DETERMINATION OF A RIEMANNIAN MANIFOLD FROM THE DISTANCE DIFFERENCE FUNCTIONS

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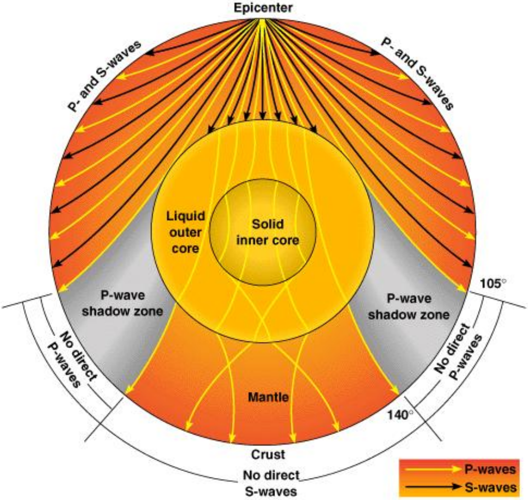
RIMS, January 21st 2016

Contents

Contents

Seismic imaging a geometric inverse problem



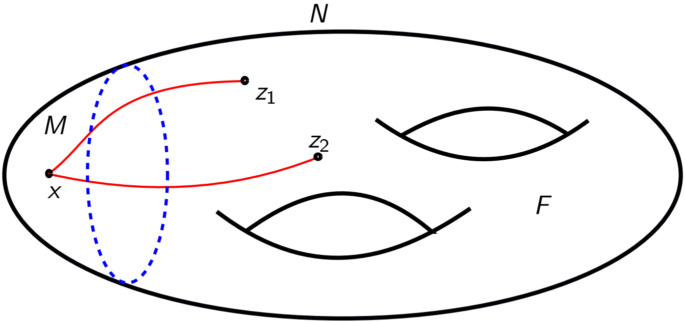
Propagation of seismic waves:
http://www.cyberphysics.co.uk/topics/earth/geophysics/SeismicWavesEarthStructure.html

Contents

Distance difference function

Let (N, g) be a Riemannian manifold, $M \subset N$ open. Denote $F := N \setminus M$. For every $x \in M$ we define

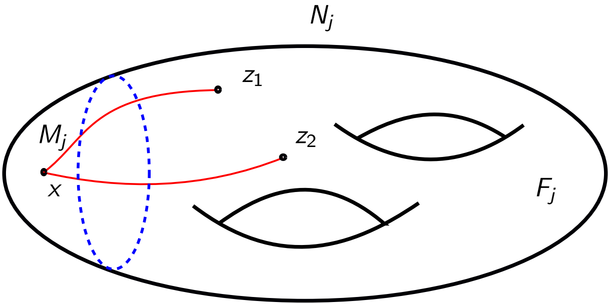
$$D_x : F \times F \to \mathbb{R},\ D_x(z_1, z_2) = d(x, z_1) - d(x, z_2).$$



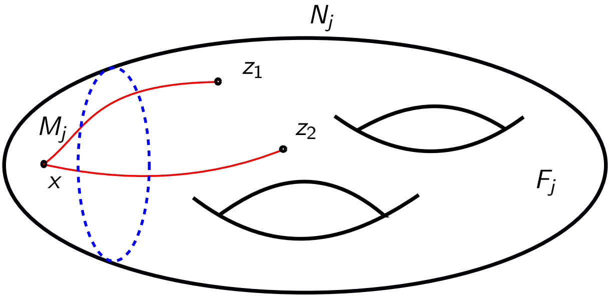
Distance difference function

Distance difference data 1

Let (N_1, g_1) and (N_2, g_2) be compact and connected $n-$dimensional Riemannian manifolds without boundary. Let $d_j(x, y)$ denote the Riemannian distance of points $x, y \in N_j$, $j = 1, 2$. Let $M_j \subset N_j$ be open sets and define closed sets $F_j = N_j \setminus M_j$. Suppose F_j are smooth $n-$ *manifolds* with smooth boundary $\partial F_j = \partial M_j$.



