DETERMINATION OF A RIEMANNIAN MANIFOLD FROM THE DISTANCE DIFFERENCE FUNCTIONS

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Abstract. Let (N, g) be a Riemannian manifold with the distance function $d(x, y)$ and an open subset $M \subset N$. For $x \in M$ we denote by D_x the distance difference function $D_x : F \times F \to \mathbb{R}$, given by $D_x(z_1, z_2) = d(x, z_1) - d(x, z_2)$, $z_1, z_2 \in F = N \setminus M$. We consider the inverse problem of determining the topological and the differentiable structure of the manifold M and the metric $g|_M$ on it when we are given the distance difference data, that is, the set F, the metric $g|_F$, and the collection $\mathcal{D}(M) = \{D_x;\ x \in M\}$. Moreover, we consider the embedded image $\mathcal{D}(M)$ of the manifold M, in the vector space $C(F \times F)$, as a representation of manifold M. The inverse problem of determining (M, g) from $\mathcal{D}(M)$ arises e.g. in the study of the wave equation on $\mathbb{R} \times N$ when we observe in F the waves produced by spontaneous point sources at unknown points $(t, x) \in \mathbb{R} \times M$. Then $D_x(z_1, z_2)$ is the difference of the times when one observes at points z_1 and z_2 the wave produced by a point source at x that goes off at an unknown time. The problem has applications in hybrid inverse problems and in geophysical imaging.

Keywords: Inverse problems, distance functions, embeddings of manifolds, wave equation.

Contents

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1. Introduction

1.1. **Motivation of the problem.** Let us consider a body in which there spontaneously appear point sources that create propagating waves. In various applications one encounters a geometric inverse problem where we detect such waves either outside or at the boundary of the body and aim to determine the unknown wave speed inside the body. As an example of such situation, one can consider the micro-earthquakes that appear very frequently near active faults. The related inverse problem is whether the surface observations of elastic waves produced by the micro-earthquakes can be used in the geophysical imaging of Earth's subsurface [23, 56], that is, to determine the speed of the elastic waves in the studied volume. In this paper we consider a highly idealized version of the above inverse problem: We consider the problem on an n dimensional manifold N with a Riemannian metric g that corresponds to the travel time of a wave between two points. The Riemannian distance of points $x, y \in N$ is denoted by $d(x, y)$. For simplicity we assume that the manifold N is compact and has no boundary. Instead of considering measurements on boundary, we assume that the manifold contains an unknown part $M \subset N$ and the metric is known outside the set M. When a spontaneous point source produces a wave at some unknown point $x \in M$ at some unknown time $t \in \mathbb{R}$, the produced wave is observed at the point $z \in N \setminus M$ at time $T_{x,t}(z) = d(z, x) + t$. These observation times at two points $z_1, z_2 \in N \setminus M$ determine the *distance difference function*

$$D_x(z_1, z_2) = T_{x,t}(z_1) - T_{x,t}(z_2) = d(z_1, x) - d(z_2, x). \tag{1}$$

Physically, this function corresponds to the difference of times at z_1 and z_2 of the waves produced by the point source at (x, t), see Fig 1. and Section 3. An assumption there is a large number point sources and that we do measurements over a long time can be modeled by the assumption that we are given the set $N \setminus M$ and the family of functions

$$\{D_x \ ; \ x \in X\} \subset C((N \setminus M) \times (N \setminus M)),$$

where $X \subset M$ is either the whole manifold M or its dense subset, see Remark 2.5.

1.2. **Definitions and the main result.** Let (N_1, g_1) and (N_2, g_2) be compact and connected Riemannian manifolds without boundary. Let $d_j(x, y)$ denote the Riemannian distance of points $x, y \in N_j$, $j = 1, 2$. Let $M_j \subset N_j$ be open sets and define closed sets $F_j = N_j \setminus M_j$. Suppose

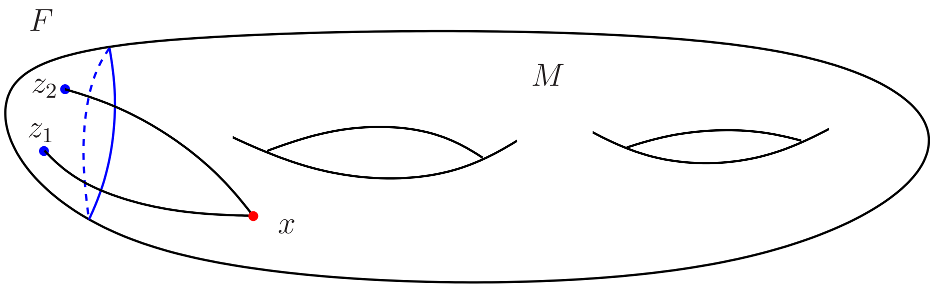

FIGURE 1. *The distance difference functions are related to observation on the closed manifold N that contains an unknown open subset M and its known complement $F = N \setminus M$. The distance difference function D_x associated to a source point $x \in M$ has, at the observation points $z_1, z_2 \in F$, the value $D_x(z_1, z_2) = d(x, z_1) - d(x, z_2)$. Consider the wave equation and a wave that is produced by a point source at x that goes off at an unknown time and that is observed on F. Then the difference of the times when the wave is observed at the points z_1 and z_2 is equal to $D_x(z_1, z_2)$. The time difference inverse problem is determine the topogy and the isometry type of (N, g) from such observations when x runs over a dense subset of M.*

$F_j^{int} \neq \emptyset$. This is a crucial assumption and we provide a counterexample for a case $F_j^{int} = \emptyset$ in the Appendix 4.

Below, we assume that we know F_j as a differentiable manifold, that is, we know the atlas of C^∞-smooth coordinates on F_j, and the metric tensor $g_j|_{F_j}$ on F_j, but we do not know the manifold $(M_j, g_j|_{M_j})$. We assume F_j to be a smooth manifold with smooth boundary $\partial F_j = \partial M_j$.

Definition 1.1. *For $j = 1, 2$ and all points $x \in N_j$ we define the distance difference function*

$$D_x^j : F_j \times F_j \to \mathbb{R},\ D_x^j(z_1, z_2) := d_j(z_1, x) - d_j(z_2, x)$$

where $F_j = N_j \setminus M_j$. Recall that here d_j is the Riemannian distance function of manifold N_j. We denote by

$$\mathcal{D}^j : N_j \to C(F_j \times F_j), \quad \mathcal{D}^j(x) = D_x^j$$

the map from a point x to the corresponding distance difference function D_x^j. The pair $(F_j, g_j|_{F_j})$ and the collection

$$\mathcal{D}^j(M_j) = \{D_x^j\,;\ x \in M_j\} \subset C(F_j \times F_j)$$

of the distance difference functions of the points $x \in M_j$ is called the distance difference data for the set M_j.

We emphasize that the above collections $\{D_x^j(\cdot,\cdot);\ x \in M_j\}$ are given as unindexed subsets of $C(F_j \times F_j)$, that is, for a given element $D_x^j(\cdot,\cdot)$ of this set we do not know what is the corresponding "index point" x.

To prove the uniqueness of this inverse problem, we assume the following:

(2) There is a diffeomorphism $\phi : F_1 \to F_2$ such that $\phi^* g_2|_{F_2} = g_1|_{F_1}$,
(3) $\{D^1_x(\cdot,\cdot)\ ;\ x \in M_1\} = \{D^2_y(\phi(\cdot),\phi(\cdot))\ ;\ y \in M_2\}$.

The following proposition states that using the small data $\mathcal{D}_j(M_j)$ we can construct the bigger data set $\mathcal{D}_j(N_j)$.

Proposition 1.2. *Assume that* (2)-(3) *are valid. Then:*

(i) *The map* $\phi : F_1 \to F_2$, *is an isometry, that is,* $d_1(z,w) = d_2(\phi(z),\phi(w))$ *for all* $z, w \in F_1$.
(ii) *The collections* $\mathcal{D}_j(N_j) = \{D^j_x(\cdot,\cdot);\ x \in N_j\} \subset C(F_j \times F_j)$ *are equivalent in the following sense*

$$\{D^1_x(\cdot,\cdot)\ ;\ x \in N_1\} = \{D^2_y(\phi(\cdot),\phi(\cdot))\ ;\ y \in N_2\}. \tag{4}$$

We postpone the proof of this proposition and the other results in the introduction and give the proofs later in the paper.

The main theorem of the paper is the following:

Theorem 1.3. *Let* (N_1, g_1) *and* (N_2, g_2) *be compact and connected Riemannian manifolds, without boundary, of dimension* $n \geq 2$. *Let* $M_j \subset N_j$ *be open sets and define closed sets* $F_j = N_j \setminus M_j$. *Suppose* $F_j^{int} \neq \emptyset$. *We assume* F_j *to be a smooth manifold with smooth boundary* $\partial F_j = \partial M_j$. *Suppose* (2)-(3) *are valid. Then the manifolds* (N_1, g_1) *and* (N_2, g_2) *are isometric.*

We prove Theorem 1.3 in Section 2. This proof is divided into 5 subsections. In the first we set notations and consider some basic facts about geodesics. In the second we prove Proposition 1.2. In the third we show that manifolds (N_j, g_j) are homeomorphic. In the fourth subsection we will construct smooth atlases with which we show that manifolds (N_j, g_j) are diffeomorphic. In fifth subsection we will use techniques developed in papers [44] and [41] to prove that manifolds (N_j, g_j) are isometric.

Finally, in Section 3 we give an example how the main result can be applied for an inverse source problem for a geometric wave equation.

1.3. **Embeddings of a Riemannian manifold.**

A classical distance function representation of a Riemannian manifold is the Kuratowski-Wojdyslawski embedding,

$$\mathcal{K} : x \mapsto \mathrm{dist}_M(x,\cdot),$$

from M to the space of continuous functions $C(M)$ on it. The mapping $\mathcal{K} : M \to C(M)$ is an isometry so that $\mathcal{K}(M)$ is an isometric representation of M in a vector space.

An other important example is the Berard-Besson-Gallot representation [10]

$$\mathcal{G} : M \to C(M \times \mathbb{R}_+), \quad \mathcal{G}(x) = \Phi_M(x, \cdot\,, \cdot\,)$$

where $(x, y, t) \mapsto \Phi_M(x, y, t)$ is the heat kernel of the manifold (M, g). The asymptotics of the heat kernel $\Phi_M(x, y, t)$, as $t \to 0$, determines the distance $d(x, y)$, and by endowing $C(M \times \mathbb{R}_+)$ with a suitable topology, the image $\mathcal{G}(M) \subset C(M \times \mathbb{R}_+)$ can be considered as an embedded image of the manifold M.

Theorem 1.3 implies that the set $\mathcal{D}(M) = \{D_x;\ x \in M\}$ can be considered as an embedded image (or a representation) of the manifold (M, g) in the space $C(F \times F)$ in the embedding $x \mapsto D_x$. Moreover, in the proof of Theorem 1.3 we show that $(F, g|_F)$ and the set $\mathcal{D}(M)$ determine uniquely an atlas of differentiable coordinates and a metric tensor on $\mathcal{D}(M)$. These structures make $\mathcal{D}(M)$ a Riemannian manifold that is isometric to the original manifold M. Note that the metric is different than the one inherited from the inclusion $\mathcal{D}(M) \subset C(F \times F)$. Hence, $\mathcal{D}(M)$ can be considered as a representation of the manifold M, given in terms of the distance difference functions, and we call it the *distance difference representation* of the manifold of M in $C(F \times F)$.

The embedding $\mathcal{D}$ is different to the above embeddings $\mathcal{K}$ and $\mathcal{G}$ in the following way that makes it important for inverse problems: With $\mathcal{D}$ one does not need to know a prori the set M to consider the function space $C(F \times F)$ into which the manifold M is embedded. Similar type of embedding have been also considered in context of the boundary distance functions, see Subsection 1.4.1.

In addition to the above tensor g on N, let us consider a sequence of metric tensors g_k, $k \in \mathbb{Z}_+$ on the manifold N and assume that $g_k|_F = g|_F$ on $F \subset N$. We denote the Riemannian manifolds $(N \setminus F, g_k|_{N \setminus F})$, having the boundary ∂F, by (M_k, g_k). Also, we denote by $\mathcal{D}(M_k) \subset C(F \times F)$ the distance difference representations of the manifolds (M_k, g_k) and let $d_H(X_1, X_2)$ denote the Hausdorff distance of sets $X_1, X_2 \subset C(F \times F)$. When $d_H(\mathcal{D}(M_k), \mathcal{D}(M)) \to 0$, as $k \to \infty$, an interesting open question is, if the manifolds (M_k, g_k) converge to (M, g) in the Gromov-Hausdorff topology. This type of questions have been studied for other representation e.g. in [2, 10], but this question is outside the context of this paper.

