ONE DIMENSIONAL INVERSE SPECTRAL BOUNDARY PROBLEM

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STUDENTS' SEMINAR

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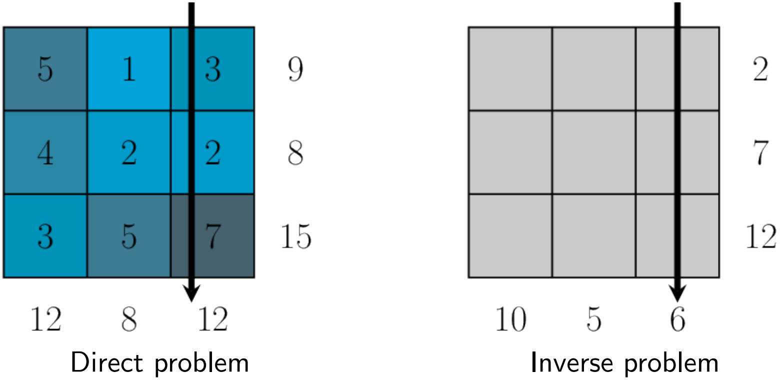
An Inverse problem



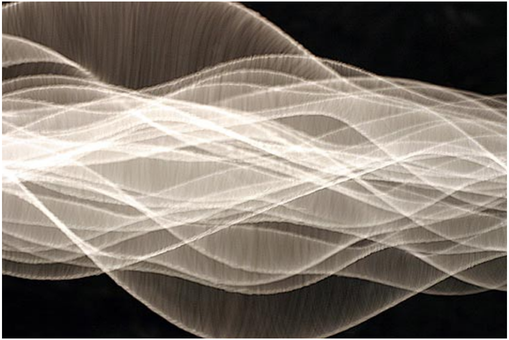
- Does this problem have a solution?
- If there is a solution, is it unique?
- Do we have some prior information about the numbers?

Vibrating string 1

Let $a, b \in \mathbb{R}$, $a \neq 0$. Recall that the 2nd order equation

$$\begin{cases} a^2 \frac{d^2}{dx^2} v(x) = 0, \ x \in (0,1) \\ u(0) = 0, \ \frac{d}{dx} u(0) = b, \end{cases}$$

descripes the the motion of a vibrating string. Here a is related to the material parameters of the string.



Vibrating string 2

Direct Problem: If numbers a, b are given then

$$\begin{cases} a^2 \frac{d^2}{dx^2} v(x) = 0, \ x \in (0,1) \\ u(0) = 0, \ \frac{d}{dx} u(0) = b, \end{cases}$$

has a unique solution.

Inverse Problem: Find a, if some information about the operator $a^2 \frac{d^2}{dx^2}$ is given.

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Definitions and domain

In this talk we will consider 2nd order differential operators that have a general form

$$A = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$$

where $x \in [0,1]$ and $a, b, c \in C^\infty([0,1]),\ a(x) > 0$ and $a(0) = 1$.

We define the domain of A

$$D(A) := H_0^1(0,1) \cap H^2(0,1) \sim \{f \in C^2(0,1) : f(0) = f(1) = 0\}$$

Then $A : D(A) \to L^2(0,1)$.

Spectrum of a differential operator

Recall that a function $\varphi \in D(A)$ is an eigen function of A, if $\varphi \neq 0$ and there exists $\lambda \in \mathbb{R}$ such that

$$A\varphi = \lambda\varphi.$$

Actually one can show that every eigen function of A is smooth.