Distance difference data 2



We asumme that:

$$\begin{cases} \exists \text{ diffeomorphism } \phi : F_1 \to F_2 \text{ s.t. } \phi^* g_2|_{F_2} = g_1|_{F_1} \\ \{D_x^1(\cdot,\cdot) \; ; \; x \in M_1\} = \{D_y^2(\phi(\cdot),\phi(\cdot)) \; ; \; y \in M_2\}. \end{cases} \quad (1)$$

Here for each $x \in M_j$

$$D_x^j(z_1,z_2) = d_j(x,z_1) - d_j(x,z_2), \; z_1, z_2 \in F_j.$$

Main result

Theorem (Lassas-S)

Let (N_1, g_1) and (N_2, g_2) be closed and connected $n-$dimensional Riemannian manifolds, $n \geq 2$. Let $M_j \subset N_j$ be open sets and define closed sets $F_j = N_j \setminus M_j$. Suppose that F_j is a smooth $n-$dimensional manifold with boundary ∂F.

If the Distance difference data of N_1 and N_2 coincide i.e. (1) is valid, then manifolds (N_1, g_1) and (N_2, g_2) are Riemannian isometric.

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Idea of the proof:

1. Recover topology
2. Recover smooth structure
3. Recover Riemannian structure

Contents

Boundary distance functions and Inverse problem

Let (M, g) be a compact n-dimensional Riemannian manifold with boundary and $x \in M$. We define a *boundary distance function of* x as

$$r_x : \partial M \to \mathbb{R}, \; r_x(z) = d(x, z).$$

Let $\mathcal{R}(M) := \{r_x : x \in M\} \subset L^\infty(\partial M)$.

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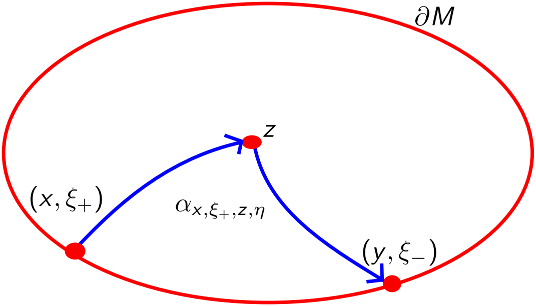
Theorem (Kurylev 97, Katchalov-Kurylev-Lassas 01)

Knowing only a Riemannian manifold $(\partial M, g|_{\partial M})$ and functions $\mathcal{R}(M) \subset L^\infty(\partial M)$ one can construct such a smooth structure to set $\mathcal{R}(M)$ that mapping $\mathcal{R} : M \to \mathcal{R}(M)$ is a diffeomorphism. In addition one can explicitly construct such a Riemannian metric tensor $\widetilde{g}$ of $\mathcal{R}(M)$ that (M, g) and $(\mathcal{R}(M), \widetilde{g})$ are isometric.

Broken scattering relation

Let (M, g) be a compact n-dimensional Riemannian manifold with boundary and $x \in M$. Let
$\Omega_+ = \{(x,\xi) \in SM : x \in \partial M,\ \langle \xi, \nu(x)\rangle > 0\}$,
$\Omega_- = \{(x,\eta) \in SM : x \in \partial M,\ \langle \eta, \nu(x)\rangle < 0\}$ and

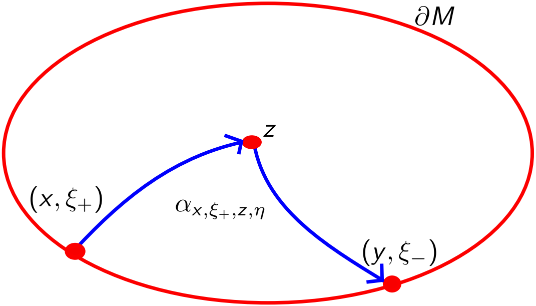
$$\alpha_{x,\xi_+,z,\eta}(t) = \begin{cases} \gamma_{x,\xi_+}(t),\ t < s,\ (x,\xi_+) \in \Omega_+ \\ \gamma_{z,\eta}(t-s),\ t \geq s,\ (z,\eta) \in SM^{int}. \end{cases}$$



Let $\ell(\alpha_{x,\xi_+,z,\eta}) > 0$ be the first time when $\alpha_{x,\xi_+,z,\eta}(t)$ hits ∂M.

