How to find the Riemannian structure from the local source-to-solution mapping of a wave equation?

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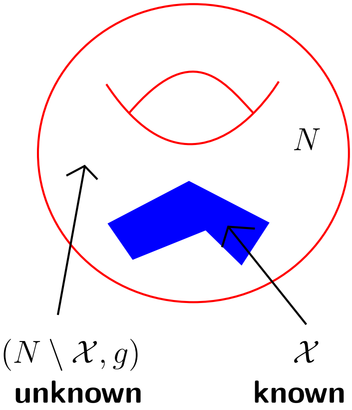
part of the paper: Correlation based passive imaging with a white noise source, JMPA, accepted, arXiv:1609:08022.

Problem setting 1

Let (N, g) be a smooth and complete Riemannian manifold without boundary and $\mathcal{X} \subset N$ open.



Model:

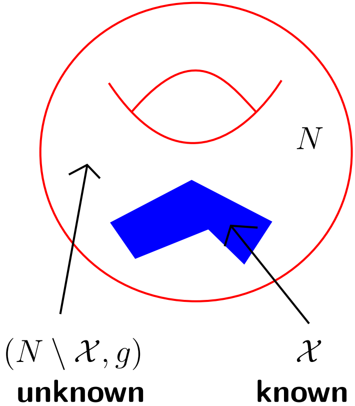
$$(\partial_t^2 - \Delta_g) w(t, x) = f, \quad \text{in } (0, \infty) \times N,$$
$$w|_{t=0} = \partial_t w|_{t=0} = 0,$$

where

$$f \in C_0^\infty((0, \infty) \times N).$$

Let Λ be the solution operator of the wave equation above. Denote $\Lambda f = w^f$.

Problem setting 2



Local source-to-solution operator:

For $f \in C_0^\infty((0,\infty) \times N)$, we define

$$\Lambda_{\mathcal{X}} f := \Lambda f|_{(0,\infty)\times N} = w^f|_{(0,\infty)\times N}.$$

What does $\Lambda_{\mathcal{X}}$ tell about (N, g)?

Main theorem

Theorem (Helin-Lassas-Oksanen-S 2016)

Let (N, g) *be a smooth and complete Riemannian manifold of dimension* $n \geq 2$*. Let* $\mathcal{X} \subset N$ *be an open and nonempty set. Consider the following initial value problem for the wave equation*

$$\begin{aligned} &\partial_t^2 w(t,x) - \Delta_g w(t,x) = f, \quad \textit{in } (0,\infty) \times N, \\ &w|_{t=0} = \partial_t w|_{t=0} = 0. \end{aligned}$$

Let $\Lambda_{\mathcal{X}} : C_0^\infty((0,\infty) \times \mathcal{X}) \to C^\infty((0,\infty) \times \mathcal{X})$ *be the local source-to-solution operator defined by*

$$\Lambda_{\mathcal{X}} f = w^f|_{(0,\infty)\times\mathcal{X}}.$$

Then the data $(\mathcal{X}, \Lambda_{\mathcal{X}})$ *determines* (N, g) *up to an isometry.*

Three components of the proof

(1) Show that the local the source-to-solution map $\Lambda_{\mathcal{X}}$ determines $d_g : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ and $g|_{\mathcal{X}}$ (**Not considered today, sorry**).

(2) Show that local the source-to-solution map $\Lambda_{\mathcal{X}}$ determines a certain family of distance functions

(3) Show that this family of distance functions determines the Riemannian manifold (N, g) (**Not considered today, sorry**).

Inverse spectral problem

Corollary

Let (N, g) be a smooth, connected and compact Riemannian manifold of dimension $n \geq 2$ without boundary. Let $\mathcal{X} \subset N$ be an open and nonempty set. Let $(\varphi_k)_{k=1}^{\infty} \subset C^{\infty}(N)$ be the collection of orthonormal eigenfunctions of operator Δ_g in $L^2(N)$. Let $(\lambda_k)_{k=1}^{\infty}$ be the collection of corresponding eigenvalues of Δ_g. Then the Spectral data

$$(\mathcal{X}, (\varphi_k|_{\mathcal{X}})_{k=1}^{\infty}, (\lambda_k)_{k=1}^{\infty})$$

determines (N, g) up to isometry.

