The $\overline{\partial}$-method for non-linear inverse problems

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Outline

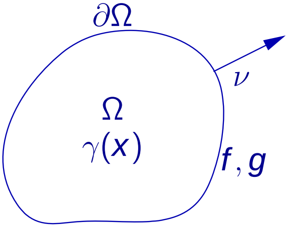
1. The inverse conductivity problem
2. Solution in 2D by the $\overline{\partial}$-method
3. Solution to the 3D problem
4. References

1. The inverse conductivity problem

The conductivity equation

Smooth bounded domain $\Omega \subset \mathbb{R}^n,\ n = 2, 3$; conductivity coefficient $\gamma \in L^\infty(\Omega),\ C^{-1} \leq Re(\gamma) \leq C$, for $C > 0$.
A voltage potential u in Ω generated by Dirichlet data f

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega.$$
$$f = u|_{\partial\Omega}$$



Corresponding Neumann data:

$$g = \gamma \partial_\nu u|_{\partial\Omega}.$$

Dirichlet to Neumann map

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$
$$f \mapsto g.$$

Inverse problem

Consider the (non-linear) mapping

$$\Lambda \colon \gamma \mapsto \Lambda_\gamma.$$

This mapping encodes the direct problem.

The Calderón problem (the inverse conductivity problem):

- Uniqueness: is Λ injective?
- Reconstruction: how can γ be computed from Λ_γ?

Applications include Electrical Impedance Tomography, emerging technology for medical imaging.

Short and incomplete history

1980 Calderón: Problem posed, uniqueness for linearized problem, and linear, approximate reconstruction algorithm

3D

1987 Sylvester and Uhlmann: Uniqueness for smooth conductivities. Implicit reconstruction algorithm

1987-88 Novikov, Nachman-Sylvester-Uhlmann, Nachman: Uniqueness for conductivities with 2 derivatives and explicit high frequency reconstruction algorithm. Multidimensional D-bar equation.

2003 Brown-Torres, Päivärinta-Panchenko-Uhlmann: Uniqueness for conductivities with $3/2$ derivatives.

2006 Cornean-Knudsen-Siltanen: Low frequency reconstruction algorithm

2010 Bikowski-Knudsen-Mueller: Numerical implementation of reconstruction algorithms

2D

1996 Nachman: Uniqueness and reconstruction for $W^{2,p}(\Omega)$ conductivities.

1997 Brown-Torres: Uniqueness for $W^{1}, p(\Omega)$ conductivities

2001 Knudsen-Tamasan: Reconstruction for $C^{1+\epsilon}$ conductivties

2005 Astala-Päivärinta: Uniqueness and reconstruction for $L^{\infty}(\Omega)$

2009 Knudsen-Lassas-Mueller-Siltanen: Regularized $\overline{\partial}$-method

Assumptions

Assume throughout that

1. Ω and γ are sufficiently smooth
2. $\gamma = 1$ near $\partial\Omega$
3. γ is extended to $\mathbb{R}^n \setminus \Omega$ by $\gamma = 1$
4. In 2D assume γ is real

Note that 1. and 4. are restrictive, but 2.-3. can be assumed WLOG.

2. Solution in 2D by the $\overline{\partial}$-method

Recall from yesterday the scattering and inverse scattering transforms

$$q \leftrightarrow S$$

associated with the system

$$(D - Q)\Psi = 0, \quad D = \begin{pmatrix} \partial_{\overline{z}} & 0 \\ 0 & \partial_z \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q \\ \overline{q} & 0 \end{pmatrix}.$$

$$S(k) = \frac{-i}{\pi} \int_{\mathbb{R}^2} e(z,k)\overline{q}(z) m_1(z,k) d\mu(z)$$

facilitated by the exponentially growing Jost solutions

$$\Psi(z,k) = e^{izk} m(z,k) = e^{izk} \begin{pmatrix} m_1(z,k) \\ m_2(z,k) \end{pmatrix}, \quad \lim_{|z|\to\infty} m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The conductivity equation as a first order system

