Iterative solution methods for inverse problems: II Nonlinear Problems and Tikhonov regularization

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overview

Nonlinear setting

We want to solve the operator equation

$$F(x) = y \tag{1}$$

given noisy data $y^\delta \in Y$ satisfying $\|y^\delta - y\| \leq \delta$..
Assume that for exact data y exact solution $x^\dagger$ exists and is unique.

Discuss some aspects of methods for nonlinear problems by means of the best investigated one:
Tikhonov regularization: x_α^δ minimizer of

$$J_\alpha(x) = \|F(x) - y^\delta\|^2 + \alpha\|x - x_0\|^2 = \min_{x \in D(F)}! \tag{2}$$

x_0 ... initial guess of $x^\dagger$, J_α ... Tikhonov functional

Well-definedness and stability

$$((\psi_n)_{n\in\mathbb{N}} \subset \mathcal{D}(F) \ \wedge\ \psi_n \hookrightarrow \psi \ \wedge\ F(\psi_n) \hookrightarrow f) \\ \Longrightarrow \quad \psi \in \mathcal{D}(F) \ \wedge\ F(\psi) = f\,. \tag{3}$$

Theorem

Let $\alpha > 0$ and assume that F is weakly closed (3) and continuous. Then the Tikhonov functional (2) has a global minimizer.

Theorem

Let $\alpha > 0$ and assume that F is weakly closed (3) and continuous. For any sequence $y^k \to y^\delta$ as $k \to \infty$ the corresponding minimizers x_α^k of (2) (with y^k in place of y^δ) converge to x_α^δ.

Convergence

Theorem

Assume that F is weakly closed (3) and continuous. Let $\alpha = \overline{\alpha}(\delta)$ be chosen such that

$$\overline{\alpha}(\delta) \to 0 \quad \text{and} \quad \delta^2/\overline{\alpha}(\delta) \to 0 \qquad \text{as } \delta \to 0. \tag{4}$$

If y^{δ_k} is some sequence in Y such that $\|y^{\delta_k} - y\| \leq \delta_k$ and $\delta_k \to 0$ as $k \to \infty$, and if $x_{\alpha_k}^{\delta_k}$ denotes a solution to (2) with $y^\delta = y^{\delta_k}$ and $\alpha = \alpha_k = \overline{\alpha}(\delta_k)$, then $\|x_{\alpha_k}^{\delta_k} - x^\dagger\| \to 0$ as $k \to \infty$.

The same result holds for the discrepancy principle $\alpha = \max \ s.t. \ \|F(x_\alpha^\delta) - y^\delta\| \leq \tau\delta \,,$ (with some $\tau > 1$)in place of the a priori choice (4)

Conditions on F and convexitiy of the Tikhonov functional

Lemma

Let the weak nonlinearity *condition [Chavent&Kunisch 1996]*

$$\forall x \in D(F)\, \forall w : \, x + w \in D(F) \; : \quad \|F''(x)[w,w]\| \leq \frac{1}{R} \|F'(x)w\|^2 \tag{5}$$

for some $R > \delta$ hold. Then for all $\alpha > 0$, the Tikhonov functional is convex in a sufficiently small neighborhood of $x^\dagger$.

Proof:

$$J''_\alpha(x)[w,w] = 2\|F'(x)w\|^2 + 2\langle F(x) - y^\delta, F''(x)[w,w]\rangle + 2\alpha\|w\|^2$$

Lemma

Let the following Taylor remainder estimate $\forall \tilde{x}, x \in D(F)$

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \leq \min\{\frac{1}{R}\|F(\tilde{x}) - F(x)\|^2 \,,\, c\|F(\tilde{x}) - F(x)\|\}$$

for some $R > \delta$, $c < 1 - \frac{2\delta}{R}$ *hold. Then for all* $\alpha > 0$, *the Tikhonov functional is convex in a sufficiently small neighborhood of* $x^\dagger$.

Proof:

$$\begin{aligned}
&\langle J'_\alpha(\tilde{x}) - J'_\alpha(x), (\tilde{x} - x)\rangle \\
&\quad = \quad \langle F(\tilde{x}) - F(x), F'(x)(\tilde{x} - x)\rangle + \langle F(\tilde{x}) - y^\delta, (F'(\tilde{x}) - F'(x))(\tilde{x} - x)\rangle \\
&\quad \geq \quad \|F(\tilde{x}) - F(x)\|^2 - c\|F(\tilde{x}) - F(x)\|^2 - \frac{2}{R}\|F(x) - y^\delta\| \, \|F(\tilde{x}) - F(x)\|^2
\end{aligned}$$

since $(F'(\tilde{x}) - F'(x))(\tilde{x} - x) =$
$F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x) + F(x) - F(\tilde{x}) - F'(\tilde{x})(x - \tilde{x})$.