Let $f \in C_0^{\infty}((0,\infty) \times N)$ and w^f be the solution of wave equation. Denote the j^{th} Fourier coefficient of w^f

$$I_j(t) := \langle w^f(t,\cdot), \varphi_j \rangle_{L^2(N)}.$$

By Greens formula and initial values of w^f we have

$$\begin{aligned} &\tfrac{d^2}{dt^2} I_j(t) - \lambda_j I_j(t) = \int_{\mathcal{X}} f(t,x)\varphi_j(x) dV_g(x) \\ &I_j(0) = \tfrac{d}{dt} I_j(0) = 0. \end{aligned}$$

History

Let (M, g) be smooth, compact manifold with boundary.

$$\begin{aligned} &\partial_t^2 w(t,x) - \Delta_g w(t,x) = 0, \quad \text{in } (0,\infty) \times M, \\ &w|_{t=0} = \partial_t w|_{t=0} = 0. \\ &u = f, \quad \text{in } (0,\infty) \times \partial M,\ f \in C_0^\infty((0,\infty) \times \partial M) \end{aligned}$$

Let Θ be the hyperbolic Dirichlet-to-Neuman map of above problem.

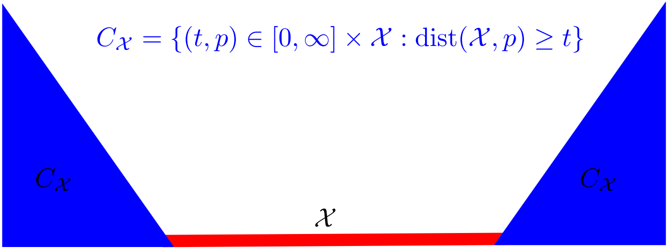
Does Θ determine (M, g)?

- The approach that we use is a modification of the Boundary Control method. This method was first developed by Belishev to the acoustic wave equation on $\mathbb{R}^n$ with an isotropic wave speed
- A geometric version of the method, suitable when the wave speed is given by a Riemannian metric tensor as in the present paper, was introduced by Belishev and Kurylev.
- Partial data problem is also considered for instance by: Katchalov–Kurylev, Lassas–Oksanen, Milne

Essential tool: Finite speed of wave propagation

Let $\mathcal{X} \subset N$ be an open and bounded set. Define

$$C_{\mathcal{X}} = \{(t,p) \in [0,\infty] \times \mathcal{X} : \mathrm{dist}(\mathcal{X},p) \geq t\}$$



$$(\partial_t^2 - \Delta_g)u = f, \ \text{ in } (0,\infty) \times N$$
$$f|_{C_{\mathcal{X}}} = 0$$
$$u|_{N \times \{t=0\}} = \partial_t u|_{N \times \{t=0\}} = 0,$$

Then

$$u|_{C_{\mathcal{X}}} = 0.$$

Essential tool: Unique continuation, by Tataru

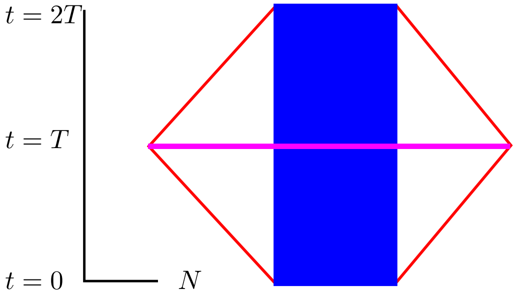
Consider an open double cone created by a cylindrical set $(0, 2T) \times \mathcal{X}$

$$C(T, \mathcal{X}) = \{(t, x) \in (0, 2T) \times N : \mathsf{dist}_g(x, \mathcal{X}) < \min\{t, 2T - t\}\}.$$

We write

$$M(T, \mathcal{X}) = \{x \in N : \mathsf{dist}_g(x, \mathcal{X}) \le T\}.$$

Let $u \in C_0^\infty(\mathbb{R} \times N)$. Suppose that $(\partial_t^2 - \Delta_g)u = 0$ in $(0, 2T) \times N$ and $u|_{(0,2T)\times\mathcal{X}} \equiv 0$. Then $u|_{C(T,\mathcal{X})} \equiv 0$.