The broken scattering relation

$$\begin{aligned}\mathcal{R} \;=\; & \{((x,\xi_+),(y,\xi_-),t) \in \Omega_+ \times \Omega_- \times \mathbb{R}_+ : \\ & t = \ell(\alpha_{x,\xi_+,z,\eta}) \text{ and } (y,\xi_-) = (\alpha_{x,\xi,z,\eta}(t), \partial_t\alpha_{x,\xi,z,\eta}(t)), \\ & \text{ for some } (z,\eta) \in SM\}\end{aligned}$$



Inverse problem of the Broken scattering relation

Theorem (Kurylev-Lassas-Uhlmann 2010)

Let (M, g) be a compact connected Riemannian manifold with a nonempty boundary of dimension $n \geq 3$. Then ∂M and the broken scattering relation $\mathcal{R}$ determine the isometry type of the manifold (M, g) uniquely.

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We define a *scattering distance* of $z \in M^{int}$ as

$$D_z^+(x, y) = d(z, x) + d(z, y), \quad x, y \in \partial M$$

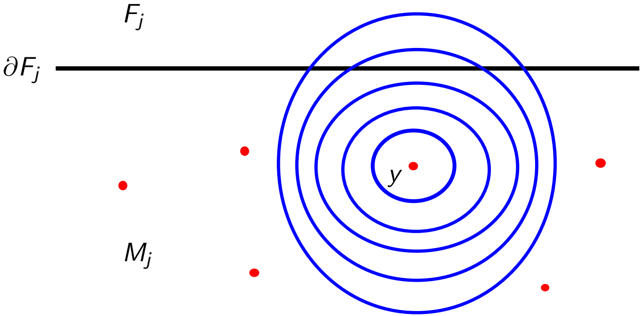
Notice that, if γ_{x,ξ_+} and $\gamma_{z,\eta}$ are distance minimizers then

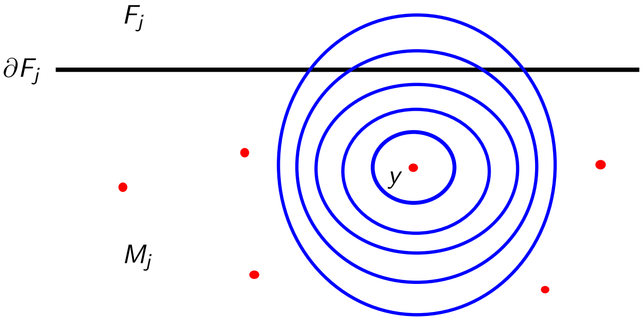
$$D_z^+(x, y) = \ell(\alpha_{x,\xi_+,z,\eta}).$$

An application for a wave equation

Let (N_j, g_j), $j = 1, 2$ be smooth Riemannian manifolds. $M_j \subset N_j$ open and $F_j := N_j \setminus M_j$ Consider a wave equation

$$\begin{cases} (\partial_t^2 - \Delta_{g_j})G_j(\cdot, \cdot, y, s) = \delta_{y,s}(\cdot, \cdot), & \text{in } N_j \times \mathbb{R},\ (y, s) \in M_j \times \mathbb{R}. \\ G_j(x, t, y, s) = 0, & \text{for } t < s,\ x \in N_j. \end{cases}$$