1.4. **Earlier results and the related inverse problems.** The inverse problem for the distance difference function is closely related to many other inverse problems. We review some results below:

1.4.1. *Boundary distance functions and the inverse problem for a wave equation.* The reconstruction of a compact Riemannian manifold (M, g) with boundary from distance information has been considered e.g. in [25, 28]. There, one defines for $x \in M$ the boundary distance function

$r_x : \partial M \to \mathbb{R}$ given by $r_x(z) = d(x,z)$. Assume that one is given the boundary ∂M and the collection of boundary distance functions corresponding to all $x \in M$ that is,

$$\partial M \quad \text{and} \quad \mathcal{R}(M) := \{r_x \in C(\partial M);\ x \in M\}. \tag{5}$$

It is shown in [25, 28] that only knowing the boundary distance data (5) one can reconstruct the topology of M, the differentiable structure of M (i.e., an atlas of C^∞-smooth coordinates), and the Riemannian metric tensor g. Thus $\mathcal{R}(M) \subset C(\partial M)$ can be considered as an isometric copy of M, and the pair $(\partial M, \mathcal{R}(M))$ is called the boundary distance representation of M, see [25, 28]. Similar results for non-compact manifolds is considered in [16]. Constructive solutions to determine the metric from the boundary distance functions have been developed in [14] using a Riccati equation [54] for metric tensor in boundary normal coordinates and in [53] using the properties of the conformal killing tensor.

The results of this paper is closely related to data (5): Knowing the distance difference functions $D_x^{\partial M} : \partial M \times \partial M \to \mathbb{R}$

$$D_x^{\partial M}(z_1, z_2) = d(x, z_1) - d(x, z_2), \quad (z_1, z_2) \in \partial M \times \partial M$$

is equivalent to knowing the boundary distance functions with error $\varepsilon(x)$, depending on $x \in M$, that is, the functions $z \mapsto r_x(z) - \varepsilon(x)$. Indeed, $r_x(z) - \varepsilon(x) = D_x^{\partial M}(z, z_2)$ when $\varepsilon(x) = d(x, z_2)$.

Physically speaking, functions r_x are determined by the wave fronts of waves produced by the delta-sources $\delta_{x,0}$ that take place at the point x at time $s = 0$. The distance difference functions $D_x^{\partial M}$ are determined by the wave fronts of waves produced by the delta-sources $\delta_{x,s}$ that take place at the point x at an unknown time $s \in \mathbb{R}$.

Many hyperbolic inverse problems with time-independent metric reduce to the problem of reconstructing the isometry type of the manifold from its boundary distance functions. Indeed, in [25, 24, 27, 29, 30, 33, 49, 50] it has been show that the boundary measurements for the scalar wave equation, Dirac equation, and for Maxwell's system (with isotropic scalar impedance) determine the boundary distance functions of the Riemannian metric associated to the wave velocity.

1.4.2. *Hybrid inverse problems.* Hybrid inverse problems are based on coupling two physical models together. In a typical setting of these problems, the first physical system is such that by controlling the boundary values of its solution, one can produce high amplitude waves, that create, e.g. due to energy absorption, a source for the second physical system. Typically, the second physical system corresponds to a hyperbolic equation with the metric

$$ds^2 = c(x)^{-2}((dx^1)^2 + \cdots + (dx^n)^2)$$

corresponding to the wave speed $c(x)$. Examples of such hybrid inverse problems are encountered in thermo-acoustic and photo-acoustic imaging see e.g. [1, 5, 6, 7, 8, 57, 59, 58, 55] and quantitative elastography [4, 20, 21]. In some cases one can use beam forming in the first physical system to make the source for the second physical system to be strongly localized, that is, to be close to a point-source, see e.g. [4, 21].

To simplify the above hybrid inverse problem, one often do approximations by assuming that the wave speed in the second physical system is either a constant or precisely known. Usually one also assumes that the time moment when the source for the second physical system is produced is exactly known. However, when these approximations are not made, the wave speed $c(x)$ needs to be determined, too. When the source of the second physical system is produced at the given time in the whole domain M, the problem is studied in [40, 60]. In the cases when the source of the second physical system are close to a point sources, one can try to determine $c(x)$ from the wavefronts that are produced by the point sources and are observed outside the domain M. This problem can be uniquely solved by Theorem 1.3 and we consider it in detail below in Section 3.

1.4.3. *Inverse problems of micro-earthquakes.* The earthquakes are produced by the accumulated elastic strain that at some time suddenly produce an earthquake. As mentioned above, the small magnitude earthquakes (e.g. the micro-earthquakes of magnitude $1 < M < 3$) appear so frequently that the surface observations of the produced elastic waves have been proposed to be used in the imaging of the Earth near active faults [23, 56]. The so-called time-reversal techniques to study the inverse source and medium problems arising from the micro-seismology have been developed in [3, 15, 22].

In geophysical studies, one often approximates the elastic waves with scalar waves satisfying a wave equation. Let us also assume that the sources of such earthquakes are point-like and that one does measurements over so long time that the source-points are sufficiently dense in the studied volume. Then the inverse problem of determining the the speed of the waves in the studied volume from the surface observations of the microearthquakes is close to the problem studied in this paper. We note that the above assumptions are highly idealized: For example, considering the system of elastic equations would lead to a problem where travel times are determined by a Finsler metric instead of a Riemannian one.

1.4.4. *Broken scattering relation.* If the sign in the definition of the distance difference functions is changed in (1), we come to distance sum functions

$$(6)\quad D_x^+(z_1, z_2) = d(z_1, x) + d(z_2, x), \quad x \in M,\ z_1, z_2 \in N \setminus M.$$

This function gives the length of the broken geodesic that is the union of the shortest geodesics connecting z_1 to x and the shortest geodesics connecting x to z_2. Also, the gradients of $D^+_x(z_1, z_2)$ with respect to z_1 and z_2 give the velocity vectors of these geodesics. The functions (6) appear in the study of the radiative transfer equation on manifold (N, g), see [13, 45, 46, 47, 52]. Also, the inverse problem of determining the manifold (M, g) from the broken geodesic data, consisting of the initial and the final points and directions, and the total length, of the broken geodesics, has been considered in [31].

2. Proof of the main result

2.0.1. *Notations and basic facts on pre geodesics.* When we are concerning only one manifold, we use the shorthand notations M, N, F, and g instead of ones with sub-indexes.

Let (N, g) be a compact and connected Riemannian n-manifold without boundary and $n \geq 2$. We assume that $M \subset N$ is an open set of N and set $F = N \setminus M$ is a compact manifold with smooth boundary. Suppose that set F contains an open set and the we know the Riemannian structure of manifold $(F, g|_F)$.

We denote the Riemannian connection of the metric g as ∇. An unit speed geodesic of (N, g) emanating from a point $(p, \xi) \in SN$ is denoted by $\gamma_{p,\xi}(t) = \exp_p(t\xi)$. Here, $SN = \{(p, \xi) \in TN;\ \|\xi\|_g = 1\}$. We use a short hand notation $D_t := \nabla_{\dot\gamma_{p,\xi}(t)}$ for the covariant differentiation in the direction $\dot\gamma_{p,\xi}$ for vector fields along geodesic $\gamma_{p,\xi}$.

Let $p \in N$ and choose some smooth coordinates (U, X) at point p. Denote the Cristoffel symbols of connection ∇ by $\Gamma^k_{i,j}$.

We say that a curve $\alpha([t_1, t_2])$ is distance minimizing if the length of this curve is equal to the distance of its end points $\alpha(t_1)$ and $\alpha(t_1)$. Also, a geodesic that is distance minimizing is called a minimizing geodesic.

We say that a curve $\alpha([t_1, t_2])$ is a *pre-geodesic*, if $\alpha(t)$ is a C^1-smooth curve such that $\dot\alpha(t) \neq 0$ on $t \in [t_1, t_2]$, and $\alpha([t_1, t_2])$ can be re-parametrized so that it becomes a geodesic.

Let us next recall some properties of the pre-geodesics. Let us consider a geodesic curve $\gamma : \mathbb{R} \to N$, satisfying in local coordinates the equation

$$D_t\dot\gamma(t) = \frac{d^2\gamma^k}{dt^2}(t) + \Gamma^k_{i,j}(\gamma(t))\frac{d\gamma^i}{dt}(t)\frac{d\gamma^j}{dt} = 0,\ k \in \{1, \dots, n\}. \tag{7}$$

We need the following result, often credited to Levi-Civita [36] :

Lemma 2.1. *Let $\kappa : \mathbb{R} \to \mathbb{R}$ be continuous and $\widetilde\gamma : \mathbb{R} \to N$ be a C^2-curve that satisfies a local equation*

$$\frac{d^2\widetilde\gamma^k}{ds^2}(s) + \Gamma^k_{i,j}(\widetilde\gamma(s))\frac{d\widetilde\gamma^i}{ds}(s)\frac{d\widetilde\gamma^j}{ds}(s) = \kappa(s)\frac{d\widetilde\gamma^k}{ds}(s),\ k \in \{1, \dots, n\}. \tag{8}$$

Then there exists a change of parameters $t : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\frac{dt}{ds}(s) = \exp\Big(\int_0^s \kappa(\tau)d\tau\Big). \tag{9}$$

such that curve $\gamma(s) := \widetilde{\gamma}(t(s))$ *solves the geodesic equation (7).*

Proof. The proof is a direct computation. □

Let us now consider a family of curves. Let $f : TU \to \mathbb{R}$ be a continuous function that satisfies

$$f(av) = af(v), \text{ for all } a \in \mathbb{R} \text{ and } v \in TU, \tag{10}$$

i.e., it is homogeneous of degree 1. Let Γ be a family of all such C^2-curves $\widetilde{\gamma} : \mathbb{R} \to N$ in U that satisfy the equation

$$\frac{d^2\widetilde{\gamma}^k}{ds^2}(s) + \Gamma^k_{i,j}(\widetilde{\gamma}(s))\frac{d\widetilde{\gamma}^i}{ds}(s)\frac{d\widetilde{\gamma}^j}{ds}(s) = f\Big(\frac{d\widetilde{\gamma}}{ds}(s)\Big)\frac{d\widetilde{\gamma}^k}{ds}(s). \tag{11}$$

By Lemma 2.1 each $\widetilde{\gamma} \in \Gamma$ is a pre-geodesic of connection ∇. Thus equations (7) and (11) are equivalent in the sense that curves satisfying the latter one, for appropriate f, are also geodesics of metric g, but parametrized in a different way.

The distance function of N is denoted by $d(x, y) = d_N(x, y)$ for $x, y \in N$. Denote by ν the interior normal vector field of ∂M. The boundary cut locus function is $\tau_{\partial M} : \partial M \to \mathbb{R}_+$,

$$\tau_{\partial M}(z) = \sup\{t > 0;\ d(\gamma_{z,\nu}(t), \partial M) = t\}. \tag{12}$$

Also, we use the cut locus function of N that is $\tau : TN \to \mathbb{R}_+$

$$\tau(x, \xi) = \sup\{t > 0;\ d(\exp_x(t\xi), x) = t\}. \tag{13}$$

Functions $\tau_{\partial M}(z)$ and $\tau(x, \xi)$ are continuous and satisfy the inequality (see Lemma 2.13 of [25])

$$\tau(z, \nu(z)) > \tau_{\partial M}(z), \quad z \in \partial M. \tag{14}$$

2.1. **Extension of data.** In this subsection we prove Proposition 1.2.

Let $z_1, z_2 \in \partial F = \partial M$. Then using the triangular inequality and that $d(z_1, z_2) = D_{z_2}(z_1, z_2)$ we see easily that

$$d(z_1, z_2) = \sup_{x \in M} D_x(z_1, z_2). \tag{15}$$

Thus $\mathcal{D}(M)$ determines the distances of the boundary points, that is, the function $d|_{\partial M \times \partial M} : \partial M \times \partial M \to \mathbb{R}$.

Lemma 2.2. *Suppose that* (2)-(3) *are valid. Then it holds that* $d_1(w, z) = d_2(\phi(w), \phi(z))$.

Proof. Let $w, z \in F_1$. Let γ be a minimizing unit speed geodesic in N_1 from z to w and denote $S = \gamma([0, d_1(w, z)]) \cap \partial M_1$. When $S = \emptyset$, the facts that $\phi^* g_2 = g_1$ and that the path $\phi(\gamma)$ connects $\phi(z)$ to $\phi(w)$ imply that $d_1(w, z) \geq d_2(\phi(w), \phi(z))$.

