{\pi}{2}\int_0^1 dr\, r = \frac{\pi}{2}\Big/_0^1 \frac{r^2}{2} = \frac{\pi}{4}.$$

$$\iint_A x\,dx\,dy = \iint_B r\cos\varphi\,|J_g(r,\varphi)|\,dr\,d\varphi = \int_0^{\pi/2} d\varphi \int_0^1 dr\, r^2\cos\varphi$$

$$= \int_0^{\pi/2} d\varphi \Big/_0^1 \frac{r^3}{3}\cos\varphi = \frac{1}{3}\int_0^{\pi/2} d\varphi \cos\varphi = \frac{1}{3}\Big/_0^{\pi/2} \sin\varphi = \frac{1}{3}.$$

$$\iint_A y\,dx\,dy = \iint_B r\sin\varphi\,|J_g(r,\varphi)|\,dr\,d\varphi = \int_0^{\pi/2} d\varphi \int_0^1 dr\, r^2\sin\varphi$$

$$= \int_0^{\pi/2} d\varphi \Big/_0^1 \frac{r^3}{3}\sin\varphi = \frac{1}{3}\int_0^{\pi/2} d\varphi \sin\varphi = \frac{1}{3}\Big/_0^{\pi/2}(-\cos\varphi) = \frac{1}{3}.$$

Siis joukon A keskiö on

$$\frac{1}{a(A)}\left(\iint_A x\,dx\,dy,\ \iint_A y\,dx\,dy\right) = \frac{4}{\pi}\left(\frac{1}{3}, \frac{1}{3}\right) = \left(\frac{4}{3\pi}, \frac{4}{3\pi}\right).$$