Suppose that the *spontanuous point source data* is valid:

$$\begin{cases} \exists \text{ diffeomorphism } \phi : F_1 \to F_2 \text{ s.t. } \phi^* g_2|_{F_2} = g_1|_{F_1} \\ W_1 = W_2 \end{cases} \tag{2}$$

$$W_1 = \{\mathrm{supp}(G_1(\cdot,\cdot,y_1,s_1)) \cap (F_1 \times \mathbb{R});\ y_1 \in M_1,\ s_1 \in \mathbb{R}\} \subset 2^{F_1 \times \mathbb{R}}$$

and

$$W_2 = \{\mathrm{supp}(G_2(\phi(\cdot),\cdot,y_2,s_2)) \cap (F_1 \times \mathbb{R}); y_2 \in M_2,\ s_2 \in \mathbb{R}\} \subset 2^{F_1 \times \mathbb{R}}$$

Theorem (Lassas-S)

Let (N_j, g_j), $j = 1, 2$ be a closed compact Riemannian n-manifolds, $n \geq 2$ and $M_j \subset N_j$ be an open set such that $F_j = N_j \setminus M_j$ are smooth n-manifolds with boundary. If the spontanuous point source data of these manifolds coincide, that is, we have (2), then (N_1, g_1) and (N_2, g_2) are isometric.

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Proof: Let $y \in M_j$, $z_i \in F_j$, $s \in \mathbb{R}$ $i, j = 1, 2$.

$$\mathcal{T}^j_{y,s}(z_i) = \sup\{t \in \mathbb{R};\ \text{the point } (z_i, t) \text{ has a neighborhood } U \subset N_j \times \mathbb{R} \text{ such that } G_j(\cdot,\cdot,y,s)\big|_U = 0\}$$

Hence one can deduce that $\mathcal{T}^j_{y,s}(z_i) = d_j(z_i, y) - s$ and therefore distance difference functions satisfy equation

$$D^j_y(z_1, z_2) = d_j(z_1, y) - d_j(z_2, y) = \mathcal{T}^j_{y,s}(z_1) - \mathcal{T}^j_{y,s}(z_2).$$

Contents

The mapping

We define mappings $\mathcal{D}^j : N_j \to L^\infty(F_j \times F_j)$, $\mathcal{D}_j(x) = D_x^j(\cdot,\cdot)$ and $\Phi : L^\infty(F_2 \times F_2) \to L^\infty(F_1 \times F_1)$, $\Phi(f) = f(\phi,\phi)$. Recall that mapping $\phi : F_1 \to F_2$ was assumed to be a diffeomorphism that pullbacks the metric.

The aim is to show that mapping

$$\Psi := \mathcal{D}_1^{-1} \circ \Phi \circ \mathcal{D}_2 : N_2 \to N_1$$

is a diffeomorphism s.t.

$$\Psi^* g_1 = g_2.$$

Notice that range of $\mathcal{D}_1$ is the same as $\Phi \circ \mathcal{D}_2$ by distance difference data (1)!

Extension of data

By Distance difference data (1) we only know $\mathcal{D}_j|_{M_j}$. That's why we have to show that we can get the following:

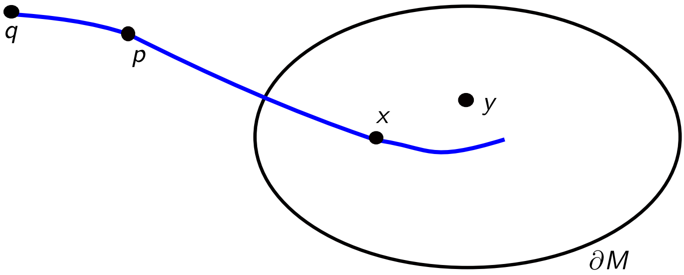
1. The map $\phi : F_1 \to F_2$, is a metric isometry, that is, $d_1(z,w) = d_2(\phi(z), \phi(w))$ for all $z, w \in F_1$.
2. $\{D_x^1(\cdot,\cdot)\ ;\ x \in N_1\} = \{D_y^2(\phi(\cdot),\phi(\cdot))\ ;\ y \in N_2\}$.