Suppose that $S \neq \emptyset$. Let $e_1, e_2 \in S$ be such that

$$d_1(w, e_1) = \min\{d_1(w, x) : x \in S\} \text{ and } d_1(z, e_2) = \min\{d_1(z, x) : x \in S\}.$$

As (2)-(3) is valid, the formula (15) implies that

$$d_1(e_1, e_2) = d_2(\phi(e_1), \phi(e_2)).$$

Since $\phi : F_1 \to F_2$ satisfies $\phi^* g_2 = g_1$, it holds that

$$\begin{aligned}
&d_1(w, z) = d_1(w, e_1) + d_1(e_1, e_2) + d_2(e_2, z) \\
&\geq d_2(\phi(w), \phi(e_1)) + d_2(\phi(e_1), \phi(e_2)) + d_2(\phi(e_2), \phi(z)) \\
&\geq d_2(\phi(w), \phi(z))
\end{aligned}$$

The claim follows by changing the roles of N_1 and N_2. □

Let us consider the case when $x \in F_1$. Then, Lemma 2.2 implies that for $z_1, z_2 \in F_1$ we have

$$\begin{aligned}
D_x^1(z_1, z_2) &= d_1(x, z_1) - d_1(x, z_2) \\
&= d_2(\phi(x), \phi(z_1)) - d_2(\phi(x), \phi(z_2)) \\
&= D^2_{\phi(x)}(\phi(x), \phi(z_2)).
\end{aligned}$$

Hence,

$$\{D_x^1(\cdot, \cdot)\ ;\ x \in F_1\} \subset \{D_y^2(\phi(\cdot), \phi(\cdot))\ ;\ y \in F_2\}. \tag{16}$$

Changing roles of N_1 and N_2 and considering $\phi^{-1} : F_2 \to F_1$ instead of the diffeomorphism $\phi : F_1 \to F_2$, we see that in formula (16) we have the equality. This and formula (3), together with Lemma 2.2, imply Proposition 1.2. □

2.2. **Manifolds N_1 and N_2 are homeomorphic.** To simplify the notations, we will next in our considerations omit the sub-indexes of sets M_1, N_1, and F_1 and just consider the sets M, N, and F.

Let $x \in N$ and define a function $D_x : F \times F \to \mathbb{R}$ by a formula

$$D_x(z_1, z_2) = d(x, z_1) - d(x, z_2).$$

Let $\mathcal{D} : N \to C(F \times F)$ be defined as $\mathcal{D}(x) = D_x$. We give the function space $C(F \times F)$ the Banach space structure with the sup-norm.

Theorem 2.3. *Image $\mathcal{D}(N) \subset C(F \times F)$ is a topological manifold homeomorphic to manifold N and especially $\mathcal{D}(M)$ is homeomorphic to M.*

Proof. The proof consists of four short steps.

Step 1 First, we want to show that $\mathcal{D}$ is continuous. Let $x, y \in N$. Using the triangular inequality we see that

$$\begin{aligned} \|D_x - D_y\|_{L^\infty(F\times F)} &= \sup_{z_1,z_2\in F} |D_x(z_1,z_2) - D_y(z_1,z_2)| \\ &\le \sup_{z_1,z_2\in F} |d(x,z_1) - d(y,z_1)| + |d(x,z_2) - d(y,z_2)| \qquad (17) \\ &\le 2d(x,y). \end{aligned}$$

Thus $\mathcal{D}$ is 2-Lipschitz and therefore continuous. Next we consider injectivity of $\mathcal{D}$.

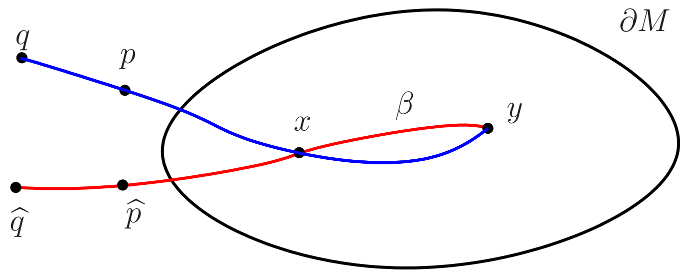

FIGURE 2. *The setting in Step 2 in the proof of Theorem 2.3. We consider points $x, y \in N$ and points p and q such that p is on a distance minimizing geodesic from q to x. Then this geodesic can be extended to a distance minimizing geodesic from q to y. Similarly, the point $\widehat{p}$ is on a distance minimizing geodesic from $\widehat{q}$ to x and this geodesic can be extended to a distance minimizing geodesic from $\widehat{q}$ to y. Then the union of the (blue) geodesic from q to x and the (red) geodesic β is a lenght minimizing curve from q to y that is not a geodesic.*

Step 2. Suppose that $x, y \in N$ are such that $D_x = D_y$ and $x \neq y$. Let $q \in F^{int}$ and denote $\ell_x = d(q,x)$ and $\ell_y = d(q,y)$. Next, without loss of generality, we assume that $\ell_x \le \ell_y$. Also, let $\eta \in S_qN$ be such that $\gamma_{q,\eta}([0,\ell_x])$ is a minimizing geodesic from q to x. Let $s_1 > 0$ be such that $s_1 < \min(\ell_x, \ell_y)$ and $\gamma_{q,\eta}([0,s_1]) \subset F^{int}$. Consider a point $p = \gamma_{q,\eta}(s)$ with $s \in [0,s_1]$. Then we see that

$$\begin{aligned} (d(q,p) + d(p,y)) - d(q,y) &= d(q,p) + D_y(p,q) \\ &= d(q,p) + D_x(p,q) \\ &= (d(q,p) + d(p,x)) - d(q,x) = 0 \end{aligned}$$

and hence p is on a minimizing geodesic from q to y.

Let us consider a minimizing geodesic α from p to y with the length $\ell_y - s$. Then the union of the geodesics $\gamma_{q,\eta}([0,s])$ and α is a distance minimizing curve from q to y and thus this union is a geodesic. This implies that α is a continuation of the geodesics $\gamma_{q,\eta}([0,s])$ and hence $y = \gamma_{q,\eta}(\ell_y)$. Summarizing, $\gamma_{q,\eta}([0,\ell_x])$ and $\gamma_{q,\eta}([0,\ell_y])$ are distance minimizing geodesics from q to x and y, respectively. Since $x \neq y$, we have $\ell_x \neq \ell_y$. Then, as we have assumed that $\ell_x \leq \ell_y$, we see that $\ell_x < \ell_y$.

Let $\widehat{q} \in F^{int}$ be a point such that $\widehat{q}$ is not on the curve $\gamma_{q,\eta}(\mathbb{R})$. Clearly, such a point exists due to measure theoretic arguments. Let $\widehat{\ell}_x = d(\widehat{q},x)$ and $\widehat{\ell}_y = d(\widehat{q},y)$. Also, let $\widehat{\eta} \in S_{\widehat{q}}N$ be such that $\gamma_{\widehat{q},\widehat{\eta}}([0,\widehat{\ell}_x])$ is minimizing geodesic from $\widehat{q}$ to x. As above, we see that then $\gamma_{\widehat{q},\widehat{\eta}}([0,\widehat{\ell}_x])$ and $\gamma_{\widehat{q},\widehat{\eta}}([0,\widehat{\ell}_y])$ are distance minimizing geodesics from $\widehat{q}$ to x and y, respectively. However, the geodesics $\gamma_{q,\eta}(\mathbb{R})$ and $\gamma_{\widehat{q},\widehat{\eta}}(\mathbb{R})$ do not coincide as point sets and hence the vectors $\dot{\gamma}_{q,\eta}(\ell_x) \in T_xN$ and $\dot{\gamma}_{\widehat{q},\widehat{\eta}}(\widehat{\ell}_x) \in T_xN$, are not parallel. Recall that $\ell_x < \ell_y$. In the case when $\widehat{\ell}_x < \widehat{\ell}_y$, let β be the geodesic segment $\gamma_{\widehat{q},\widehat{\eta}}([\widehat{\ell}_x,\widehat{\ell}_y])$ conneting x to y. In the case when $\widehat{\ell}_x > \widehat{\ell}_y$, let β be the geodesic segment $\gamma_{\widehat{q},\widehat{\eta}}([\widehat{\ell}_y,\widehat{\ell}_x])$ connecting x to y.

Then we see that the union of the paths $\gamma_{q,\eta}([0,\ell_x])$ and β is a distance minimizing path from q to y. As the vectors $\dot{\gamma}_{q,\eta}(\ell_x)$ and $\dot{\gamma}_{\widehat{q},\widehat{\eta}}(\widehat{\ell}_x)$ are not parallel, we see that the union of these curves is not a geodesic. This is contradiction and hence there are no $x, y \in N$ such that $D_x = D_y$ and $x \neq y$. Thus, $\mathcal{D} : N \to C(F \times F)$ is an injection.

Step 3. So far we have proved the continuity and injectivity of mapping $\mathcal{D}$. Since the domain N of the mapping $\mathcal{D}$ is compact and $(C(F \times F), \|\cdot\|_\infty)$ is a Hausdorff space as a metric space, it holds by basic results of topology that mapping $\mathcal{D} : N \to \mathcal{D}(N)$ is a homeomorphism.

Step 4. By assumption $M \subset N$ is open and therefore mapping $\mathcal{D} : M \to \mathcal{D}(M)$ is open. This proves that the mapping $\mathcal{D} : M \to \mathcal{D}(M)$ is a homeomorphism. □

Define a mapping

$$\Phi : C(F_2 \times F_2) \to C(F_1 \times F_1),\ \Phi(f) = f \circ (\phi \times \phi). \tag{18}$$

Here $f \times g : X \times X \to Y \times Y$ is defined as $(f \times g)(x_1,x_2) = (f(x_1), g(x_2)) \in Y \times Y$ for mappings $f, g : X \to Y$. Sometimes, to simplify the notations, we denote $\mathcal{D}^j = \mathcal{D}_j$, $j = 1, 2$, see Def. 1.1.

Theorem 2.4. *Suppose that Riemannian manifolds* (N_1, g_1) *and* (N_2, g_2) *are as in section 1.2 and the assumptions of the Proposition 1.2 are valid. Then mapping*

$$\Psi := \mathcal{D}_1^{-1} \circ \Phi \circ \mathcal{D}_2 : N_2 \to N_1 \tag{19}$$

is a homeomorphism. In addition it holds that $\Psi^{-1}|_{F_1} = \phi$.

Proof. Due the Theorem 2.3 we only have to prove that mapping Φ is a homeomorphism. Note that mapping Φ has an inverse mapping

$g \mapsto g \circ (\phi^{-1} \times \phi^{-1})$. Let $(x, y) \in F_1 \times F_1$ and $f, g \in C(F_2 \times F_2)$ then it follows

$$|(\Phi(f) - \Phi(g))(x, y)| = |f(\phi(x), \phi(y)) - g(\phi(x), \phi(y))| \leq \|f - g\|_\infty.$$

This proves the continuity of Φ. A similar argument where ϕ is replaced by ϕ^{-1} proves that mapping Φ is a homeomorphism.

Let $x \in F_1$ and denote $y = \phi(x)$. Then

$$\begin{aligned} \Psi^{-1}(x) &= (\mathcal{D}_2^{-1} \circ \Phi^{-1} \circ \mathcal{D}_1)(x) = \mathcal{D}_2^{-1}(D_x^1(\phi^{-1}(\cdot) \times \phi^{-1}(\cdot))) \\ &\overset{(4)}{=} \mathcal{D}_2^{-1}(D_y^2) = y. \end{aligned}$$

□

Remark 2.5. *As the map $\mathcal{D} : M \to \mathcal{D}(M)$, $x \mapsto D_x$, is a homeomorphism, we see that for a dense set $X \subset M$ we have*

$$\mathcal{D}(M) = cl(\mathcal{D}(X)) = cl\{D_x \ ; \ x \in X\} \subset C((N \setminus M) \times (N \setminus M))\},$$

where the closure cl is taken with respect to the topology of $C((N \setminus M) \times (N \setminus M))$. This means that the distance difference functions corresponding to x in a dense set X determine the distance difference functions corresponding to the points in the whole set M.

2.3. **Manifolds N_1 and N_2 are diffeomorphic.** Our next goal is to construct such smooth atlases for manifolds N_i that homeomorphism $\Psi : N_2 \to N_1$ of Theorem 2.4 is a diffeomorphism.

Lemma 2.6. *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space of dimension n and $v \in E, v \neq 0$. Then there exists a basis $v_1, \ldots, v_n$ of E be such that $\|v_j\| = 1$ for all j and $v = b^1 v_1 + b^2 v_2$, $b^i \neq \frac{\|v\|^2}{\langle v_i, v \rangle}$ and $b^i \neq 0$. Moreover, for such vectors there exists $\varepsilon > 0$ such that the vectors*

$$\frac{v + tv_1}{\|v + tv_1\|} - \frac{v}{\|v\|}, \ldots, \frac{v + tv_n}{\|v + tv_n\|} - \frac{v}{\|v\|}$$

are linearly independent for any $t \in (0, \epsilon)$.

