Note: This exercise contains problems that should be solved using the Matlab software. That's why the exercise session is in class C128.

Problems on this page should be computed by habd as before, and the problems on the next two pages are Matlab problems.

1. Let
$$A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}.$$
   (a) Find eigenvalues of A.
   (b) Find eigenvectors of A.
   (c) Write A in the form $A = PDP^{-1}$, where D is diagonal.

2. Let
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$
   (a) Find eigenvalues and eigenvectors of A.
   (b) Write A in the form $A = PDP^{-1}$, where D is diagonal.
   (c) Compute A^{2008} using (b).

(M1) Define the following matrices in Matlab. Example: 2×2-unit matrix can be defined as $[[1,0];[0,1]]$.

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$D = \begin{bmatrix} 7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Check the sizes of your matrices using command `size`.

(M2) Consider the equations

$$\begin{aligned} x + 2y &= 1 \\ -x - y &= -2 \end{aligned}$$

Form the coefficient matrix A and right hand side $\vec{b}$. Enter them to Matlab as `A` and `b`. Solve the system of linear equations by command `inv(A)*b`.

(M3) Check the validity of the following formula in Matlab:

$$A = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}.$$

Examine all four matrices using the routine `kissa.m` available at the course web page. Clarify to yourself how the linear mapping defined by A can be interpreted as changing coordinates, stretching along the horizontal axis by the factor of 2, and returning to original coordinates.

(M4) We study determining the dominant (i.e. greatest in absolute value) eigenvalue and the corresponding eigenvector using the so-called *power method* (Poole, Section 4.5).

Set

$$A = \begin{bmatrix} 5 & -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & 4 & 0 & 0 & 1 \end{bmatrix}.$$

This is the power method:

(i) Take as initial guess a nonzero vector $\vec{x}_0 = \vec{y}_0 \in \mathbb{R}^5$ whose elements are not greater than 1 in absolute value.

(ii) Repeat the following steps for $k = 1, 2, 3, \ldots$:

  (a) Compute $\vec{x}_k = A\vec{y}_{k-1}$.

  (b) Denote by m_k the element of $\vec{x}_k$ which is greatest in absolute value.

  (c) Set $\vec{y}_k = (1/m_k)\vec{x}_k$.

Now with most choices of $\vec{x}_0$ the number m_k converges to the dominating eigenvalue and $\vec{y}_k$ converges to a corresponding eigenvector when $\to \infty$.

(a) Implement the power method for the matrix A using Matlab.

(b) Use the command `[P,D]=eig(A)` to check your result.

(c) Can you find such an initial guess $\vec{x}_0$ that the method fails?