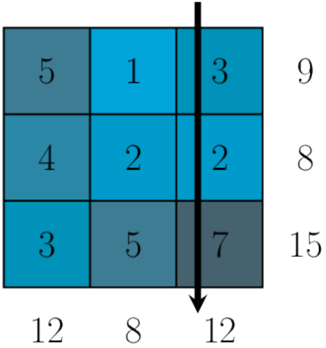
LECTIO PRAECURSORIA: Reconstruction of Riemannian manifold from boundary and interior data

Teemu Saksala

UNIVERSITY OF HELSINKI
FACULTY OF SCIENCE

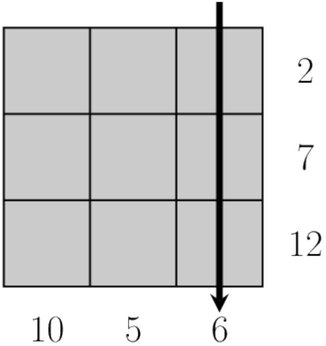
1. What is an inverse problem?
2. What is Riemannian geometry?
3. What is contained in this dissertation?

Direct problem



Sum the integers in each line and column.

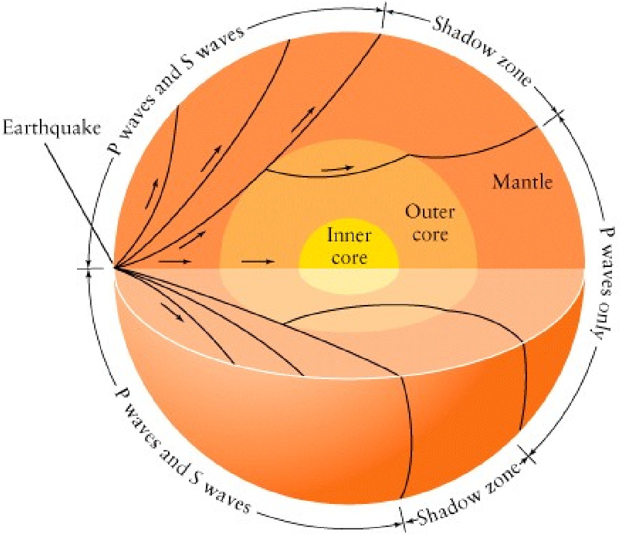
Inverse problem



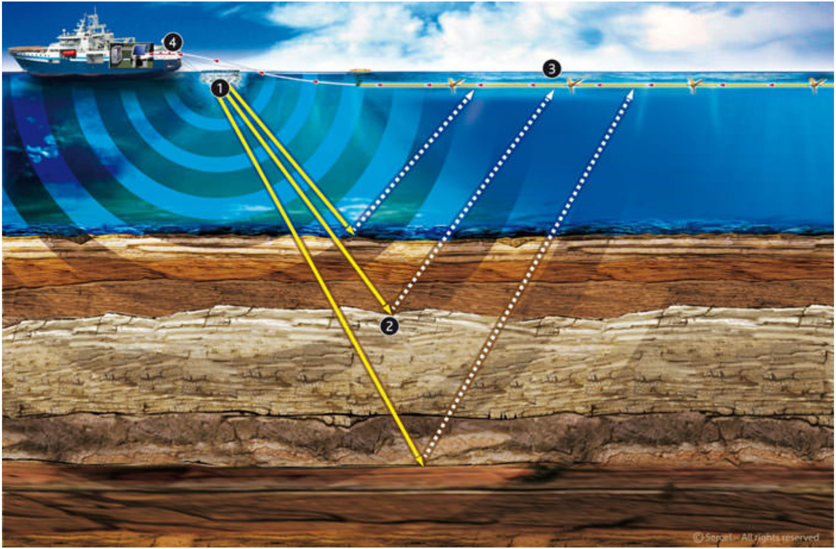
- Do we have some *a priori* information about the numbers? Are they positive integers, integers, rational?
- *A priori* information $\{$numbers are postive integers$\}$ + measurements $\{10, 5, 6, 12, 7, 2\}$ = data
- Is there a solution for given data?
- Is the solution unique for the given data?

X-ray imaging

Earthquakes and seismic waves



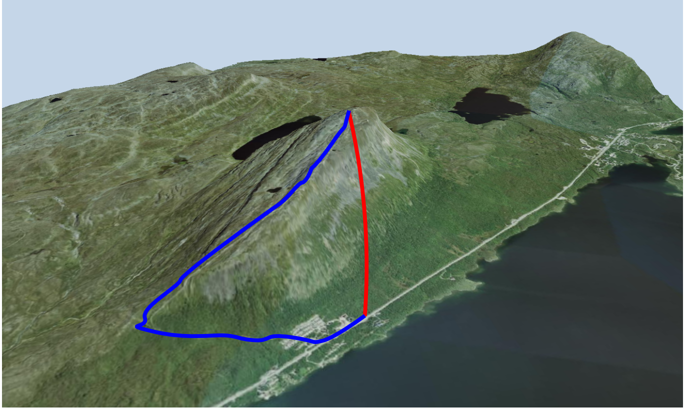
How to see inside the Earth?



2D Riemannian manifold

3D Riemannian manifold

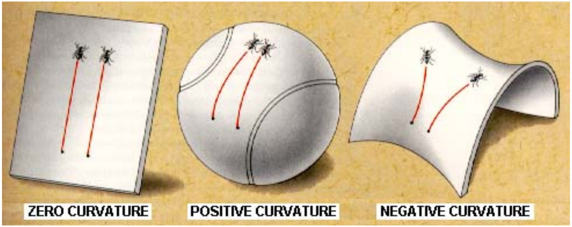
How to measure distances?