Proof. Let $v^\perp \in E$ be such that

$$\langle v, v^\perp \rangle = 0 \text{ and } \|v\| = \|v^\perp\|. \tag{20}$$

We define

$$v_i = \frac{v + (-1)^i v^\perp}{\sqrt{2}\|v\|}, \ i \in \{1, 2\}.$$

Choose $b^i = \frac{\|v\|}{\sqrt{2}}$, $i \in \{1, 2\}$ and complete the set $\{v_1, v_2\}$ to be a basis of E. This basis satisfies the first claim of the lemma.

Let us denote $b^i = 0$ for $i \in \{3, \ldots, n\}$ so that $v = \sum_{i=1}^n b^i v_i$. Define functions $f_i : \mathbb{R} \to \mathbb{R}$, $i \in \{1, \ldots, n\}$ by

$$f_i(t) := t\|v\| + b^i(\|v\| - \|tv_i + v\|).$$

By the choice of numbers b_i we have

$$\frac{df_i}{dt}(0) = \|v\| - b^i \frac{\langle v_i, v\rangle}{\|v\|} \neq 0, \text{ for all } i \in \{1,\dots,n\}.$$

This implies that there exists $\epsilon > 0$ such that for all $t \in (0,\epsilon)$ and $i = 1,2,\dots,n$ we have $f_i(t) \neq 0$.

Let $t \in (0,\epsilon)$ and let $a^i \in \mathbb{R}$, $i \in \{1,\dots,n\}$ be such that

$$0 = \sum_{i=1}^n a^i \Big(\frac{v+tv_i}{\|v+tv_i\|} - \frac{v}{\|v\|}\Big) = \sum_{i=1}^n a^i \Big(\frac{t+b^i}{\|v+tv_i\|} - \frac{b^i}{\|v\|}\Big) v_i. \tag{21}$$

We are done, if we can show that equation (21) implies that $a^i = 0$ for every $i \in \{1,\dots,n\}$. Since $(v_i)_{i=i}^n$ is a basis it holds by (21) that each product $a^i\left(\frac{t+b^i}{\|v+tv_i\|} - \frac{b^i}{\|v\|}\right) = 0$. For the latter term the following holds.

$$\Big(\frac{t+b^i}{\|v+tv_i\|} - \frac{b^i}{\|v\|}\Big) = 0$$

if and only if

$$f_i(t) = 0. \tag{22}$$

By the choice of ϵ the equation (22) is not valid. Therefore $a^i = 0$ for every $i \in \{1,\dots,n\}$. The claim is proved. □

Lemma 2.7. *Let (N,g) be a compact Riemannian manifold of dimension n, $x \in N$ and $\xi \in T_xN$, $\|\xi\|_g = 1$. Let $\gamma_{x,\xi} : [0,\ell] \to N$ be a distance minimizing geodesic. Let $0 < h < \ell$, $z = \gamma_{x,\xi}(h)$, and $\theta = \dot\gamma_{x,\xi}(h) \in T_zN$. Then there exists a basis $\{\eta_i : i = 1,2,\dots,n\}$ of T_zN and $\epsilon > 0$ such that for all $s \in (0,\varepsilon)$ there is a neighborhood $W \subset N$ of x such that the function*

$$H : W \to \mathbb{R}^n,\ H(y) = (d(y,z_i) - d(y,z))_{i=1}^n,\ z_i = \gamma_{z,\eta_i}(s)$$

is a smooth coordinate mapping.

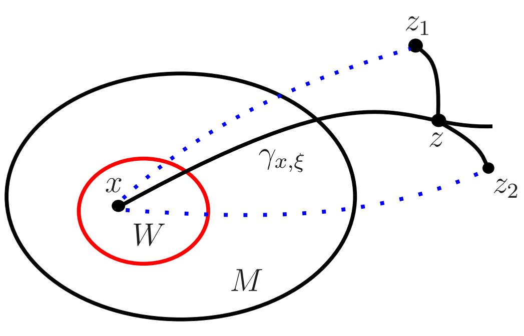

FIGURE 3. *A schematic picture of the coordinate system H.*

Proof. Since the geodesic $\gamma_{x,\xi}([0,\ell])$ is distance minimizing, the geodesic segment $\gamma_{x,\xi}([0,h])$ from x to z has no cut points. Moreover, there exist neighborhoods U_x and U_z of x and z such that the mapping $(p,q) \mapsto d(p,q)$ is smooth on $U_x \times U_z$. As the geodesic $\gamma_{x,\xi}([0,h])$ has no cut points, the differential of $\exp_x$ at $v := h\xi \in T_xN$ is invertible. Choose vectors $v_1, v_2 \in T_xN$ as in Lemma 2.6 and let $(v_i)_{i=1}^n$ be a basis of T_xN. By Lemma 2.6 there exists $\delta > 0$ such that for all $t \in (0,\delta)$ the vectors

$$\frac{v+tv_1}{\|v+tv_1\|} - \frac{v}{\|v\|}, \dots, \frac{v+tv_n}{\|v+tv_n\|} - \frac{v}{\|v\|}$$

are linearly independent. We define vectors

$$\eta_i = D(\exp_x)|_v v_i, \ i = 1, 2, \dots, n.$$

Notice that this is a basis of T_zN. Consider curves $c_i(t) := \exp_x^{-1}(\gamma_{z,\eta_i}(t))$ in tangent space T_xN. These curves have the following properties

$$c_i(0) = v \text{ and } \dot{c}(0) = \frac{d}{dt}(\exp_x^{-1}(\gamma_{z,\eta_i}(t)))|_{t=0} = D(\exp_x^{-1})|_z \eta_i = v_i. \tag{23}$$

Next we will show that there exists $\epsilon > 0$ such that for each $0 < t < \epsilon$ the vectors $\{\frac{c_i(t)}{\|c_i(t)\|} - \frac{v}{\|v\|} : i = 1, \dots, n\}$ are linearly independent. By equation (23) it follows that for each $i = 1, \dots, n$ the curves

$$t \mapsto \frac{c_i(t)}{\|c_i(t)\|} - \frac{v}{\|v\|} \text{ and } t \mapsto \frac{v+tv_i}{\|v+tv_i\|} - \frac{v}{\|v\|}$$

have the same initial point and velocity. Since vectors $\{\frac{v+tv_i}{\|v+tv_i\|} - \frac{v}{\|v\|} : i = 1, \dots, n\}$ are linearly independent for each $0 < t < \delta$ the sought $\epsilon \in (0,\delta)$ exists by the Taylor expansion of $t \mapsto \frac{c_i(t)}{\|c_i(t)\|} - \frac{v}{\|v\|}$.

By the preparations made above, it holds for all $s \in (0,\epsilon)$ that gradients

$$\nabla(d(\cdot, z_i) - d(\cdot, z))|_x = \frac{c_i(s)}{\|c_i(s)\|} - \frac{v}{\|v\|}$$

are linearly independent, where $z_i := \exp_x(c_i(s)) = \gamma_{z,\eta_i}(s) \in U_z$. Then due to the Inverse function theorem it follows that there exists such a neighborhood W of x that function

$$H : W \to \mathbb{R}^n, \ H(y) = (d(y, z_i) - d(y, z))_{i=1}^n$$

is a smooth coordinate mapping. □

Next we consider the homeomorphism $\Psi : N_2 \to N_1$ of Theorem 2.4.

Theorem 2.8. *Suppose that Riemannian manifolds (N_1, g_1) and (N_2, g_2) are as in section 1.2 and Proposition 1.2 is valid. Then mapping $\Psi : N_2 \to N_1$ of formula (19), is a diffeomorphism.*

Proof. Note that for any $p \in N_2$ and all $q, r \in F_2$ holds

$$D_p^2(q,r) = D_{\Psi(p)}^1(\Psi(q), \Psi(r)).$$

Let $x \in N_2$, $y \in F_2^{int}$ and denote $\widetilde{x} = \Psi(x)$ and $\widetilde{y} = \Psi(y)$. Let $h \in (0, d_2(x,y))$ be such that $z := \gamma_{x,\xi_2}(h) \in F_2^{int}$ and $\gamma_{x,\xi_2}([0,h]) \subset F_2^{int}$, where γ_{x,ξ_2} is a minimizing unit speed geodesic from x to y and $\widetilde{z}(h) = \Psi(z(h)) \in F_1^{int}$. Note that by the choice of z it holds that it is not a cut point of x on curve γ_{x,ξ_2}. Therefore mapping $p \mapsto D_p^2(r,q)$ is smooth, if p is close to x and r,q are close to z. Since

$D_x^2(y,z) = D_{\widetilde{x}}^1(\widetilde{y},\widetilde{z}), d_2(z,y) \geq d_1(\widetilde{z},\widetilde{y})$ and $d_2(x,y) = d_2(x,z)+d_2(z,y)$, we deduce using the triangle inequality that

$$d_1(\widetilde{x},\widetilde{y}) = d_1(\widetilde{x},\widetilde{z}) + d_1(\widetilde{z},\widetilde{y}).$$

Therefore there exists an unit speed distance minimizing geodesic $\gamma_{\widetilde{x},\xi_1}$ from $\widetilde{x}$ to $\widetilde{y}$ that also goes trough $\widetilde{z}$. Therefore mapping $\widetilde{p} \mapsto D_{\widetilde{p}}^1(\widetilde{r},\widetilde{q})$ is smooth, when $\widetilde{p}$ is close to $\widetilde{x}$ and $\widetilde{r},\widetilde{q}$ are close to $\widetilde{z}$

Let us denote $\widetilde{h} = d_1(\widetilde{x},\widetilde{z})$. Let $v = h\xi_2 \in T_x N_2$ and $\theta = \dot{\gamma}_{x,\xi_2}(h) \in T_z N_2$. Note that assumptions of Lemma 2.7 are valid for x, h and γ_{x,ξ_2}. By data (1.2) it holds that $\phi^{-1} : F_2^{int} \to F_1^{int}$ is such a diffeomorphism that $g_1 = (\phi^{-1})^* g_2$. Therefore it holds that $(\phi^{-1})_* \theta = \dot{\gamma}_{\widetilde{x},\xi_1}(\widetilde{h}) \in T_{\widetilde{z}} N_1$ and it also holds that

$$A\xi_2 := D(\exp_{\widetilde{x}}^{-1})|_{\widetilde{z}} \circ (\phi^{-1})_* \circ D(\exp_x)|_v \xi_2 = \xi_1. \tag{24}$$

Let $v^\perp \in T_x N_2$ be as in formula (20). By [12], formula (II.7.2)

$$D(\exp_x)|_v v^\perp = D(\exp_x)|_{h\xi_2} v^\perp = h^{-1} J(h).$$

where J is a Jacobi field along γ_{x,ξ_2} satisfying the following initial conditions

$$J(0) = 0 \text{ and } \nabla_t J(0) = v^\perp.$$

Since by [39], formula (10.6) it holds that J is orthogonal to $\dot{\gamma}_{x,\xi}$, we conclude that $\langle Av^\perp, \xi_1 \rangle_{g_1} = 0$. We denote $\widetilde{v} := \widetilde{h}\xi_1$. Let vectors $v_i := \frac{v+(-1)^i v^\perp}{\sqrt{2}h}$ for $i \in \{1,2\}$, c.f. the proof of 2.6. Then

$$\widetilde{v} = \frac{\widetilde{h}}{\sqrt{2}}(Av_1 + Av_2) \text{ and } \frac{\|\widetilde{v}\|_1^2}{\langle \widetilde{v}, Av_i \rangle_1} = \sqrt{2}\,\widetilde{h}.$$

Thus we can find bases $(v_i)_{1=1}^n \subset T_x N_2$ and $(\widetilde{v}_i)_{i=1}^n \subset T_{\widetilde{x}} N_1$ as described in Lemma 2.6.

Let $\eta_i := D(\exp_x)|_v v_i$. Hence by Lemma 2.7 there exists $\epsilon > 0$ such that for any $s \in (0,\epsilon)$ mapping $H(p) := (D_p^2(z_i,z))_{i=1}^n$, $z_i := \gamma_{z,\eta_i}(s) \in F_2^{int}$ is a smooth coordinate mapping in a neighbourhood of x. Moreover, if ϵ is small enough, then points $\widetilde{z}_i := \phi(z_i) \in F_1^{int}$ are close enough to $\widetilde{z}$ that $\widetilde{H}(\widetilde{q}) := (D_{\widetilde{q}}^1(\widetilde{z}_i,\widetilde{z}))_{i=1}^n$ is smooth near $\widetilde{x}$. Notice also that $\widetilde{z}_i = \gamma_{\widetilde{z},\widetilde{\eta}_i}(s) = \phi^{-1}(\gamma_{z,\eta_i}(s))$, where $\widetilde{\eta}_i = (\phi^{-1})_* \eta_i$. Thus Lemma 2.7 implies that when ε is small enough, then also $\widetilde{H}$ is a smooth coordinate mapping in some neighborhood $\widetilde{W}$ of $\widetilde{x}$. Thus we have shown that

$$\widetilde{H} \circ \Psi \circ H^{-1} = Id,$$

in some neighborhood of x. Since the point $x \in N_2$ was an arbitrary one and also H and $\widetilde{H}$ are smooth coordinate mappings for x and $\widetilde{x}$ we have proved that Ψ is a diffeomorphism.