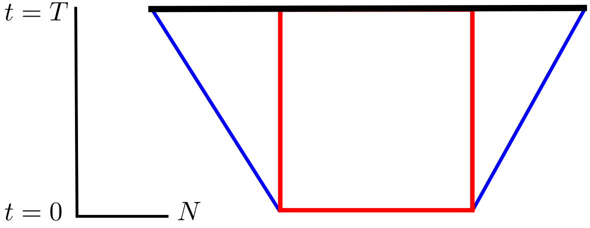


Essential tool: Approximate controllability

Let $T > 0$. The collection

$$\mathcal{W}_T := \{w^f(T,\cdot) : f \in C_0^\infty((0,T) \times \mathcal{X})\}$$

is dense in Hilbert space $L^2(M(T,\mathcal{X}))$.



Essential tool: Blagovestchenskii identity

Let $T > 0$ and $f, h \in C_0^\infty((0,2T) \times \mathcal{X})$, then

$$\langle w^f(T,\cdot), w^h(T,\cdot)\rangle_{L^2(N)} = \langle f, (J\Lambda_{\mathcal{X}} - \Lambda_{\mathcal{X}}^* J)h\rangle_{L^2((0,T)\times N)}$$

where

$$J : L^2(0,2T) \to L^2(0,T),\ J\phi(t) = \frac{1}{2}\int_t^{2T-t} \phi(s)\, ds.$$

From $\Lambda_{\mathcal{X}}$ to distance functions 1

Lemma

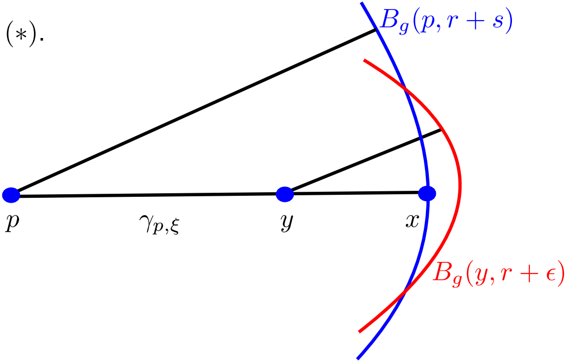
Let $(p,\xi) \in S\mathcal{X}$. *The data* $(\mathcal{X}, \Lambda_{\mathcal{X}}, d_g|_{\mathcal{X}\times\mathcal{X}})$ *determines* $\tau(p,\xi) := \sup\{t > 0 : d_g(p, \gamma_{p,\xi}(t)) = t\}$.

Let $s > 0$ so small that $\gamma_{p,\xi}([0,s]) \subset \mathcal{X}$. Denote $y = \gamma_{p,\xi}(s)$. Denote $x = \gamma_{p,\xi}(s+r)$. Let $\epsilon > 0$. If $r + s < \tau(p,\xi)$,
then for every $\epsilon > 0$ holds

$B_g(y, r+\epsilon) \setminus B_g(p, r+s) \neq \emptyset \quad (*)$.

Claim:

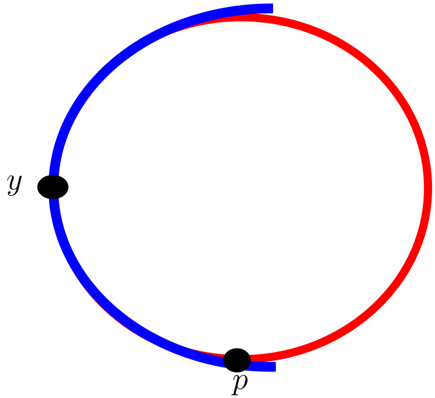
$\tau(y,\xi) = \{s + r > 0 : r > 0,$
$(*)$ holds for every $\epsilon > 0).\}$



Riemannian balls are not nice!