Theorem

There exists a L^2-orthonormal sequence $(\varphi_j)_{j=1}^{\infty} \subset D(A)$ of eigen functions of differential operator A such that

- $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \ldots \to \infty$
- $\varphi_1(x) \neq 0, x \in (0,1)$
- *For any $f \in L^2(0,1)$, $f(x) = \sum_{j=1}^{\infty}(f|\varphi_j)_2\varphi_j(x)$*

Proof: Take a course PDE 2 (Spring 2017) or Spectral theorem (Fall 2016)

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Spectral boundary data

We say that the spectral boundary data (SBD) of differential operator A is the collection

$$\{(\lambda_j)_{j=1}^{\infty},\ (\dot{\varphi}_j(0))_{j=1}^{\infty}\}.$$

Problem (Inverse spectral boundary problem)

Let

$$A = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x)$$

where $x \in [0,1]$ and $a, b, c \in C^{\infty}([0,1]),\ a(x) > 0,\ a(0) = 1.$
Suppose that the spectral boundary data

$$\{(\lambda_j)_{j=1}^{\infty},\ (\dot{\varphi}_j(0))_{j=1}^{\infty}\}$$

is given. Can you find functions a, b, c?

Gauge transformations

Let $\kappa \in C^\infty([0,1])$ such that

$$\kappa(0) = 1, \text{ and } \kappa(x) > 0, \ x \in [0,1].$$

We define a Gauge transformation A_κ of differential operator A by formula

$$A_\kappa f = \kappa A\left(\frac{f}{\kappa}\right), \ f \in D(A).$$

Let $\varphi \in D(A)$ be an eigen function of A w.r.t. eigen value λ. Then function $\varphi_\kappa := \kappa\varphi$ satisfies

$$A_\kappa \varphi_\kappa = \lambda \varphi_\kappa \text{ and } \frac{d}{dx}\varphi_\kappa(0) = \dot{\kappa}(0)\varphi(0) + \kappa(0)\dot{\varphi}(0) = \dot{\varphi}(0).$$

Therefore any Gauge transform of operator A preserves the SBD.

Changes of Coordinates 1

Let $\ell > 0$, $X : [0, \ell] \to [0, 1]$ be a smooth function such that

$$\dot{X}(y) > 0,\ X(0) = 0,\ \dot{X}(0) = 1 \text{ and } X(\ell) = 1,$$

Any such a function is called a change of coordinates. Recall that in these new coordinates we have

$$\frac{d}{dx} = \left(\frac{dX}{dy}\right)^{-1} \frac{d}{dy}$$

and

$$\frac{d^2}{dx^2} = \left(\frac{dX}{dy}\right)^{-2} \left[\frac{d^2}{dy^2} - \frac{d^2X}{dy^2}\left(\frac{dX}{dy}\right)^{-1} \frac{d}{dy}\right].$$

Changes of Coordinates 2

Thus operator A transforms to operator A^X defined as

$$A^X f(y) := a_X(y)\frac{d^2}{dy^2}f(y) + b_X(y)\frac{d}{dy}f(y) + c_X(y)f(y),$$

where

$$a_X(y) = a(X(y))\dot{X}(y)^{-2}$$
$$b_X(y) = -a(X(y))\dot{X}(y)^{-3}\ddot{X}(y) + \dot{X}(y)^{-1}b(X(y))$$
$$c_X(y) = c(X(y)).$$

Let $\varphi \in D(A)$ be an eigen function of A w.r.t. eigen value λ. Define $\varphi_X := \varphi \circ X$. Then

$$A_X\varphi_X = \lambda\varphi_X \text{ and } \dot{\varphi}_X(0) = \dot{\varphi}(X(0))\dot{X}(0) = \dot{\varphi}(0).$$

Thus a change of coordinates preserves SBD

The invariance of the spectral boundary data

Theorem

Let A and B be two second order differential operators as before. Then SBD of A coincides with SBD of B if and only if there exists a change of coordinates X and a Gauge transform κ such that

$$B = (A^X)_\kappa.$$

Main theorem

We consider 2nd order differential operators with special forms

$$A := -\frac{d^2}{dx^2} + q(x) \text{ and } B := -a(x)^2 \frac{d^2}{dx^2}$$

where $q, a \in C^\infty([0,1])$, $a(x) > 0$, $x \in [0,1]$ and $a(0) = 1$.