Proof: Omitted

Reconstruction of topology

It suffices to show that $\mathcal{D} : N \to L^\infty(F \times F)$ is continuous and 1-to-1. **Proof:** By Δ-inequality it holds that $\mathcal{D}$ is 2-Lipschitz.

Suppose that $x, y \in N$, $x \neq y$ s.t. $D_x = D_y$. Let $q \in F^{int}$, $\ell_x = d(x, q)$ and $\ell_y = d(q, y)$. Let $\gamma_{q,\eta} : [0, \ell_x] \to N$ be a minimizing geodesic from q to x. Let $s > 0$ be s.t. $s < \min(\ell_x, \ell_y, \mathrm{inj}(q))$ and $\gamma_{q,\eta}([0, s]) \subset F^{int}$. Denote $p = \gamma_{q,\eta}(s)$.

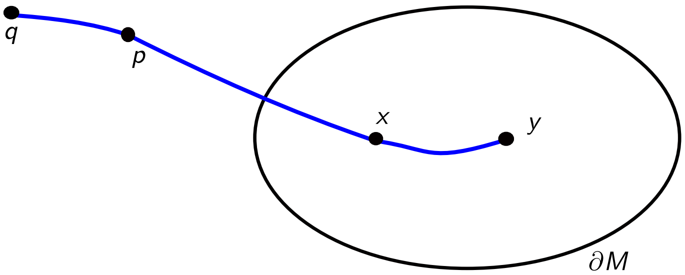


$\gamma_{q,\eta}$ = blue

By $D_x = D_y$,

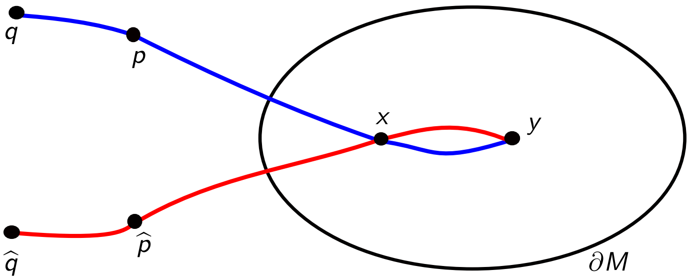
$$d(q,p) + (d(p,y) - d(q,y)) = d(q,p) + (d(p,x) - d(q,x)) = 0.$$

Thus p is on a minimizing geodesic from q to y and therefore $\gamma_{q,\eta}$ is also a minimizing geodesic from q to y. Since we assumed $x \neq y$ we can assume $\ell_x < \ell_y$.



$\gamma_{q,\eta} =$ blue

Since F^{int} is open, we can choose point $\widehat{q} \in F^{int}$ s.t. $\widehat{q} \notin \gamma_{q,\eta}(-\infty,\infty)$. Let β be a minimizing geodesic from $\widehat{q}$ to x.



β = red, $\gamma_{q,\eta}$ = blue

Now $\beta \cup \gamma_{q,\eta} = (\textit{red} \cup \textit{blue})$ would be a distance minimizing curve from $\widehat{q}$ to y. This a contradiction and $x = y$. Thus $\mathcal{D}$ is 1-to-1.

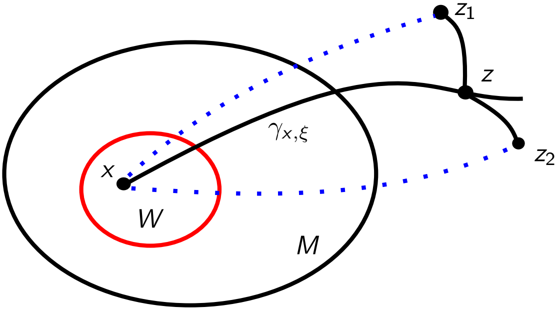
Smooth structure: Suitable atlases

For each $x \in M_j$ there exist a neighborhood W_j of x, point $z \in F_j^{int}$ and $s > 0$ s.t.

$$H_j : W_j \to \mathbb{R}^n,\ H_j(y) = (d_j(y, z_i) - d_j(y, z))_{i=1}^n,\ z_i = \gamma_{z,\eta_i}(s)$$

is a smooth coordinate system.