□

2.4. **Riemannian metrics g_1 and $\Psi_* g_2$ coincide in N_1.** In this section we will show that manifolds (N_1, g_1) and (N_2, g_2) that give the data (2)-(3), are isometric.

Definition 2.9. *Let $z_1 \in F$ and $\xi \in S_{z_1}N$. Define a set*

$$\omega(z_1,\xi) := \{x \in N \;\; ; \;\; \exists z_2 \in F \text{ such that } D_x(\cdot, z_2) \text{ is } C^1\text{-smooth, near } z_1 \text{ and } \nabla D_x(\cdot, z_2)|_{z_1} = \xi\} \cup \{z_1\}. \tag{25}$$

Lemma 2.10. *Let $z_1 \in F$ and $\xi \in S_{z_1}N$. Then it follows*

$$\omega(z_1,\xi) = \gamma_{z_1,-\xi}(\{s \; ; \; 0 \le s < \tau(z_1,-\xi)\}), \tag{26}$$

This means that using data (4) *we can see the unparametrized geodesics of N.*

Proof. First we recall that for all $x \in N$ the distance function $d(\cdot, x)$ is not smooth near $y \in N \setminus \{x\}$ if and only if point y is in a cut locus of x. This holds due Lemma 2.1.11 and Theorem 2.1.14 of [26]. The Lemma 2.1.11 of [26] states that every cut point is either a conjugate point or an ordinary cut point. Being an ordinary cut point means that there exist two different distance minimizing unit speed geodesics from x to y. Therefore the gradient of distance function $d(\cdot, x)$ is not continuous at ordinary cut points. The Theorem 2.1.14 of [26] states that the complement of the cut locus of x is the maximal open set with property that each y in this set can be joined to x with exactly one unit speed distance minimizing geodesic. Therefore any conjugate point that is not an ordinary cut point is a cluster point of ordinary cut points.

If $x \in \omega(z_1,\xi) \setminus \{z_1\}$, it follows that x is not in a cut locus of z_1, since by the definition of $\omega(z_1,\xi)$ distance function $d(\cdot, x)$ is smooth near z_1. Therefore there exists an unique distance minimizing unit speed geodesic from x to z_1. Since this geodesic has a velocity

$$\nabla d(\cdot, x)|_{z_1} = \nabla D_x(\cdot, z_2)|_{z_1} = \xi$$

at z_1, it follows that $x \in \gamma_{z_1,-\xi}(\{s \; ; \; 0 \le s < \tau(z_1,-\xi)\})$.

If $x \in \gamma_{z_1,-\xi}(\{s \; ; \; 0 \le s < \tau(z_1,-\xi)\}) \setminus \{z_1\}$ we know that $D_x(\cdot, z_1)$ is smooth near z_1 and

$$\nabla D_x(\cdot, z_1)|_{z_1} = \nabla d(\cdot, x)|_{z_1} = \dot\gamma(d(x, z_1)) = -\dot\gamma_{z_1,-\xi}(0) = \xi.$$

Here γ stands for an unique distance minimizing unit speed geodesic from x to z_1. □

The Lemma 2.10 will be the key element to prove that the mapping Ψ is an isometry.

Definition 2.11. *Let N be a smooth manifold with metric tensors g and $\widetilde{g}$. We say that metrics g and $\widetilde{g}$ are geodesically equivalent, if for all geodesics $\gamma : I_1 \to N$ of metric g and $\widetilde{\gamma} : \widetilde{I}_1 \to N$ of metric $\widetilde{g}$ there exist changes of parameters $\alpha : I_2 \to I_1$ and $\widetilde{\alpha} : \widetilde{I}_2 \to \widetilde{I}_1$ such that*

$$\gamma \circ \alpha \text{ is a geodesic of metric } \widetilde{g}$$

and

$$\widetilde{\gamma} \circ \widetilde{\alpha} \text{ is a geodesic of metric } g.$$

A trivial example of two geodesically equivalent Riemannian metrics are g and cg, where $c > 0$. Couple more interesting examples are:

(1) Plane $\mathbb{R}^2$ and the Southern hemisphere of the Riemann sphere that are mapped to each other in a gnomonic projection. I.e. great circles are mapped to straight line.
(2) Unit disc in $\mathbb{R}^2$ and the Beltrami-Klein model of a hyperbolic plane.

Our first goal is to show that from our data (2)-(3) we can deduce that manifolds (N_1, g_1) and $(\Psi(N_2), \Psi_* g_2)$ must be geodesically equivalent. By Lemma 2.10 we know all the geodesics of N_1 that exit unknown region M_1, as point sets. Next we will show that this information is enough to deduce the geodesic equivalence of manifolds.

Since mapping Ψ is diffeomorphism, it holds that each geodesic of (N_2, g_2) is mapped to some smooth curve of (N_1, g_1). By formula (4) and Lemma 2.10, it holds that sets $\omega(z, \xi)$ with $z \in F_1$ and $\xi \in S_z N_1$ are also images of geodesics of (N_2, g_2) in mapping Ψ. Note that the segments of geodesics of N_1 we know as non-parametrized curves are not self-intersecting, since cut points occur before a geodesic stops to be one-to-one.

Let $z \in F_2$, $\xi \in S_z N_2$ and $t_2 = \tau_2(z, \xi)$. Then curve $\Psi(\gamma^2_{z,\xi}(t)) : [0, t_2) \to N_1$ is smooth and not self-intersecting and by Proposition 1.2 and Theorem 2.4 we have

$$\Psi(\gamma^2_{z,\xi}([0, t_2)) = \omega(\Psi(z), \Psi^*(\xi)) = \omega(\phi^{-1}(z), (\phi^{-1})^* \xi).$$

Set $\phi^{-1}(z) = w$ and $(\phi^{-1})^* \xi = \eta$. Then by Lemma 2.10 we have $\omega(w, \eta) = \gamma^1_{w,-\eta}(\{s;\ 0 \leq s < t_1\})$, $t_1 = \tau_1(w, -\eta)$. Furthermore, it is easy to see that there is a re-parametrization

$$s : [0, t_1) \to [0, t_2) \text{ such that } \gamma^1_{w,-\eta}(t) = \Psi(\gamma^2_{z,\xi}(s(t))),\ t \in [0, t_1). \tag{27}$$

Let $a < b$ and define a collection $C_{a,b}$ of geodesics of (N_1, g_1) as

$$C_{a,b} = \{c : [a, b) \to N_1 \quad ; \quad c \text{ is a geodesic, there exists } z \in F_1^{int} \text{ and } \xi \in T_z N_1 \text{ such that } c([a, b)) = \omega(z, \xi)\}.$$

Observe that here $c([a, b)) = \omega(z, \xi)$ means that the sets $c([a, b)) \subset N$ and $\omega(z, \xi) \subset N$ are the same, or equivalently, that $c([a, b))$ and $\omega(z, \xi)$ are the same as unparametrized curves.

Let

$$\mathcal{C} = \bigcup_{a<b} C_{a,b}.$$

For a moment we consider only metric g_1. Assume that p is a point in N_1 and q is point of F_1^{int} such that $q = \gamma_{p,\xi}(\ell)$, $\ell > 0$ and the geodesic $\gamma_{p,\xi}([0,\ell])$ has no cut points. Then there is a neighborhood $U \subset F_1^{int}$ of q and a neighborhood $V \subset T_pN$ of $\ell\xi$ such that $\exp_p : V \to U$ is a diffeomorphism. Assuming that the neighborhood V is small enough, we see that for any $v \in V$ the geodesics $\gamma_{p,v}([0,\ell])$ has no cut points. Then, $\gamma^1_{p,v}([0,\ell)) \in \mathcal{C}$. This proves that set

$$\begin{aligned}\Omega_p := \{v \in T_pN_1 \quad ; \quad & \text{there are } c \in \mathcal{C} \text{ and } t_p \in \mathrm{dom}(c) \text{ such that} \\ & c(t_p) = p \text{ and } \dot c(t_p) \text{ is proportional to } v\}\end{aligned}$$

contains a non-empty open double cone Σ_p, that is, an open set that satisfies $rv \in \Sigma_p$ for all $v \in \Sigma_p$ and $r \in \mathbb{R} \setminus \{0\}$. Note that the complement of Ω_p in T_pN_1 is non-empty if in manifold M there are closed geodesics, or geodesics that are trapping in both directions, that go through the point p.

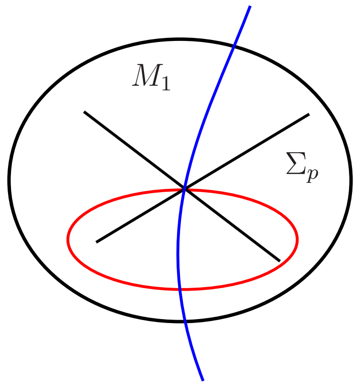

FIGURE 4. *For all $p \in M_1$ there exists an open conic set $\Sigma_p \subset T_pN_1$ such that for every $\xi \in \Sigma_p$ the geodesic $\gamma_{p,\xi}$ of (N_1, g_1) can be extended to a distance minimizing geodesic (blue curve in the figure) that enters the set $F = N \setminus M$. These geodesics are known to be pregeodesic also with respect to the metric $\Psi^* g_2$. Note that there may be g_1-geodesics emanating from p to directions $\xi \notin \Sigma_p$ that does not intersect the set F. Such geodesics can be e.g. closed loops in M_1 (red curve).*

Let point $p \in N_1$ and (U, X) be coordinates near p, that is $X : U \to \mathbb{R}^n$ and denote $X(q) = (x^j(q))_{j=1}^n$. Recall that a pre-geodesic $\widetilde\gamma$ on (N_1, g_1) satisfies the formula (11) that is,

$$\Big[\frac{d^2\widetilde\gamma^k}{ds^2}(s) + \Gamma^k_{i,j}(\widetilde\gamma(s))\frac{d\widetilde\gamma^i}{ds}(s)\frac{d\widetilde\gamma^j}{ds}(s)\Big]\Big|_{s=s_p} = f\Big(\frac{d\widetilde\gamma}{ds}\Big)\frac{d\widetilde\gamma^k}{ds}(s)\Big|_{s=s_p},$$

$k \in \{1, \dots, n\}$. Here $\gamma(s_p) = p$ and f is some function that is homogeneous of degree 1.

Next, we change the point of view and consider this equation as a system of equations for the "unknown" (Γ, f) with the given coefficients $\frac{d\widetilde{\gamma}}{ds}(s)|_{s=s_p} \in \Omega_p$ and $\frac{d^2\widetilde{\gamma}}{ds^2}(s)|_{s=s_p}$ where $\widetilde{\gamma} \in \mathcal{C}$. Here Γ stands for a collection of Cristoffel symbols $\Gamma^k_{i,j}$ and $f : TU \to \mathbb{R}$ is a continuous function that satisfies equation (10).

Suppose that we also have another Riemannian connection $\widetilde{\Gamma}^k_{i,j}$ which Critoffel symbols in the (U, X)-coordinates have the form

$$\widetilde{\Gamma}^k_{i,j} = \Gamma^k_{i,j} + \delta^k_i \varphi_j + \delta^k_j \varphi_i, \tag{28}$$

for some smooth functions $\varphi_i : U \to \mathbb{R}$, $i = 1, 2, \dots, n$. Here, δ^k_i is one when $k = i$ and zero otherwise. Let $\varphi(x) = \varphi_i(x)dx^i$ be a smooth 1-form that has functions $(\varphi_i)_{i=1}^n$ as the coefficients. We need the following consequence of Lemma 2.1:

Lemma 2.12. *If the Christoffel symbols $\widetilde{\Gamma}$ and Γ satisfy the equation* (28) *for some 1-form φ and pair (f, Γ), f is homogenous of degree 1, is a solution of* (11) *for all $\widetilde{\gamma} \in \mathcal{C}$, then pair $(\widetilde{\Gamma}, \widetilde{f})$ where*

$$\widetilde{f}(v) = f(v) + 2\varphi(v). \tag{29}$$

is also a solution of (11) *for all $\widetilde{\gamma} \in \mathcal{C}$.*

Proof. Let $\widetilde{\gamma} \in \mathcal{C}$. A direct computation shows that

$$\begin{aligned} (\delta^k_i \varphi_j + \delta^k_j \varphi_i)\tfrac{d\widetilde{\gamma}^i}{ds}(s)\tfrac{d\widetilde{\gamma}^j}{ds}(s) &= \varphi_j \tfrac{d\widetilde{\gamma}^k}{ds}(s)\tfrac{d\widetilde{\gamma}^j}{ds}(s) + \varphi_i \tfrac{d\widetilde{\gamma}^i}{ds}(s)\tfrac{d\widetilde{\gamma}^k}{ds}(s) \\ &= 2\tfrac{d\widetilde{\gamma}^k}{ds}(s)\Big(\varphi_i \tfrac{d\widetilde{\gamma}^i}{ds}(s)\Big) = 2\tfrac{d\widetilde{\gamma}^k}{ds}(s)\varphi\Big(\tfrac{d\widetilde{\gamma}}{ds}(s)\Big) \end{aligned} \tag{30}$$

Use this and substitute equation (28) into equation (11) to obtain

$$\frac{d^2\widetilde{\gamma}^k}{ds^2}(s) + \widetilde{\Gamma}^k_{i,j}(p)\frac{d\widetilde{\gamma}^i}{ds}(s)\frac{d\widetilde{\gamma}^j}{ds}(s)\bigg|_{s=s_p} = \frac{d\widetilde{\gamma}^k}{ds}(s)\Big[f\Big(\frac{d\widetilde{\gamma}}{ds}(s)\Big) + 2\varphi\Big(\frac{d\widetilde{\gamma}}{ds}(s)\Big)\Big]\bigg|_{s=s_p}$$

that proves the claim. □

The following lemma gives the converse result for Lemma 2.12. It is obtained by using, in a quite straightforward way, results of V. Matveev [44, Sec. 2] for general affine connections on pseudo-Riemannian manifolds. However, for the convenience of the reader, we give a detailed proof for the lemma and analyze at the same time the smoothness of the 1-form $x \mapsto \varphi(x)$ in a local coordinate neighbourhood $U \subset M$.