Let u be a solution to the conductivity equation.
Then $(v, w) = \gamma^{1/2}(\partial u, \overline{\partial} u)$ solves in $\mathbb{R}^2$

$$\begin{array}{ll} \overline{\partial} v & = qw \\ \partial w & = \overline{q} v \end{array} \Leftrightarrow (D - Q)(v, w)' = 0,$$

where

$$q = -\gamma^{-1/2}\partial\gamma^{1/2}$$
$$\partial = (\partial_{x_1} - i\partial_{x_2})/2, \quad \overline{\partial} = (\partial_{x_1} + i\partial_{x_2})/2.$$

Consequence for conductivty equation: there is a unique complex geometrical optics solution φ

$$\nabla \cdot \gamma \nabla \varphi(z, k) = 0, \qquad e^{-izk}\varphi(z, k) \rightarrow_{|z|\rightarrow\infty=1}$$

Reconstruction algorithm

Reconstruction is based on the decomposition

$$\Lambda_\gamma \stackrel{1}{\to} S(k) \stackrel{2}{\to} q(\gamma)$$

Facts

1. S is computable from the boundary measurements Λ_γ
2. The second step is facilitated by the inverse scattering transform

$\Lambda_\gamma \to S$

From previous lecture

$$S(k) = \frac{-i}{\pi} \int_{\mathbb{R}^2} e(z,k)\overline{q}(z) m_1(z,k) d\mu(z)$$

Implies

$$\begin{aligned} S(k) &= \frac{-i}{\pi} \int_{\mathbb{R}^2} \partial(e(z,k) m_2(z,k)) d\mu(z) \\ &= \frac{-i}{2\pi} \int_{\partial\Omega} m_2(z,k)(\nu_1 + i\nu_2) d\sigma(z). \end{aligned}$$

In terms of φ the formula becomes

$$S(k) = \frac{-1}{2k} \int_{\partial\Omega} e^{i\overline{zk}} (\Lambda_\gamma - \Lambda_1)\varphi(\cdot,k) d\sigma(z)$$

where φ is the complex geometrical optics solution ($\varphi(z,k) \sim e^{izk}$) to the conductivity equation.

How to compute $\varphi|_{\partial\Omega}$?

Let S_k denote the single layer potential with Faddeev's Green's function G_k for $-\Delta$:

$$S_k f(x) = \int_{\partial\Omega} G_k(x-y)f(y)d\sigma(y).$$

Then $\varphi|_{\partial\Omega}$ is the unique solution to

$$\varphi(z,k) = e^{izk} - S_k(\Lambda_\gamma - \Lambda_1)\varphi,$$

This is a Fredholm equation of the second kind; uniqueness for homogeneous problem follows from uniuqness of Jost solutions (complex geometrical optics).

The algorithm

1. Solve for $z \in \partial\Omega$ and $k \in \mathbb{C}$

$$\varphi(z,k) = e^{izk} - S_k(\Lambda_\gamma - \Lambda_1)\varphi,$$

   and compute

$$S(k) = \frac{-1}{2k}\int_{\partial\Omega} e^{i\overline{zk}}(\Lambda_\gamma - \Lambda_1)\varphi(\cdot,k)d\sigma(z)\ k \in \mathbb{C}$$

2. Could go for q by inverse scattering transform. However, it turns out that

$$\gamma(z) = Re(m^+(z,k))^2$$

   with m^+ the solution to the $\partial_{\overline{k}}$-equation

$$\partial_{\overline{k}} m^+(z,k) = \overline{S(-k)}e(z,-k)\overline{m^+(z,k)}, \quad \lim_{|k|\to\infty} = 1.$$

Step 1. is severely ill-posed but step 2. is well-posed.

Regularization of the algorithm

In practice we cut off the spectral scattering data $S(k)$ for $k > R$. This is a regularization strategy.