Proof: Omitted.



Riemannian structure: Geodesically equivalent metrics

Definition

Let N be smooth manifold with two Riemannian metric tensors g_1 and g_2. We say that metrics g_1 and g_2 are geodesically equivalent, if for all geodesics γ_1 of metric g_1 there exists a reparametrization α_1 s.t. $\gamma_1 \circ \alpha_1$ is a geodesic of g_2 and vice versa.

Examples of geodesic equivalence:

$\mathbb{R}^n$ with Euclidean metric e and ce, where $c > 0$ is constant.

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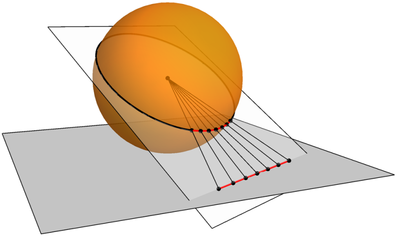

Gnomonic projection F of Sphere:
Wikipedia,
Consider metrics $F_* g_{S^2}$ and
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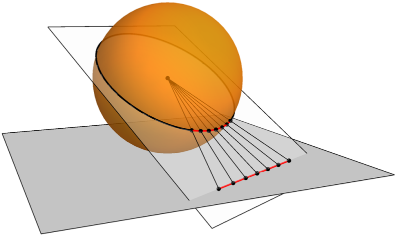

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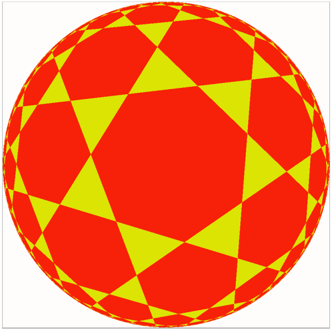
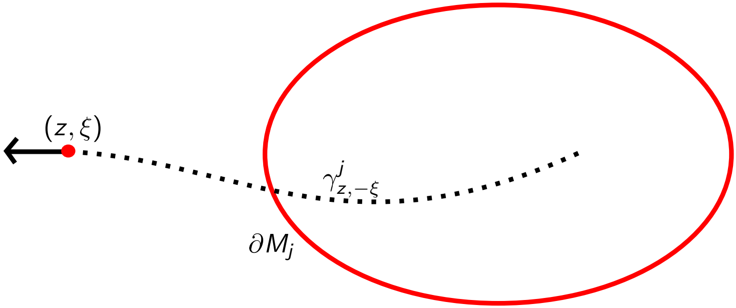


Beltrami-Klein model for disc: Wikipedia, Hyperbolic disc

Our goal is first to show that g_1 and $g_2 := \Psi_* g_2$ are geodisically equivalent on N_1.

Let $z \in F_j$ and $\xi \in S_z N_j$. Define a set

$$\omega_j(z,\xi) := \{x \in N_j \quad ; \quad \exists w \in F_j \text{ such that } D^j_x(\cdot, w) \text{ is } C^1\text{-smooth, near } z \text{ and } \nabla D^j_x(\cdot, w)|_z = \xi\} \cup \{z\}.$$



Then

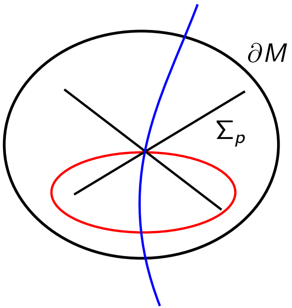
$$\omega_j(z,\xi) = \gamma^j_{z,-\xi}(\{s \; ; \; 0 \le s < \tau_j(z,-\xi)\}).$$

Let $z \in F_2$, $\xi \in S_z N_2$. Then curve $\Psi(\gamma^2_{z,\xi}(\cdot)) : [0, \tau_2(z,\xi)) \to N_1$ is smooth and non self-intersecting. By distance difference data we have

$$\Psi(\gamma^2_{z,\xi}([0, \tau_2(z,\xi))) = \Psi(\omega_2(z,\xi)) = \omega_1(\phi^{-1}(z), (\phi^{-1})^* \xi).$$

Thus we know that "quite many" qeodesics of g_2 are also geodesics of g_1.