Let $N = S^2$, $p = (0, -1), s = \pi/2$, $y = (-1, 0)$ and $r > \pi/2$. for every $\epsilon > 0$ we have

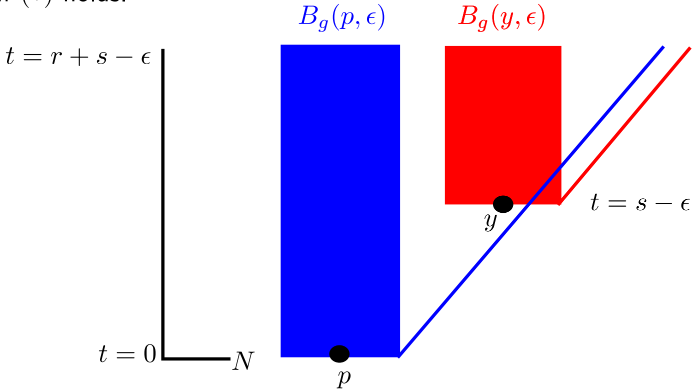
$$B_g(y, r + \epsilon) \setminus B_g(p, r + s) = \emptyset$$



From $\Lambda_{\mathcal{X}}$ to distance functions 2

Let $\epsilon > 0$. If $r + s < \tau(p, \xi)$. Then $B_g(y, r + \epsilon) \setminus B_g(p, r + s) \neq \emptyset$ $(*)$.

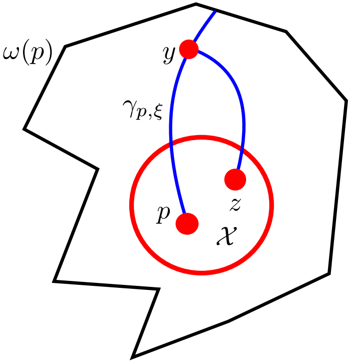
Using The approximate controllability and the Blagovestchenskii identity we can test if $(*)$ holds.



From $\Lambda_{\mathcal{X}}$ to distance functions 3

Lemma

Let $p, z \in \mathcal{X}$, $\xi \in T_p\mathcal{X}$, $\|\xi\| = 1$ *and* $\widetilde{r} < \tau(y, \xi)$. *Then data* $(\mathcal{X}, \Lambda_{\mathcal{X}}, d_g|_{\mathcal{X}\times\mathcal{X}})$ *determines* $d_g(y, z)$, *where* $y = \gamma_{p,\xi}(\widetilde{r})$.

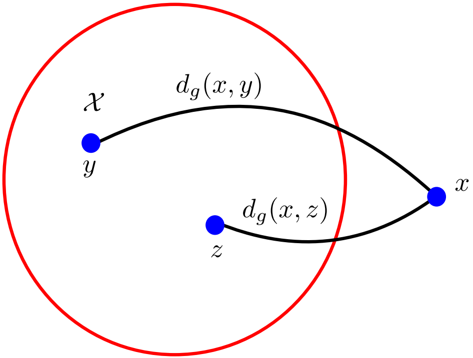


From $\Lambda_{\mathcal{X}}$ to distance functions 4

Theorem

Let (N, g) be a complete Riemannian manifold. Then the local source-to-solution data $(\mathcal{X}, \Lambda_{\mathcal{X}}, d_g|_{\mathcal{X}\times\mathcal{X}})$ determines the following family of distance functions

$$R_{\mathcal{X}}(N) := \{d_g(x, \cdot)|_{\mathcal{X}} : x \in N\} \subset C(\mathcal{X}).$$



From distance functions to reconstruction of (N, g)

Theorem

Let (N, g) be a complete smooth Riemannian manifold without a boundary. Let $U \subset N$ be open, bounded and have a smooth boundary. Suppose that the topological and smooth structure of U are known, and $g|_U$ is also known. Then

$$R(N) := \{d_g(\cdot, x)|_{\overline{U}} : x \in N\} \subset C(\overline{U})$$

determines, topological, smooth and Riemannian structure of (N, g) up to isometry.

Thank you for your attention!