Lemma 2.13. *Let functions $f : TU \to \mathbb{R}$ and $\widetilde{f} : TU \to \mathbb{R}$ be homogeneous of degree 1. Suppose that pairs (f, Γ) and $(\widetilde{\Gamma}, \widetilde{f})$ both solve at all points $p \in U$ the system* (11) *for all such coefficients $\frac{d\gamma}{ds}(s)|_{s=s_p} \in \Omega_p$ and $\frac{d^2\gamma}{ds^2}(s)|_{s=s_p}$ that $\gamma \in \mathcal{C}$ and $\gamma(s_p) = p$. Then Cristoffel symbols Γ and $\widetilde{\Gamma}$ satisfy equation* (28) *in U with a C^∞-smooth 1-form φ in U.*

Proof. Define a pair $(\overline{f}, \overline{\Gamma})$ as

$$\overline{f} = f - \widetilde{f} \text{ and } \overline{\Gamma}^k_{i,j} = \Gamma^k_{i,j} - \widetilde{\Gamma}^k_{i,j}.$$

As a difference of two connection coefficients, $\overline{\Gamma}$ is a tensor. By substitution of pairs (f, Γ) and $(\widetilde{\Gamma}, \widetilde{f})$ into equation (11) and by subtracting the obtained equation from the other, we obtain at $p \in U$

$$\overline{\Gamma}^k_{i,j} v^i v^j = \overline{f}(v) v^k, \text{ for every } v \in \Omega_p. \tag{31}$$

Note that (31) defines a smooth extension of $\overline{f}|_{\Omega_p}$ to $T_pN \setminus \{0\}$, given by

$$\overline{f}(v) = \frac{\overline{f}(v) v^k g_{k\ell} v^\ell}{g(v,v)} = \frac{\overline{\Gamma}^k_{i,j}(p) v^i v^j g_{k\ell}(p) v^\ell}{g_{ab}(p) v^a v^b}. \tag{32}$$

Here, the rightmost term is smooth in $T_pN \setminus \{0\}$.

Recall that Ω_p contains an open double cone $\Sigma_p \subset \Omega_p$. Our next goal is to show that there exist a linear function $\varphi : T_pN \to \mathbb{R}$ such that the restriction of function $\overline{f}$, to $\Sigma_p \subset \Omega_p$, is equal to $2\varphi|_{\Sigma_p}$. Define a family of symmetric bi-linear mappings

$$\sigma^k : T_pN \times T_pN \to \mathbb{R},\ \sigma^k(u,v) = \overline{\Gamma}^k_{i,j} v^i u^j,\ k \in \{1, \dots, n\}.$$

Since mappings σ^k are symmetric, the parallelogram equation

$$0 = \sigma^k(u+v, u+v) + \sigma^k(u-v, u-v) - 2\sigma^k(u,u) - 2\sigma^k(v,v)$$

holds.

Next, let $u \in \Sigma_p$, $u \neq 0$. Then there is $\varepsilon = \varepsilon(u) > 0$ such that, if $v \in T_pN$ satisfies $\|v\| < \varepsilon$, then $u - v \in \Sigma_p$.

Let us next consider $v \in \Sigma_p$ with $\|v\| < \varepsilon$. Then $u - v, u + v \in \Sigma_p \subset \Omega_p$. By the parallelogram equality of mapping σ^k and (31) we have

$$\begin{aligned} 0 &= \overline{f}(u+v)(u+v) + \overline{f}(u-v)(u-v) - 2\overline{f}(u)u - 2\overline{f}(v)v \\ &= (\overline{f}(u+v) + \overline{f}(u-v) - 2\overline{f}(u))u + (\overline{f}(u+v) - \overline{f}(u-v) - 2\overline{f}(v))v. \end{aligned} \tag{33}$$

If vectors u and v are linearly independent, we get a system

$$\begin{cases} \overline{f}(u+v) + \overline{f}(u-v) - 2\overline{f}(u) = 0 \\ \overline{f}(u+v) - \overline{f}(u-v) - 2\overline{f}(v) = 0. \end{cases} \tag{34}$$

Sum up these two equations to get

$$\overline{f}(u+v) = \overline{f}(u) + \overline{f}(v). \tag{35}$$

If vector $v = \lambda u$, $\lambda \in \mathbb{R}$, we note that the system (34) is still valid. Recall that the mappings f and $\widetilde{f}$ are solutions of (11) and therefore they satisfy the equation (10), i.e., they commute with scalar multiplication in Ω_p.

So far we have proved that $\overline{f}(u+\cdot)$ and $\overline{f}(u)+\overline{f}(\cdot)$ coincide in set $B_p(0,\epsilon)\cap\Sigma_p$. Since $\overline{f}$ is homogeneous of degree 1 it holds by (35) that

$$\overline{f}(u+av)=\overline{f}(u)+a\overline{f}(v),\ v\in B_p(0,\epsilon)\cap\Sigma_p,\ -1<a<1. \tag{36}$$

We define a linear function

$$2\varphi:T_pN\to\mathbb{R},\ 2\varphi(v)=\lim_{r\to 0}\frac{\overline{f}(u+rv)-\overline{f}(u)}{r}=\nabla_u\overline{f}(u)\cdot v. \tag{37}$$

If $v\in\Sigma_p$ and r is small enough, then $rv\in B_p(0,\varepsilon)\cap\Sigma_p$ and therefore by formula (36) it holds that

$$2\varphi(v)=\overline{f}(v)\quad\text{for every } v\in\Sigma_p.$$

As Σ_p is open, and φ and $\overline{f}$ are linear, this holds for all $v\in T_pN$ and thus $\varphi(v)$ given by the formula (37) is independent on the choice of used $u\in\Sigma_p$. In local coordinates we have

$$\varphi(\frac{\partial}{\partial x^\ell})=\frac{1}{2}\sum_{i,k,j=1}^{n}\frac{1}{g_{\ell\ell}(x)}\overline{\Gamma}^k_{i,j}(x)\delta^i_\ell\delta^j_\ell g_{k\ell}(x).$$

This defines a C^∞-smooth 1-form $x\mapsto\varphi(x)$ in U.

Define a connection

$$\widehat{\Gamma}^k_{i,j}:=\widetilde{\Gamma}^k_{i,j}+\delta^k_i\varphi_j+\delta^k_j\varphi_i,$$

and choose $v=\frac{d}{ds}\gamma(s)|_{s=s_p}\in\Sigma_p$. Since pairs (f,Γ) and $(\widetilde{\Gamma},\widetilde{f})$ are both solutions of (11) the following holds due the reasoning done so far

$$\begin{aligned}
&\Big[\frac{d^2\gamma^k}{ds^2}(s)+\widetilde{\Gamma}^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big]\Big|_{s=s_p}=\Big[f\Big(\frac{d\gamma}{ds}(s)\Big)\frac{d\gamma^k}{ds}(s)\Big]\Big|_{s=s_p}\\
=\;&\frac{d\gamma^k}{ds}(s)\Big[2\varphi\Big(\frac{d\gamma}{ds}(s)\Big)+\widetilde{f}\Big(\frac{d\gamma}{ds}(s)\Big)\Big]\Big|_{s=s_p}\\
=\;&\Big[\frac{d^2\gamma^k}{ds^2}(s)+\widetilde{\Gamma}^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big]\Big|_{s=s_p}+\frac{d\gamma^k}{ds}(s)\Big[2\varphi\Big(\frac{d\gamma}{ds}(s)\Big)\Big]\Big|_{s=s_p}\\
\overset{(30)}{=}\;&\Big[\frac{d^2\gamma^k}{ds^2}(s)+\widehat{\Gamma}^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big]\Big|_{s=s_p}.
\end{aligned}$$

Therefore we have

$$\Gamma^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big|_{s=s_p}=\widehat{\Gamma}^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big|_{s=s_p}. \tag{38}$$

Thus we have proved that for all $v\in\Sigma_p$ the equation

$$\Gamma^k_{i,j}(p)v^iv^j=\widehat{\Gamma}^k_{i,j}(p)v^iv^j \tag{39}$$

is valid. Since set Σ_p is open, it holds that

$$\Gamma^k_{\ell,m}(p)=\partial_{v^\ell v^m}\Gamma^k_{i,j}(p)v^iv^j=\partial_{v^\ell v^m}\widehat{\Gamma}^k_{i,j}(p)v^iv^j=\widehat{\Gamma}^k_{\ell,m}(p).$$

As above $p\in U$ is arbitrary, this proves the claim. □

Proposition 2.14. *Suppose that Riemannian manifolds (N_1, g_1) and (N_2, g_2) are as in Section 1.2 and (2)-(3) are valid. Let $p \in N_1$ and (U, X) be coordinates for p. Then it holds that the Cristoffel symbols Γ and $\widetilde{\Gamma}$ of metrics g_1 and $\Psi_* g_2$, respectively, satisfy equation (28) in U with some 1-form φ, where Ψ is as in (19).*

Proof. Let $p \in N_1$ and (U, X) be coordinates for p. Let $c_1 \in \mathcal{C}$ be a curve that passes through p. By definition of $\mathcal{C}$ and equation (27) it holds that there is a reparametrization s of c_1 such that for curves c_1 and $c_2 = c_1 \circ s$ holds

$$\begin{cases} \ddot{c}_1(t) + \dot{c}_1^i(t)\dot{c}_1^j(t)\Gamma_{i,j}^k(c_1(t)) = 0 \\ \ddot{c}_2^k(t) + \dot{c}_2^i(t)\dot{c}_2^j(t)\widetilde{\Gamma}_{i,j}^k(c_2(t)) = 0 \end{cases}$$

Use the chain rule and write the latter equation as

$$\ddot{c}_1^k(t) + \dot{c}_1^i(t)\dot{c}_1^j(t)\widetilde{\Gamma}_{i,j}^k(c_1(s(t))) = -\frac{\ddot{s}(t)}{\dot{s}(t)^2}\dot{c}_1^k(t).$$

Define a mapping $f : TU \to \mathbb{R}$ by setting for $(q, v) \in TU$, with $v \neq 0$,

$$f(q, v) = -\frac{\ddot{s}(0)}{\dot{s}(0)^2},$$

where $s(t)$ is a re-parametrization of the geodesic for which we have $\gamma_{q,v}^1(t) = \Psi(\gamma^2_{\phi^{-1}(q),\Psi^* v}(s(t)))$, so that $s(0) = 0$. Also, we define $f(q, v)|_{v=0} = 0$. Note that function f satisfies the equation (10), since geodesic equation (7) is preserved under affine re-parametrizations. Therefore it holds that pairs $(\Gamma, 0)$ and $(\widetilde{\Gamma}, f)$ both solve the system (11) for all such coefficients $\frac{d\gamma}{ds}(s)|_{s=s_p} \in \Omega_p$ and $\frac{d^2\gamma}{ds^2}(s)|_{s=s_p}$ that $\gamma \in \mathcal{C}$ and $\gamma(s_p) = p$. By the Lemma 2.13 the claim follows. □

Lemma 2.15. *Suppose that Γ and $\widetilde{\Gamma}$ are connections satisfying the equation (28) with 1-form φ. If $t \mapsto \gamma(t)$, $\gamma \in \mathcal{C}$ is a geodesic of connection Γ, then there exists a change of parameters $s \mapsto t(s)$ such that $s \mapsto \gamma(t(s))$ is a geodesic of $\widetilde{\Gamma}$, hence metrics g and $\widetilde{g}$ are geodesically equivalent.*

Proof. Since γ is a geodesic of Γ it satisfies the geodesic equation (7). Substitute Γ with $\widetilde{\Gamma}$ into (7) to get the equation

$$\frac{d^2\gamma^k}{dt^2}(t) + \widetilde{\Gamma}_{i,j}^k(\gamma(t))\frac{d\gamma^i}{dt}(t)\frac{d\gamma^j}{dt}(t) = 2\frac{d\gamma^k}{dt}(t)\varphi\Big(\frac{d\gamma}{dt}(t)\Big).$$

Write $\kappa(t) = 2\varphi(\dot{\gamma}(t))$ and use Lemma 2.1 to show that appropriate $s \mapsto t(s)$ exists. □

By the Lemma 2.15, we know that our data (2)-(3) proves the geodesic equivalence of metrics g and $\Psi_* g_2$ on N_1. In the following theorem that shows that metrics g and $\Psi_* g_2$ coincide also in N_1, we will use the implications of the Matveev-Topalov theorem [41]. Their result is also concerned in the appendix of the extended preprint version of this paper [35] and its generalizations have been considered in [11, 61].