Since F_1 contains an open set, it holds that for each $p \in M_1$ there exists an open conic neighbourhood $\Sigma_p \subset T_p N_1$ s.t. for each $\xi \in \Sigma_p$ geodesic $\gamma_{p,\xi}$ is a pre-geodesic of g_2.



There can be a trapped geodesic in M. red curve

Use a modified version of a results of V. Matveev to show that this is enough

Invariants of Geodesic flow

We define a function $I_0 : TN_1 \to \mathbb{R}$ as

$$I_0((x,v)) = \left(\frac{\det(g^1(x))}{\det(g^2(x))}\right)^{\frac{2}{n+1}} g^2_{ij}(x) v^i v^j.$$

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Theorem (Matveev-Topalov 2003)

Let N be a smooth manifold with geodesically equivalent Riemannian metrics g_1 and g_2. Then function I_0 is constant on curves $t \mapsto (\gamma^1(t), \dot\gamma^1(t))$. Where γ^1 is any geodesic of metric g_1.

Metrics g_1 and g_2 coincide on N_1

Let $p \in M$ and $z \in F_1^{int}$ and denote by γ an unit speed geodesic of g_1 from z to p. By distance difference data (1) it holds that

$$1 = l_0(z, \dot{\gamma}) = l_0(p, \dot{\gamma}).$$

and therefore

$$g^1_{ij}(p)\dot{\gamma}^i\dot{\gamma}^j = 1 = \left(\frac{\det(g1(p))}{\det(g^2(p))}\right)^{\frac{2}{n+1}} g^2_{ij}(p)\dot{\gamma}^i\dot{\gamma}^j. \qquad (3)$$

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Since metrics are bilinear, $\exists$ an open conic set $W \subset T_pN$ s.t. $\forall\ v \in W$ equation (3) holds. Thus

$$g^1_{ij}(p) = \left(\frac{\det(g^1(p))}{\det(g^2(p))}\right)^{\frac{2}{n+1}} g^2_{ij}(p), \forall\ i,j = 1,\dots,n.$$

And therefore

$$\left(\frac{\det(g_1(p))}{\det(g_2(p))}\right)^{\frac{2n}{n+1}-1} = 1.$$

Since we assumed $n > 1$, it holds that

$$\frac{\det(g_1(p))}{\det(g_2(p))} = 1.$$

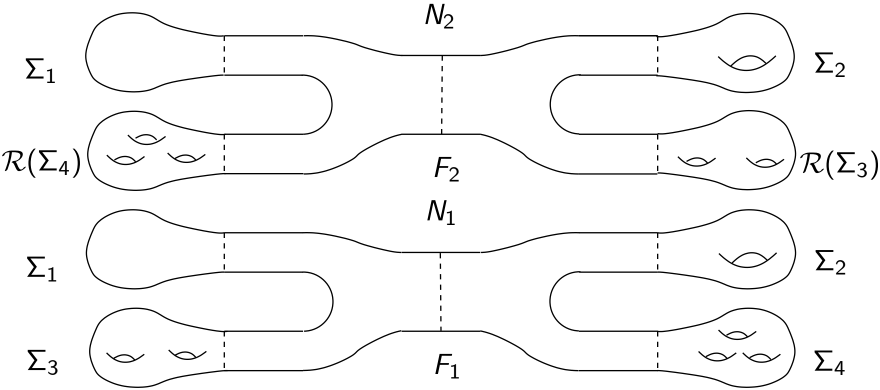
Then we can conclude that

$$g_1(p) = g_2(p).$$

Thus we have proved that $\Psi : N_2 \to N_1$ is a Riemannian isomorphism.

Contents

Why did we assume that $F^{int} \neq \emptyset$



Thank you for your attention!