Linear algebra and matrices I, fall 2009 Exercise 4 (5.10.-9.10.2009)
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1. Let A be a $m \times n$ matrix and let $\vec{e}_i \in \mathbb{R}^m$ be the standard basis vector of size $1 \times m$. Show that $\vec{e}_i A$ is the ith row of the matrix A.

2. Matrix A is *symmetric* if $A = A^T$. Let B be an arbitrary matrix. Prove that BB^T ja B^TB are well-defined and symmetric.

3. Construct 6×6 matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ satisfying the following conditions:

$$a_{ij} = \begin{cases} i+j, & \text{jos } i \leq j, \\ 0, & \text{jos } i > j. \end{cases} \qquad b_{ij} = \begin{cases} 1, & \text{jos } |i-j| \leq 1, \\ 0, & \text{jos } |i-j| > 1. \end{cases}$$

4. Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   (a) Show that $A^2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$.

   (b) Prove by induction that $A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$ for $n \geq 1$.

   Hint: an induction proof has two parts. In the first part one shows that the claim holds for $n = 1$. The second part assumes that the claim holds for $n - 1$ and proves that this implies the claim for n.

5. Let A be an invertible matrix.

   (a) Show that $(cA)^{-1} = \frac{1}{c}A^{-1}$, when $c \in \mathbb{R}$ and $c \neq 0$.

   (b) Show that $(A^T)^{-1} = (A^{-1})^T$.

6. There are k chicken and p pigs on a farm. It is known that the total number of feet is 38 and the total number of heads is 16.

   (a) Denote $\vec{x} = [k \quad p]^T$ and write the problem in the form $A\vec{x} = \vec{b}$ where A is a 2×2 matrix.

   (b) Calculate by hand the inverse matrix A^{-1}.

   (c) Find the number of chicken and pigs using formula $\vec{x} = A^{-1}\vec{b}$.