Suppose we measure noisy data $\tilde{\Lambda}_\gamma = \Lambda_\gamma + \mathcal{E}$ where $\|\mathcal{E}\|_{B(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))} = \epsilon$. Then there is a choice of $R(\epsilon)$ such that if we solve

$$\tilde{\varphi}(z,k) = e^{izk} - S_k(\tilde{\Lambda}_\gamma - \Lambda_1)\tilde{\varphi}, \quad |k| < R(\epsilon)$$

compute

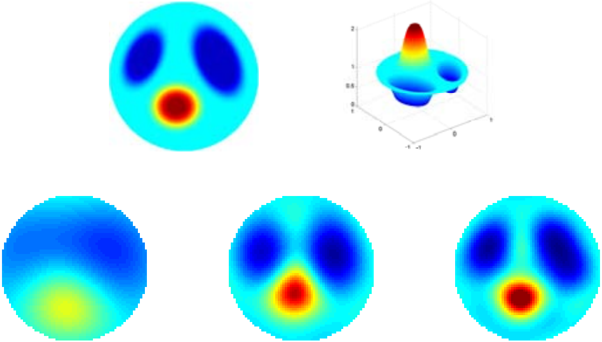
$$\tilde{S}(k) = \frac{-1}{2k}\int_{\partial\Omega} e^{i\overline{zk}}(\tilde{\Lambda}_\gamma - \Lambda_1)\tilde{\varphi}(\cdot,k)d\sigma(z), \quad |k| < R(\epsilon)$$

and solve the $\overline{\partial}$-equation with this compactly supported $\tilde{S}$, then

$$\tilde{\gamma}(z) \to \gamma(z) \text{ for } \epsilon \to 0.$$

Exact regularization algorithm for a non-linear inverse problem.

Numerical results



Reconstructions with noiselevel $10^{-2}, 10^{-4}$ and 10^{-6}. Error in approximation is $52\%, 14\%$ and 12% respectively.

3. Solution to the 3D problem

Transformation to Schrödinger equation

Suppose u solves

$$\nabla \cdot \gamma \nabla u = 0 \text{ in } \Omega, \qquad u|_{\partial\Omega} = f.$$

Then $v = \gamma^{-1/2} u$ solves

$$(\Delta + q) v = 0 \text{ in } \Omega \qquad v|_{\partial\Omega} = \gamma^{-1/2} f,$$

with $q = -\Delta\gamma^{1/2}/\gamma^{1/2} \Leftrightarrow (\Delta + q)\gamma^{1/2} = 0$.

Dirichlet to Neumann map $\Lambda_q f = \partial_\nu v$. If $\gamma = 1$ near $\partial\Omega$ then $\Lambda_q = \Lambda_\gamma$.

The operator $(\Delta + q)$ plays the same role in 3D as $(D - Q)$ in 2D.

Complex geometrical optics (CGO)

Let $\zeta \in \mathbb{C}^3$ such that $\zeta \cdot \zeta = 0$. For sufficiently large ζ there is a unique CGO solution to the problem

$$(\Delta + q)\psi(x,\zeta) = 0 \text{ in } \mathbb{R}^3,$$
$$\psi(x,\zeta) \sim e^{ix\cdot\zeta} \text{ for large } |x| \text{ or } |\zeta|.$$

Lippmann-Schwinger-Faddeev (LSF) equation

$$\psi(x,\zeta) = e^{ix\cdot\zeta} + \int_\Omega G_\zeta(x-y)q(y)\psi(y,\zeta)dx, \qquad \Delta G_\zeta = \delta, \quad G_\zeta \sim e^{ix\cdot\zeta}.$$

Moreover, $\psi|_{\partial\Omega}$ satisfies the solvable Fredholm equation

$$\psi(x,\zeta) + \int_{\partial\Omega} G_\zeta(x-y)(\Lambda_\gamma - \Lambda_1)\psi(y,\zeta)d\sigma(y) = e^{ix\cdot\zeta}, \quad x \in \partial\Omega.$$