Lemma 2.16. *Suppose that manifold N satisfies assumptions of Section 1.2 and it has two metrics g and $\widetilde{g}$. Suppose that these metrics are geodesically equivalent on manifold N and coincide in set $F^{int} \neq \emptyset$. Then $g = \widetilde{g}$ in whole N.*

Proof. Define a smooth mapping $I_0 : TN \to \mathbb{R}$ as

$$I_0((x,v)) = \Big(\frac{\det(g_x)}{\det(\widetilde{g}_x)}\Big)^{\frac{2}{n+1}} \widetilde{g}_x(v,v), \tag{40}$$

where $\widetilde{g}_x(v,v) = \widetilde{g}_{jk}(x)v^j v^k$. Note that the function $x \mapsto \frac{\det(g_x)}{\det(\widetilde{g}_x)}$ is coordinate invariant.

Let γ_g be a geodesic of metric g. Define a smooth path β in TN as $\beta(t) = (\gamma_g(t), \dot{\gamma}_g(t))$. Then β is an integral curve of the geodesic flow of metric g. The Matveev-Topalov theorem [41] states that if g and $\widetilde{g}$ are geodesically equivalent, then there are several invariants related to the tensor $G = g^{-1}\widetilde{g}_{ik}$, given in local coordinates by $G^j_k(x) = g^{ji}(x)\widetilde{g}_{ik}(x)$, that are constants along integral curves $\beta(t)$. In particular, the function $t \mapsto I_0(\beta(t))$ is a constant.

A corollary of this theorem, [41, Cor. 2] (see also [42, Cor. 2] and [11, Thm. 3]), is that the number $n(x)$ of the different eigenvalues of the map $G(x) : T_xN \to T_xN$ is constant at almost every point $x \in N$. Since $G(x) = I$ for $x \in F^{int}$, so that $n(x) = 0$ in the set F^{int} having a positive measure. This implies that $n(x) = 0$ for almost all $x \in N$. Hence for almost all $x \in N$ there is $c(x) \in \mathbb{R}_+$ such that we have $G(x) = c(x)I$, so that $\widetilde{g}_{ik}(x) = c(x)g_{ik}(x)$. As G is continuous, this holds for all $x \in N$. Summarising, the first implication of the Matveev-Topalov theorem is that g and $\widetilde{g}$ are conformal on the whole manifold N.

Let x_0 be a point of N. Since we assumed that metrics g and $\widetilde{g}$ coincide in set F, we have for any point $z \in F$ and vector $v \in T_zN$ that formula (40) has form

$$I_0(z,v) = \widetilde{g}_z(v,v) = g_z(v,v). \tag{41}$$

Let $\gamma(t) := \gamma^g_{z,\xi}(t)$, $\xi \in S_zN$, $z \in F$ be a g-geodesic passing through x_0, that is $x_0 = \gamma(t_0)$, for some t_0. In particular we see that $I_0((z,\xi)) = 1$. By the Matveev-Topalov theorem, I_0 is constant along the integral curves of geodesic flow of g. Thus, we have

$$I_0(x_0, \dot{\gamma}(t_0)) = I_0(z,\xi) = 1. \tag{42}$$

Define W_{x_0} to be the set of all unit vectors of $T_{x_0}N$ with respect to metric g, such that every vector in W_{x_0} is a velocity of some geodesic starting from F and passing trough x_0. Recall that set $W^{int}_{x_0} \subset S_{x_0}N$ is not empty.

Let $X = (x^1, \dots, x^n)$ be any coordinate chart at x_0. Formula (42) shows that for every $\xi \in W_{x_0}$ we have

$$g_{ij}(x_0)\xi^i\xi^j = 1 = I_0(x_0,\xi) = \Big(\frac{\det(g_{x_0})}{\det(\widetilde{g}_{x_0})}\Big)^{\frac{2}{n+1}}\widetilde{g}_{ij}(x_0)\xi^i\xi^j. \tag{43}$$

Consider an open cone

$$W^{int}_{x_0}\cdot\mathbb{R}_+ := \{tw \in T_{x_0}N : t>0, w\in W^{int}_{x_0}\}.$$

Since metrics are bilinear, we know that equation (43) holds for any vector $\xi \in W^{int}_{x_0}\cdot\mathbb{R}_+$. Since set $W^{int}_{x_0}\cdot\mathbb{R}_+$ is open and both sides of equation (43) are smooth in ξ, we obtain the equation

$$g_{ij}(x_0) = \Big(\frac{\det(g_{x_0})}{\det(\widetilde{g}_{x_0})}\Big)^{\frac{2}{n+1}}\widetilde{g}_{ij}(x_0), \text{ for all } i,j\in\{1,\dots,n\}, \tag{44}$$

as a second order derivative with respect to ξ of equation (43).

Let $f(p) := \frac{\det(g(p))}{\det(\widetilde{g}(p))}$. With given notations we have shown that

$$(f(x_0))^{\frac{2}{n+1}}\,\widetilde{g}_{jk}(x_0) = g_{jk}(x_0), \text{ for all } j,k\in\{1,\dots,n\}. \tag{45}$$

We see from equation (45) that

$$(f(x_0))^{\frac{2n}{n+1}}\det(\widetilde{g}) = \det(g).$$

Therefore it holds

$$(f(x_0))^{\frac{2n}{n+1}-1} = 1. \tag{46}$$

Since we assumed the dimension of manifold N to be at least 2 we see from equation (46) that $f(x_0) = 1$. By formula (45) this implies $g = \widetilde{g}$ also on M. □

Theorem 1.3 follows now from Theorems 2.4 and 2.8 and Lemmas 2.15 and 2.16. □

3. Application for an inverse problem for a wave equation

Here we consider the application of Theorem 1.3 for an inverse problem for a wave equation with spontaneous point sources.

3.0.1. *Support sets of waves produced by point sources.* Let (N,g) be a closed Riemannian manifold. Denote the Laplace-Beltrami operator of metric g by Δ_g. We consider a wave equation

$$\begin{cases} (\partial_t^2 - \Delta_g)G(\cdot,\cdot,y,s) = \kappa(y,s)\delta_{y,s}(\cdot,\cdot), & \text{in } \mathcal{N} \\ G(x,t,y,s) = 0, & \text{for } t<s,\ x\in N. \end{cases} \tag{47}$$

where $\mathcal{N} = N\times\mathbb{R}$ is the space-time. The solution $G(x,t,y,s)$ is the wave produced by a point source located at the point $y\in M$ and time $s\in\mathbb{R}$ having the magnitude $\kappa(y,s)\in\mathbb{R}\setminus\{0\}$. Above, we have $\delta_{y,s}(x,t) = \delta_y(x)\delta_s(t)$ corresponds to a point source at $(y,s)\in\mathcal{N}$.

3.0.2. *Inverse coefficient problem with spontaneous point source data.* Assume that there are two manifolds (N_1, g_1) and (N_2, g_2) satisfying the assumptions given in Section 1.2 and

$$\text{There exists an isometry } \phi : F_1 \to F_2 \tag{48}$$

$$W_1 = W_2 \tag{49}$$

where W_1 and W_2 are collections of supports of waves produced by point sources taking place at unknown points at unknown time, that is,

$$W_1 = \{\mathrm{supp}\,(G^1(\cdot,\cdot,y_1,s_1)) \cap (F_1 \times \mathbb{R});\ y_1 \in M_1,\ s_1 \in \mathbb{R}\} \subset 2^{F_1\times\mathbb{R}}$$

and

$$W_2 = \{\mathrm{supp}\,(G^2(\phi(\cdot),\cdot,y_2,s_2)) \cap (F_1 \times \mathbb{R});\ y_2 \in M_2,\ s_2 \in \mathbb{R}\} \subset 2^{F_1\times\mathbb{R}}$$

where functions G^j, $j = \{1,2\}$ solve equation (47) on manifold N_j. Here $2^{F_j\times\mathbb{R}} = \{F';\ F' \subset F_j \times \mathbb{R}\}$ is the power set of $F_j \times \mathbb{R}$. Roughly speaking, W_j corresponds to the data that one makes by observing, in the set F_j, the waves that are produced by spontaneous point sources that that go off, at an unknown time and at an unknown location, in the set M_j.

Earlier, the inverse problem for the sources that are delta-distributions in time and localized also in the space has been studied in [15] in the case when the metric g is known. Theorem 1.3 yields the following result telling that the metric g can be determined when a large number of waves produced by the point sources is observed:

Proposition 3.1. *Let (N_j, g_j), $j = 1,2$ be a closed compact Riemannian n-manifolds, $n \geq 2$ and $M_j \subset N_j$ be an open set such that $F_j = N_j \setminus M_j$ have non-empty interior. If the spontanuous point source data of these manifolds coincide, that is, we have* (48)-(49)*, then (N_1, g_1) and (N_2, g_2) are isometric.*

Proof. Let us again omit the sub-indexes of N, M, and F. For $y \in M$, $s \in \mathbb{R}$, and $z \in F$ we define a number

$$\begin{aligned}\mathcal{T}_{y,s}(z) \;=\; &\sup\{t \in \mathbb{R};\ \text{the point } (z,t) \text{ has a neighborhood}\\ &\qquad U \subset \mathcal{N} \text{ such that } G(\cdot,\cdot,y,s)|_U = 0\}\end{aligned}$$

which tells us, what is the first time when the wave $G(\cdot,\cdot,y,s)$ is observed near the point z. Using the finite velocity of the wave propagation for the wave equation, see [19], we see that the support of $G(\cdot,\cdot,y,s)$ is contained in the future light cone of the point $q = (y,s) \in \mathcal{N}$ given by

$$J^+(q) = \{(y',s') \in \mathcal{N};\ s' \geq d(y',y) + s\}.$$

Next, for $\xi = \xi^j \frac{\partial}{\partial x^j} \in T_yN$ we denote the corresponding co-vector by $\xi^\flat = g_{jk}(y)\xi^j dx^j$. Then the results of [17] and [18] on the propagation of singularities for the real principal type operators, in particular for the wave operator, imply that in the set $\mathcal{N}\setminus\{q\}$ Green's function $G(\cdot,\cdot,y,s)$ is a Lagrangian distribution associated to the Lagranian sub-manifold

$$\Sigma_0 = \{(\gamma_{y,\eta}(t), s+t; \dot\gamma_{y,\eta}(t)^\flat, dt) \in T^*\mathcal{N};\ \eta \in S_yN,\ t>0\}$$

and its principal symbol on Σ_0 is non-zero. In particular, [18, Prop. 2.1] implies that $\Sigma = \Sigma_0 \cup (T^*_qM \setminus \{0\}$ coincides with the wave front set $\mathrm{WF}(u)$ of the solution $u = G(\cdot,\cdot,y,s)$. This means that a wave emanating from a point source (y,s) propagates along the geodesics of manifold (N,g). The image of $\mathrm{WF}(u)$ in the projection $\pi : T^*\mathcal{N} \to \mathcal{N}$ coincides the singular support of u. Hence, we see that

$$\begin{aligned}(50)\qquad & \mathrm{singsupp}(G(\cdot,\cdot,y,s)) = S(q), \quad \text{where}\\ & S(q) = \{(\exp_y(t\eta), s+t) \in \mathcal{N}; \eta \in S_yN,\ t\geq 0\}.\end{aligned}$$

Since the Riemannian manifold N is complete, the space-time $\mathcal{N}$ is a globally hyperbolic Lorentzian manifold and we have $\partial J^+(q) = S(q)$, see [48]. Summarizing, the above implies that the function $G(\cdot,\cdot,y,s)$ vanishes outside $J^+(q)$ and is non-smooth, and thus non-zero, in a neighbourhood of arbitrary point of $\partial J^+(q)$. Thus, for $z \in F$ we have $\mathcal{T}_{y,s}(z) = d(z,y) - s$. Hence the distance difference functions satisfy equation

$$D_y(z_1,z_2) = \mathcal{T}_{y,s}(z_1) - \mathcal{T}_{y,s}(z_2). \tag{51}$$

Thus, when formulas (48)-(49) are valid, we see using equation (51). that the distance difference data of the manifolds N_1 and N_2 coinside, that is, we have (2)-(3). Hence, the claim follows from Theorem 1.3. □

Finally, we note that tets W_j are closely related to the light-observation sets studied in [32] in the study of the inverse problems for non-linear hyperbolic problems with a time-dependent metric. The light-observation set $P_U(q)$ corresponding to a source point $q = (y,s)$ and the observation set U is the intersection of U and the future light cone emanating from q. In fact, the formula (50) implies that in the space time $\mathcal{N} = N \times \mathbb{R}$ the sets W_j coincide with the light-observation sets $P_U(q)$ corresponding to a source point $q = (y,s)$ and the observation set $U = F \times R$.