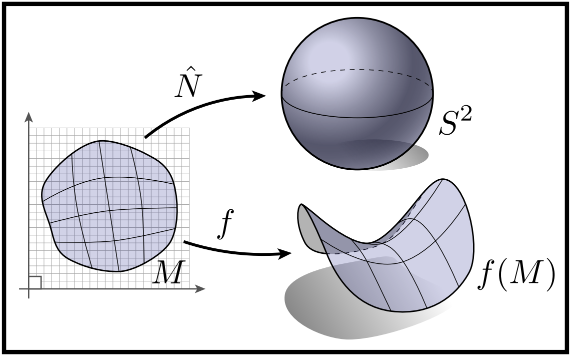


$d(\text{a,b}) =$ shortest travel time from a to b

A path that locally minimizes the distance is called a **geodesic**.

Global vs local

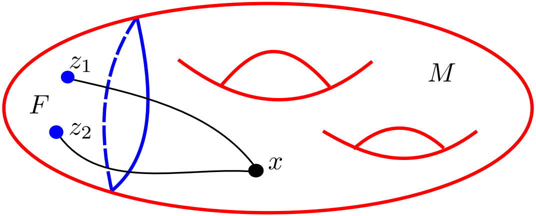




Article I

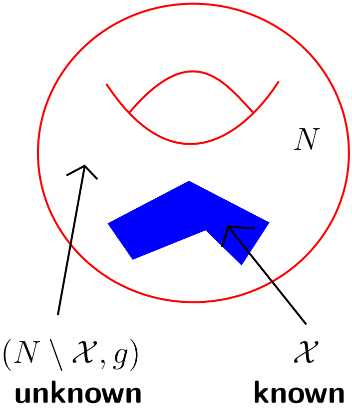
Let (N, g) be a smooth compact Riemannian manifold, $M \subset N$ open. Denote $F := N \setminus M$. For every $x \in M$ we define a distance difference function (DDF)

$$D_x : F \times F \to \mathbb{R},\ D_x(z_1, z_2) = d(x, z_1) - d(x, z_2).$$



If F contains an open set, $(F, g|_F)$ as smooth Riemannian manifold is given and for every $x \in M$ the corresponding DDF D_x is given, then we can recover, topological, smooth and Riemannian structure of (N, g).

Let (N, g) be a smooth and complete Riemannian manifold without a boundary and $\mathcal{X} \subset N$ open.



Model:

$$(\partial_t^2 - \Delta_g) w(t,x) = f, \quad \text{in } (0,\infty) \times N,$$
$$w|_{t=0} = \partial_t w|_{t=0} = 0,$$

where $f \in C_0^\infty((0,\infty) \times \mathcal{X})$.

Denote $\Lambda_{\mathcal{X}} f = w^f|_{(0,\infty)\times\mathcal{X}}$. Then $\Lambda_{\mathcal{X}} : C_0^\infty((0,\infty) \times \mathcal{X}) \to C^\infty((0,\infty) \times \mathcal{X})$ determines (N, g).

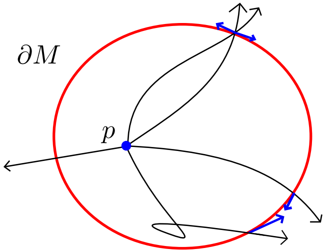
Article III

Let $(\overline{M}, g)$ be a smooth compact Riemannian manifold with a smooth boundary ∂M. We also assume that ∂M is strictly convex and M is non-trapping in the sense that for every $(p, \xi) \in SM$ the first exit time function

$$\tau_{exit}(p, \xi) := \inf\{t > 0 : \gamma_{p,\xi}(t) \in \partial M\} < \infty.$$

For every $p \in M$ we define the scattering set of point source p as

$$R_{\partial M}(p) := \{(\gamma_{p,\xi}(\tau_{exit}(p, \xi)), (\dot{\gamma}_{p,\xi}(\tau_{exit}(p, \xi)))^T) \in T\partial M : \xi \in S_pM\}.$$



Denote $R_{\partial M}(M) = \{R_{\partial M}(p) : p \in M\}$.

If g satisfies a certain generic property, then $\{\partial M, R_{\partial M}(M)\}$ determine $(\overline{M}, g)$.

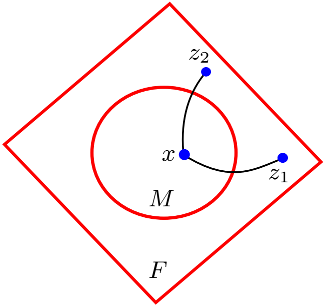
Article IV

Let (N, g) be a smooth complete Riemannian manifold. Let $M \subset N$ be open, relatively compact and let ∂M be smooth. Suppose that there exists a relatively compact open set U, with a smooth boundary ∂U, such that $M \subset \overline{U}$ and $F := U \setminus \overline{M}$ is not empty.

Suppose that $\overline{U}$ is convex in the sense that for all $p, q \in \overline{U}$ any distance minimizing geodesic from p to q is contained in $\overline{U}$.

For every $x \in M$ we define a distance difference function (DDF)

$$D_x : F \times F \to \mathbb{R},\ D_x(z_1, z_2) = d(x, z_1) - d(x, z_2).$$



Denote

$$\mathcal{D}(M) = \{D_x : x \in M\}.$$

Then $\{\mathcal{D}(M), (F, g|_F)\}$ determine $(\overline{U}, g|_{\overline{U}})$.