The scattering transform

The key intermediate object, the *non-physical scattering transform,*

$$\mathbf{t}(\xi,\zeta) = \int_\Omega e^{-ix\cdot(\xi+\zeta)} q(x)\psi(x,\zeta)dx$$
$$= \int_{\partial\Omega} e^{-ix\cdot(\xi+\zeta)} (\Lambda_\gamma - \Lambda_1)\psi(x,\zeta)|_{\partial\Omega} d\sigma(x), \qquad (\xi+\zeta)^2 = 0.$$

t satisfies the estimate

$$|\hat{q}(\xi) - \mathbf{t}(\xi,\zeta)| = \mathcal{O}(1/|\zeta|)$$

Sets up a scattering inverse scattering transform

$$q \leftrightarrow \mathbf{t}.$$

Non-linear direct reconstruction algorithm

$$\Lambda_\gamma \to \mathbf{t}(\xi,\zeta) \to q(x) \to \gamma(x)$$

Steps

1. $\psi|_{\partial\Omega}$ can be computed from boundary measurements by solving

$$\psi + S_\zeta(\Lambda_\gamma - \Lambda_1)\psi = e^{ix\cdot\zeta}, \qquad x \in \partial\Omega$$

   and $\mathbf{t}$ can be computed from boundary data and $\psi|_{\partial\Omega}$

2. q can be computed from $\mathbf{t}$ using

$$\lim_{\zeta\to\infty} \mathbf{t}(\xi,\zeta) = \hat{q}(\xi)$$

3. γ can be computed from q by solving

$$(\Delta + q)\gamma^{1/2} = 0 \text{ in } \Omega, \gamma^{1/2}|_{\partial\Omega} = 1$$

Connection to Calderón's linearized reconstruction

Near-field scattering transform:

$$\mathbf{t}^{\mathrm{exp}}(\xi,\zeta) = \left\langle (\Lambda_\gamma - \Lambda_1) e^{ix\cdot\zeta}, e^{-ix\cdot(\zeta+\xi)} \right\rangle = \int_\Omega (\gamma(x)-1)\nabla u^{\mathrm{exp}}(x,\zeta)\cdot\nabla e^{-ix\cdot(\xi+\zeta)}dx,$$

with $\nabla\cdot\gamma\nabla u^{\mathrm{exp}} = 0$ in Ω and $u^{\mathrm{exp}}|_{\partial\Omega} = e^{ix\cdot\zeta}$.
Replacing in Ω u^{exp} by $e^{ix\cdot\zeta}$ gives

$$\mathbf{t}^{\mathrm{exp}}(\xi,\zeta) \approx -\frac{1}{2}|\xi|^2 \widehat{(\gamma-1)}(\xi).$$

This algorithm was proposed by Calderón in 1980.

In 2D (Siltanen-Isaacson-Mueller, 2001) $\mathbf{t}$ was replaced by $\mathbf{t}^{\mathrm{exp}}$ before $\overline{\partial}$-equation was solved. In 3D similar substitution can be done.

$\overline{\partial}$-equation in 3D

As in 2D we can apply a differential operator in the spectral paramater ζ to the special solutions $\psi(x,\zeta)$. Let us write

$$\mu(x,\zeta) = e^{ix\cdot\zeta}\psi(x,\zeta).$$

Then it turns out that

$$w\cdot\overline{\partial}_\zeta\mu(x,\zeta) = \frac{-1}{(2\pi)^{n-1}}\int_{B_\zeta} e^{ix\cdot\xi}\mathbf{t}(\xi,\zeta)\mu(x,\zeta+\xi)(w\cdot\xi)d\sigma(\xi)$$

where $B_\zeta = \{\xi\in\mathbb{R}^n : (\xi+\zeta)^2 = 0\}$ is the ball in the plane $\xi\cdot Im(\zeta) = 0$ centred at $c = -Re(\zeta)$ with radius $r = |Re(\zeta)|$.

The $\overline{\partial}$-equation can be solved and a reconstruction can be obtained by evaluating $\gamma(x) = \mu(x,0)^2$.

This approach can be made rigorous when γ is sufficiently close to constant.