Theorem (Inverse spectral boundary problem)

Suppose that the boundary spectral data

$$\{(\lambda_j)_{j=1}^\infty, (\dot{\varphi}_j(0))_{j=1}^\infty\}$$

of operator A (respectively B) is given. Then we can reconstruct the potential q (respectively the wave speed a).

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We will provide a proof for the case

$$A := \frac{d^2}{dx^2} + q(x).$$

All we need to do is to recover the first eigen function φ_1 since then

$$q(x) = \frac{\frac{d^2}{dx^2}\varphi_1(x) + \lambda_1\varphi_1(x)}{\varphi_1(x)}.$$

Recall that we know that $\varphi_1(x) \neq 0$.

Initial/Boundary value problem of Wave equation

To solve the Inverse spectral boundary problem we will employ one dimensional wave equation

$$(*)\begin{cases} \left(\frac{d^2}{dt^2} - \frac{d^2}{dx^2} + q(x)\right)u(t,x) = 0, \ (t,x) \in (0,1)\times(0,1) \\ u(t,0) = f(t), \ u(t,1) = 0 \\ u(0,x) = \frac{\partial}{\partial t}u(0,x) = 0, \end{cases}$$

where $f \in C_0^\infty(0,1)$ is called a boundary source.

Theorem

Let $f \in C_0^\infty(0,1)$. Then there exists a unique $u^f(t,x) \in C^\infty((0,1)\times(0,1))$ that solves $(*)$.

Proof: Take course PDE 1 next fall!

A series representation of waves

Recall that $(\varphi_j)_{j=1}^{\infty} \subset C^{\infty}((0,1))$ is an ON basis of $L^2(0,1)$. Therefore for every boundary source $f \in C_0^{\infty}((0,1))$ we can write

$$u^f(t,x) = \sum_{j=1}^{\infty} u_j^f(t)\varphi_j(x),$$

where the Fourier coefficients are given by

$$u_j^f(t) := (u^f(t,\cdot)|\varphi_j)_{L^2(0,1)} = \int_0^1 u^f(t,x)\varphi_j(x)dx.$$

Theorem (Fourier coefficients of waves)

For any $f \in C^{\infty}(0,1)$ we can find the Fourier coefficients $u_j^f(t)$ from SBD.

Finding the Fourier coefficients from SBD 1

Since u^f is smooth we can differentiate under the integral to get

$$\frac{d^2}{dt^2}u_j^f(t) = \int_0^1 \frac{\partial^2}{\partial t^2}u^f(t,x)\varphi_j(x)dx$$

$$= \int_0^1 \left[\frac{\partial^2}{\partial x^2}u^f(t,x) - q(x)u^f(t,x)\right]\varphi_j(x)dx$$

$$= \int_0^1 u^f(t,x)\underbrace{\left[\frac{\partial^2}{\partial x^2}\varphi_j(x) - q(x)\varphi_j(x)\right]}_{=-\lambda_j\varphi_j(x)} dx$$

$$+\frac{\partial}{\partial x}u^f(1,t)\underbrace{\varphi_j(1)}_{=0} - \underbrace{u^f(1,t)}_{=0}\frac{\partial}{\partial x}\varphi_j(1)$$

$$-\frac{\partial}{\partial x}u^f(0,t)\underbrace{\varphi_j(0)}_{=0} + \underbrace{u^f(0,t)}_{=f(t)}\frac{\partial}{\partial x}\varphi_j(0)$$

Finding the Fourier coefficients from SBD 2

Thus we obtain the following initial value problem:

$$\begin{cases} \frac{d^2}{dt^2} u_j^f(t) = -\lambda_j u_j^f(t) + \dot{\varphi}_j(0) f(t) \\ u_j^f(0) = \frac{d}{dt} u_j^f(0) = 0. \end{cases}$$

Solution: Take courses ODE 1 and ODE 2 (Spring 2017).