An example

Estimate the diffusion coefficient a in

$$\begin{aligned} -(a(s)u(s)_s)_s = f(s)\,, \qquad s \in (0,1)\,, \\ u(0) = 0 = u(1)\,, \end{aligned} \tag{6}$$

where $f \in L^2$; subscript s ... derivative

$$\begin{aligned} F : \mathcal{D}(F) := \{a \in H^1[0,1] \,:\, a(s) \geq \underline{a} > 0\} &\rightarrow L^2[0,1] \\ a &\mapsto F(a) := u(a)\,, \end{aligned}$$

$$\begin{aligned} F'(a)h &= A(a)^{-1}[(hu_s(a))_s]\,, \\ F'(a)^*w &= -B^{-1}[u_s(a)(A(a)^{-1}w)_s]\,, \end{aligned}$$

$$\begin{aligned} A(a) : H^2[0,1] \cap H_0^1[0,1] &\rightarrow L^2[0,1] \\ u &\mapsto A(a)u := -(au_s)_s \end{aligned}$$

$$B : \mathcal{D}(B) := \{\psi \in H^2[0,1] \,:\, \psi'(0) = \psi'(1) = 0\} \rightarrow L^2[0,1]$$

$$\begin{aligned}
&\langle\, F(\tilde a) - F(a) - F'(a)(\tilde a - a), w)\,\rangle_{L^2} \\
&\qquad = \langle\, (\tilde u - u) - A(a)^{-1}[((\tilde a - a)u_s)_s], w\,\rangle_{L^2} \\
&\qquad = \langle\, A(a)(\tilde u - u) - ((\tilde a - a)u_s)_s, A(a)^{-1} w\,\rangle_{L^2} \\
&\qquad = \langle\, ((\tilde a - a)(\tilde u_s - u_s))_s, A(a)^{-1} w\,\rangle_{L^2} \\
&\qquad = -\langle\, (\tilde a - a)(\tilde u - u)_s, (A(a)^{-1} w)_s\,\rangle_{L^2} \\
&\qquad = \langle\, F(\tilde a) - F(a), ((\tilde a - a)(A(a)^{-1} w)_s)_s\,\rangle_{L^2}\,.
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\ & \|F(\tilde a) - F(a) - F'(a)(\tilde a - a)\|_{L^2} \\
&\quad \leq\ C\,\|F(\tilde a) - F(a)\|_{L^2}\,\|\tilde a - a\|_{H^1}\,.
\end{aligned}$$

Convergence rates and source conditions

Theorem

Assume that F' is Lipschitz

$$\|F'[x] - F'[\tilde{x}]\| \leq L\|x - \tilde{x}\| \tag{7}$$

for all $x, \tilde{x} \in D(F)$. Moreover, assume that the source condition

$$x^{\dagger} - x_0 = F'(x^{\dagger})^* w, \text{ with } \|w\| < \frac{1}{L}$$

is satisfied for some $w \in Y$ and that α is chosen according to the discrepancy principle $\alpha = \max \; s.t. \; \|F(x_\alpha^\delta) - y^\delta\| \leq \tau\delta$.
Then there exists a constant $C > 0$ independent of δ such that

$$\|x_\alpha^\delta - x^{\dagger}\| \leq C\sqrt{\delta}, \quad \|F(x_\alpha^\delta) - y\| \leq C\delta.$$

Theorem

Assume that F satisfies the Scherzer condition *[Scherzer 1995]*

$$\|F(\tilde{x}) - F(x) - F'(x)(\tilde{x} - x)\| \leq c\|F(\tilde{x}) - F(x)\| \qquad (8)$$

for all $x, \tilde{x} \in D(F)$. Moreover, assume that the source condition with $\mu \in (0, \frac{1}{2})$

$$x^\dagger - x_0 = (F'(x^\dagger)^* F'(x^\dagger)^\mu w$$

is satisfied for some $w \in X$ and that α is chosen according to the discrepancy principle $\alpha = \max \; s.t. \; \|F(x_\alpha^\delta) - y^\delta\| \leq \tau\delta$.
Then there exists a constant $C > 0$ independent of δ such that

$$\|x_\alpha^\delta - x^\dagger\| \leq C\delta^{\frac{2\mu}{2\mu+1}}, \quad \|F(x_\alpha^\delta) - y\| \leq C\delta.$$

Literature:

- stability and convergence: [Seidman&Vogel 1989]
- convergence rates [Engl&Kunisch&Neubauer 1989] [Neubauer 1999] [Hofmann&Scherzer 1998]
- analysis in Banach space: [Burger&Osher 2004], [Hofmann&Kaltenbacher&Pöschl&Scherzer 2007] [Hein&Hofmann&Kindermann&Neubauer&Tautenhahn 2009]
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