4. Implementation details and numerical results

Implementation details $\mathbf{t}^{\exp}$

1. Solve numerically using comsol multiphysics (FEM)

$$\nabla \cdot \gamma \nabla u^{\exp} = 0 \text{ with } u^{\exp}|_{\partial\Omega} = e^{ix\cdot\zeta}$$

2. Integrate numerically

$$\mathbf{t}^{\exp}(\xi, \zeta) = \int_{\Omega} (\gamma - 1) \nabla u^{\exp} \cdot \nabla e^{ix\cdot(\xi+\zeta)} dx.$$

Implementation details **t**

Computation of Green's function $G_\zeta(x) = e^{ix\cdot\zeta} g_\zeta(x)$:

$$g_{e_1+ie_2}(x) = \frac{e^{-r+x_2-ix_1}}{4\pi r} - \frac{1}{4\pi}\int_s^1 \frac{e^{-ru+x_2-ix_1}}{\sqrt{1-u^2}} J_1(r\sqrt{1-u^2})du, \quad |x| < 2R$$

from [Newton, 1989] + symmetry.

Computation of ψ : technique of Vainikko for solving Lippman-Schwinger eq.

$$\mu(x,\zeta) = \psi(x,\zeta)e^{-ix\cdot\zeta}$$
$$g_\zeta(x) = G_\zeta(x)e^{-ix\cdot\zeta}.$$

Then

$$\mu(x,\zeta) = 1 + \int_\Omega g_\zeta(x-y)q(y)\mu(y,\zeta)dy.$$

Implementation details **t**

$$\mu(x,\zeta) - \int_\Omega g_\zeta(x-y)q(y)\mu(y,\zeta)dy = 1.$$

Note

- RHS is periodic
- Integral is on bounded domain (compact support of q)

Periodic equation for μ^{p} :

$$\mu^{\mathrm{p}}(x,\zeta) - \int_{\mathbb{R}^3} g^{\mathrm{p}}_\zeta(x-y)q^{\mathrm{p}}(y)\mu^{\mathrm{p}}(y,\zeta)dy = 1.$$

- Periodic equation is uniquely solvable and on Ω $\mu^{\mathrm{p}}(x,\zeta) = \mu(x,\zeta)$
- Solved efficiently using FFT and GMRES

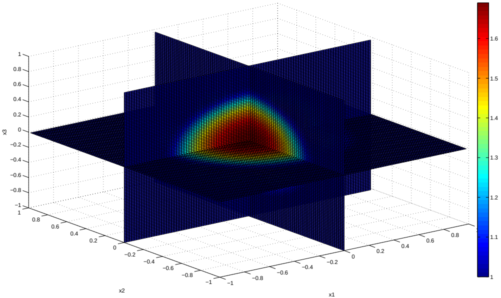
Numerical integration

$$\mathbf{t}(\xi,\zeta) = \int_\Omega e^{-ix\cdot(\xi+\zeta)}q(x)\psi(x,\zeta)dx.$$

Example 1: radially symmetric conductivity

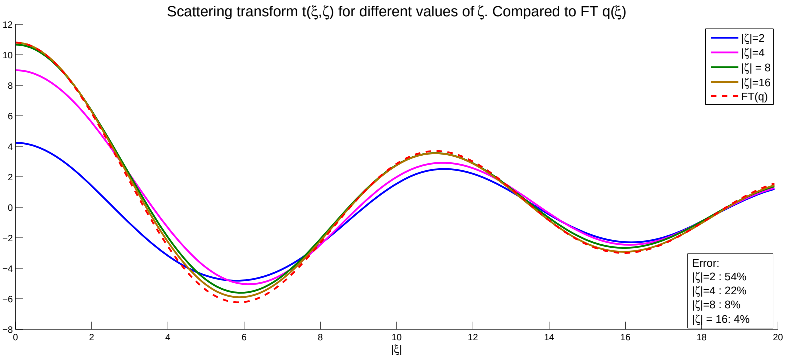
Take $\Omega = B(0,1)$ and define for $\alpha \in \mathbb{R}_+$ and $0 < d < 1$

$$\gamma(x) = \begin{cases} \left(1 + \alpha e^{-\frac{|x|^2}{(|x|^2 - d^2)^2}}\right)^2, & |x| \leq d \\ 1, & d < |x| \leq 1. \end{cases}$$