Thus we conclude that for all $f, h \in C_0^\infty(0,1)$ we have recovered the Fourier coefficients

$$(u_j^f(t))_{j=1}^\infty, \text{ of the wave } u^f(t,x).$$

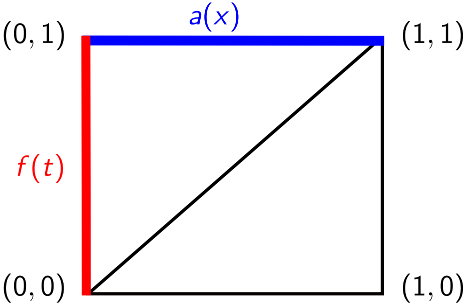
and the inner products

$$(u^f(t,\cdot)|u^h(t,\cdot))_{L^2(0,1)} = \sum_{j=1}^\infty u_j^f(t) u_j^h(t)$$

This is the Parseval identity (Funktionaali analyysin peruskurssi Spring 2017).

Controllability

Next we ask can we control the end state of a wave. I.e.



Theorem (Controllability)

Let $a \in C^\infty(0,1)$. There exists a unique $f \in C^\infty(0,1)$ such that

$$u^f(1,x) = a(x).$$

Projectors

Let $t \in [0,1]$ then we define a projection

$$P_t : L^2(0,1) \to L^2(0,1),\ P_t(f) = \chi_{[0,t]} f.$$

Define a function $M_{jk} : [0,1] \to \mathbb{R}$ by formula

$$M_{jk}(t) = (P_t \varphi_j | \varphi_k)_{L^2(0,1)} = \int_0^t \varphi_j(x)\varphi_k(x)dx.$$

Suppose that function M_{11} is known then

$$\frac{d}{dt} M_{11}(t) = \varphi_1(t)^2 \Rightarrow \text{ eigen function } \varphi_1 \text{ is recovered.}$$

Recovery of matrix valued mapping $t \mapsto M_{jk}(t)$

Let $t_0 \in [0, 1]$.

- Choose any smooth orthogonal basis $(g_k)_{k=1}^\infty$ of $L^2(0, t_0)$. By controllability theorem

$$\mathrm{span}(u^{g_k}(t_0, \cdot))_{k=1}^\infty \subset L^2(0, t_0) \text{ is dense.}$$

- Use Gram-Schmidt to orthonormalise $u^{g_k}(t_0, \cdot)$ to orthonormal basis $(v_k)_{k=1}^\infty$ of $L^2(0, t_0)$.
- Since solution mapping $f \mapsto u^f$ is linear it holds that

$$v_k(x) = u^{f_k}(x, t_0), \; f_k(t) := \sum_{j=1}^{k} d_{jk} g_j(t), \; d_{jk} \in \mathbb{R}.$$

- Since $(v_k)_{k=1}^{\infty}$ of $L^2(0,t_0)$ is ON-basis it holds that

$$P_{t_0}\varphi_j = \sum_{\ell=1}^{\infty} (\varphi_j | v_\ell)_{L^2(0,t_0)} v_\ell$$

Thus

$$M_{jk}(t_0) = (P_{t_0}\varphi_j | \varphi_k)_{L^2(0,1)} = \sum_{\ell=1}^{\infty} (\varphi_j | v_\ell)_{L^2(0,1)} (\varphi_k | v_\ell)_{L^2(0,1)}$$

- Notice that $(\varphi_j | v_\ell)_{L^2(0,1)}$ is a Fourier coefficient of v_ℓ w.r.t basis $(\varphi_j)_{j=1}^{\infty}$ i.e

$$(\varphi_j | v_\ell)_{L^2(0,1)} = u_j^{f_\ell}(t_0).$$

By the Theorem for the Fourier coefficients of waves, we can recover these from SBD.

Thank you for your attention!