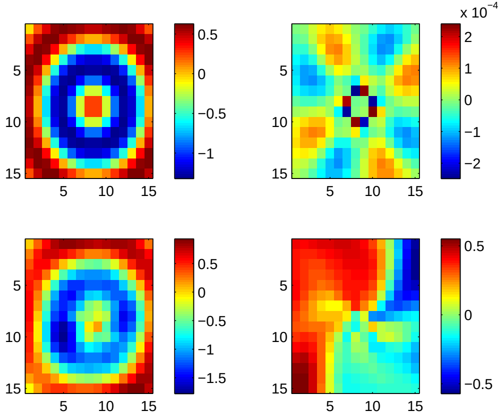
Convergence of scattering transform

Particular example with $\alpha = 2,\ d = .9$:



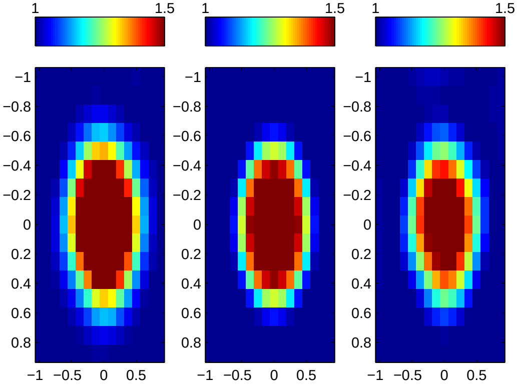
Scattering data

With $\alpha = .3$ *d* $= .9$. Crossection through plane $\xi_3 = 0$. Upper row real and imaginary part of **t**. Lower row $\mathbf{t}^{\exp}$.



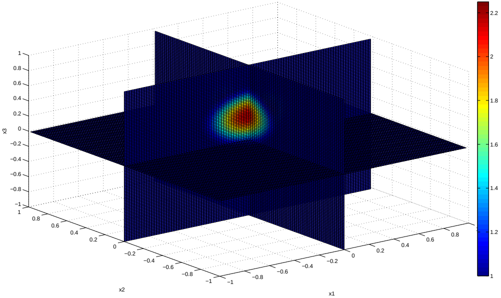
Reconstructions

With $\alpha = .3$ *d* $= .9$. Crossection through plane $\xi_3 = 0$. Left reconstruction based on $\mathbf{t}^{\exp}$, middle true conductivity, right Calderón's method.



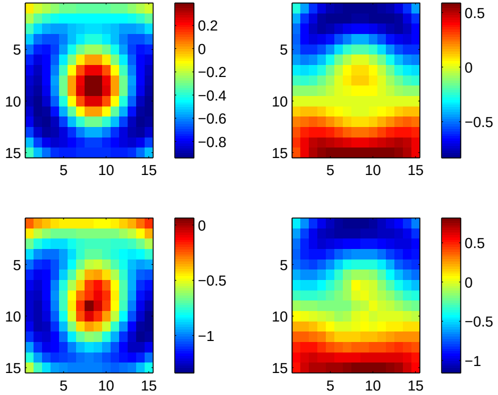
Example 2: Non-radially symmetric conductivity:

Take $\Omega = B(0,1)$. Conductivity has uniform background 1 and contains inclusion centered at $(0, .1, .3)$ with radius .6.



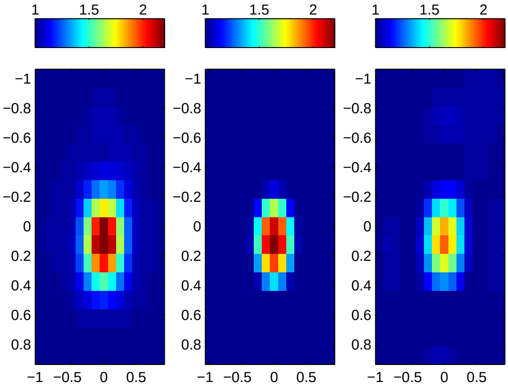
Scattering data

Crossection through plane $\xi_3 = 0$ Upper row real and imaginary part of **t**. Lower row $\mathbf{t}^{\mathrm{exp}}$.



Reconstructions

Crossection through plane $x_3 = .3$ Left reconstruction based on $\mathbf{t}^{\mathrm{exp}}$, middle true conductivity, right Calderón's method.

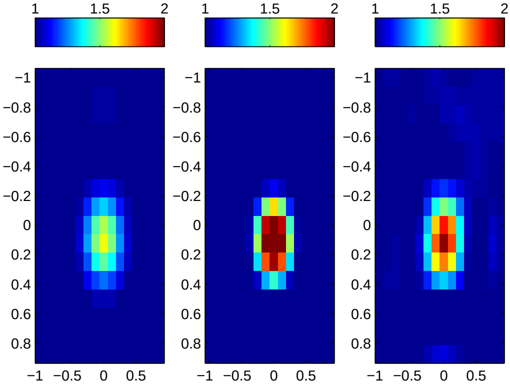


Example 3: Complex conductivity

Take $\Omega = B(0,1)$. Conductivity is a complex superposition of previous two

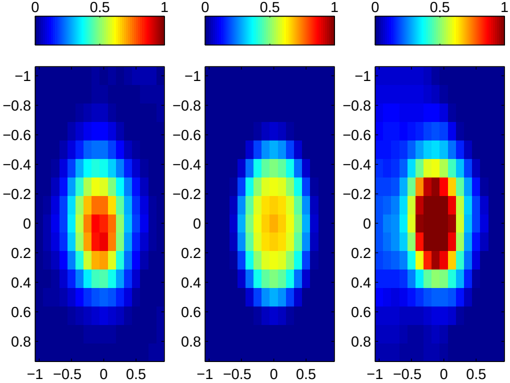
Reconstruction real part

Crossection through plane $x_3 = .3$ Left reconstruction based on $\mathbf{t}^{\mathrm{exp}}$, middle true conductivity, right Calderón's method.



Reconstruction imaginary part

Crossection through plane $x_3 = .3$ Left reconstruction based on $\mathbf{t}^{\mathrm{exp}}$, middle true conductivity, right Calderón's method.



Conclusion

- Solution of the inverse conductivty problem in 2D by the $\overline{\partial}$-method
- Similar ideas apply in 3D
- Nunmerical implementations in 2D and 3D
- Especially in 3D there are open ends:
    - Can $\overline{\partial}$-equation be solved for general conductivities
    - Numerical implementation of the full non-linear inversion

References

Non-linear evolution equations:

- A. S. Fokas, *Lax pairs: a novel type of separability*, Inverse problems 25 (2009)
- R. Beals and R.R. Coifman, *The spectral problem for the Davay-Stewartson and Ishimori hierarchies*, in Non-linear evolution equations, Integrability and Spectral methods (1988)

Inverse conductivity problem

- A. Calderón, *On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980), Soc. Brasil. Mat., Rio de Janeiro, 1980*
- J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem,* Ann. of Math. 125 (1987)
- R. Novikov, *A multidimensional inverse spectral problem for the equation (v(x) — Eu(x)) = 0*, Funktsional. Anal. i Prilozhen. 22 (1988), no. 4, translation in Funct. Anal. Appl., 22 (1988), no. 4.
- A. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem,* Ann. of Math. (2) 143 (1996), no. 1
- R. Brown and G. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions*, Comm. Partial Differential Equations 22 (1997), no. 5- 6
- K. Astala and L. Päivärinta, *Calderónfs inverse conductivity problem in the plane,* Ann. of Math. 163 (2006)
- K. Knudsen, M. Lassas, J. Mueller, S. Siltanen, *Regularized D-BAR method for the inverse conductivity problem*, Inverse problems and Imaging, 3(4), 2009
- J Bikowski, K. Knudsen, J. Mueller, *Direct numerical reconstruction of conductivities in 3-D*, submitted

Thank you