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4. Appendix A: Extensions of data

Assume that we are given the set $F = N \setminus M$ and the metric $g|_F$, but instead the function $D_x : F \times F \to \mathbb{R}$ we know only its restriction on the boundary $\partial F = \partial M$, that is, the map

$$D_x|_{\partial F\times\partial F} : \partial F \times \partial F \to \mathbb{R},\ D_x|_{\partial F\times\partial F}(z_1, z_2) := d_N(z_1, x) - d_N(z_2, x).$$

Lemma 4.1. *The manifold $F = N \setminus M$, the metric $g|_F$, and the restriction $D_x|_{\partial F\times\partial F}$ of the distance difference function corresponding to $x \in M$ determine the distance difference function $D_x : F \times F \to \mathbb{R}$.*

Proof. We can determine the map $D_x : F \times F \to \mathbb{R}$ by the formula

$$D_x(z_1, z_2) = \inf_\alpha \sup_\beta \Big(\mathcal{L}(\alpha) + D_x|_{\partial F\times\partial F}(\alpha(1), \beta(1)) - \mathcal{L}(\beta)\Big),$$

where the infimum is taken over the smooth curves $\alpha : [0, 1] \to F$ from z_1 to $\alpha(1) \in \partial F$ and the supremum is taken over the smooth curves $\beta : [0, 1] \to F$ from z_2 to $\beta(1) \in \partial F$. □

This raises the question, if the manifold (N, g) can be reconstructed when we are given a submanifold of codimension 1, e.g. the boundary of the open set M considered above, and the distance difference functions on this submanifold. To consider this, assume that we are given a submanifold $\widetilde{F} \subset N$ of dimension $(n-1)$, the metric $g|_{\widetilde{F}}$ on $\widetilde{F}$, and the collection

$$\{D^x_{\widetilde{F},N};\ x \in N\} \subset C(\widetilde{F} \times \widetilde{F}),$$

where $D^x_{\widetilde{F},N}(z_1, z_2) = d_N(x, z_1) - d_N(x, z_2)$ for $z_1, z_2 \in \widetilde{F}$. The following counterexample shows that such data do not uniquely determine the isometry type of (N, g).

Example A1. Let $C_r(y) = \{(x_1, x_2) \in \mathbb{R}^2;\ |x_1 - y_1|^2 + |x_2 - y_2|^2 = r^2\}$ be a circle of radius r centered at $y = (y_1, y_2)$. Let $p_1 = (2, 0)$, $p_2 = (-2, 0)$, $L > 3$, and

$$S_0 = C_1(0) \times [-1, 1],$$
$$S_1 = C_1(p_1) \times [2, L],$$
$$S_2 = C_1(p_2) \times [2, L],$$

and $K \subset \mathbb{R}^2 \times [1, 2]$ be a 2-dimensional surface which boundary has three components, $C_1(0) \times \{1\}$, $C_1(p_1) \times \{2\}$, and $C_1(p_2) \times \{2\}$, such that the union $S_0 \cup K \cup S_1 \cup S_2$ is a smooth surface in $\mathbb{R}^3$. Moreover, let $\mathcal{R} : (x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$ denote the reflection in the x_3-variable. Observe that then $\mathcal{R}(S_0) = S_0$. We define a smooth surface

$$\Sigma_0 = S_0 \cup K \cup S_1 \cup S_2 \cup \mathcal{R}(K) \cup \mathcal{R}(S_1) \cup \mathcal{R}(S_2).$$

The boundary of Σ_0 consists of 4 circles, namely $\Gamma_1 = C_1(p_1) \times \{L\}$, $\Gamma_2 = C_1(p_1) \times \{-L\}$, $\Gamma_3 = C_1(p_2) \times \{L\}$, and $\Gamma_4 = C_1(p_2) \times \{-L\}$. Let us consider four embedded Riemannian surfaces $\Sigma_j \subset \mathbb{R}^3$, $j = 1, 2, 3, 4$,

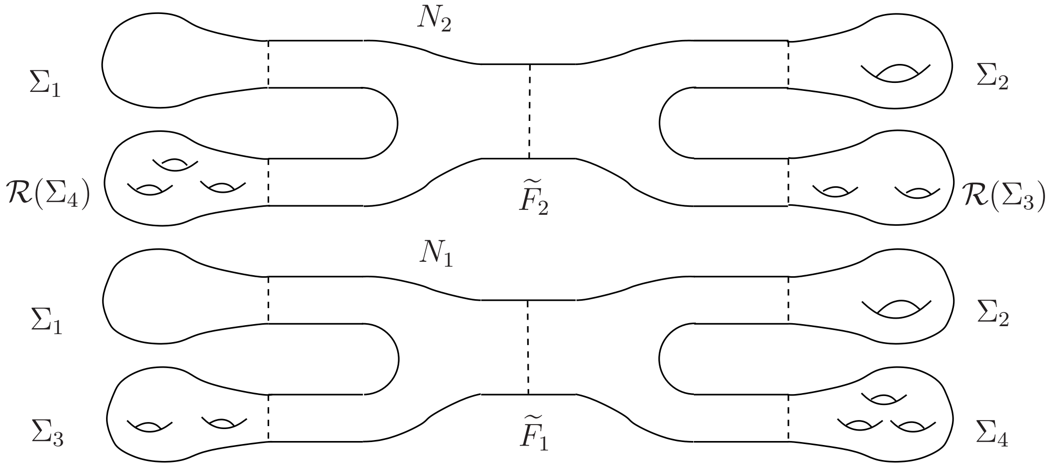

FIGURE 5. An illustration of manifolds N_1 and N_2 in Example $A1$. When $(n-1)$-dimensional submanifolds $\widetilde{F}_1 = \widetilde{F}_2 = \widetilde{F}$ are identified, the distance difference functions $\{\mathcal{D}^x_{\widetilde{F},N_1};\ x \in N_1\}$ and $\{\mathcal{D}^x_{\widetilde{F},N_2};\ x \in N_2\}$ coincide.

with boundaries $\partial\Sigma_j$ are equal to Γ_j. Assume that near $\partial\Sigma_j$ the surfaces Σ_j are isometric to the Cartesian product of Γ_j and an interval $[0,\varepsilon]$ with $\varepsilon > 0$, and that the genus of Σ_j is equal to $(j-1)$. Also, assume that $\Sigma_j \cap \Sigma_k = \emptyset$ for $j,k = 1,2,3,4$ and $\Sigma_0 \cap \Sigma_j = \Gamma_j$ for $j = 1,2,3,4$.

First, let us construct a manifold N_1 by gluing surfaces Σ_0 with $\Sigma_1, \Sigma_2, \Sigma_3$, and Σ_4 such that the boundaries Γ_j are glued with $\partial\Sigma_j$, $j \in \{1,2,3,4\}$.

Second, we construct a manifold N_2 by gluing surfaces Σ_0 with $\Sigma_1, \Sigma_2, \mathcal{R}(\Sigma_3)$, and $\mathcal{R}(\Sigma_4)$ such that the boundaries Γ_j are glued with $\partial\Sigma_j$ with $j \in \{1,2\}$ but Γ_3 is glued with $\mathcal{R}(\partial\Sigma_4)$ and Γ_4 is glued with $\mathcal{R}(\partial\Sigma_3)$, see Fig. 5. For both manifolds N_1 and N_2 we give the induced Riemannian metric from $\mathbb{R}^3$. Let $\widetilde{F} = \widetilde{F}_1 = \widetilde{F}_2 = S_0 \cap (\mathbb{R}^2 \times \{0\})$.

Let us assume that L above is larger than diam $(K) + 10$. Then on N_ℓ, $\ell = 1,2$ a minimizing geodesic from $x \in \Sigma_j$, $j \geq 1$ to $z \in \widetilde{F}$ does not intersect the other sets Σ_k with $k \in \{1,2,3,4\} \setminus \{j\}$. Using this we see that the sets $\{D^x_{\widetilde{F},N_\ell};\ x \in N_\ell\} \subset C(\widetilde{F} \times \widetilde{F})$ are the same for $\ell = 1,2$. As the manifolds N_1 and N_2 are not isometric, this implies that the data $(\widetilde{F}, g|_{\widetilde{F}})$ and $\{D^x_{\widetilde{F},N};\ x \in N\}$ do not determine uniquely the manifold (N,g).

5. Appendix B: Integrals of the geodesic flow

In this appendix we consider Matveev-Topalov theorem [41] in detail. The motivation to write this rather long appendix is that we thought that the methods used in [41], [42] and [43] are not familiar to the general audience in the field of geometric inverse problems. The appendix is self contained in many ways, but the reader is assumed to be familiar with the basics of Riemannian geometry. The notations and theory used here are mostly from [38].

5.1. **Part I, Matrix representations.** Let M be a smooth n-manifold with Riemannian metric tensors $g = g_{ij}$ and $\widetilde{g} = \widetilde{g}_{ij}$.

Denote by $G : TM \to TM$ a fiberwise linear mapping given by 1-covariant 1-contravariant tensor $G^i_j = g^{i\alpha}\widetilde{g}_{\alpha j}$. For any $x \in M$ and $v \in T_xM$ this is defined as

$$G_x(v) = g^{i\alpha}\widetilde{g}_{\alpha j}v_j.$$

Let $\chi_G := \det(G - tId_{TM})$ be a characteristic polynomial of G. Since M is n-dimensional, we can write χ_G in form

$$\chi_G(t) = c_0 t^n + c_1 t^{n-1} + \ldots + c_n, \tag{52}$$

where coefficients $c_1, \ldots, c_n$ are smooth functions on M and $c_0 \equiv (-1)^n$. Define mappings $S_k : TM \to TM$, $k \in \{0, \ldots, n-1\}$ by formula

$$S_k((x,v)) := \Big(\frac{\det g_x}{\det \widetilde{g}_x}\Big)^{\frac{k+2}{n+1}} \sum_{i=0}^{k} c_i G^{k-i+1}(v). \tag{53}$$

For every $k \in \{0, \ldots, n-1\}$ we finally define functions $I_k : TM \to \mathbb{R}$ with formula

$$I_k((x,v)) = g_x(S_k((x,v)), v).$$

After these preparations we can state the main result of [41].

Theorem 5.1 (Matveev-Topalov)**.** *Let $M, n \geq 2$ be a smooth manifold with geodesically equivalent Riemannian metrics g and $\widetilde{g}$. Let θ be the geodesic flow of Riemannian manifold (M, g). Then functions I_k are integrals of flow θ. This means that each function I_k is constant on each orbit of geodesic flow θ.*

We will provide the proof given by the Matveev and Topalov in detail, reviewing techniques used in [41], [42], and [43], but we first have to do some preparations. We start with finding a formula for mapping S_k. Let $x \in M$.

We first show that there exists such a basis $(w_i)_{i=1}^{j}$ of T_xM such that in this basis

$$g = \mathrm{diag}(1, \ldots, 1) \text{ and } \widetilde{g} = \mathrm{diag}(\rho_1, \ldots, \rho_n)$$

for some $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n > 0$. Let (U, φ) be the Riemannian normal coordinates at x with respect to g. Then it holds that $g_{ij}(x) = \langle \partial\varphi_k, \partial\varphi_j \rangle_g = \delta_{jk}$. Let $v_k = \partial\varphi_k(x)$. Write

$$a_{jk} := \widetilde{g}_x(v_j, v_k) = \widetilde{g}_x(v_k, v_j) \text{ and } A = [a_{jk}]_{j,k=1}^n.$$

Since metric tensor $\widetilde{g}$ is positive definite, it holds that there exists a set of vectors $r_1, \ldots, r_n \in \mathbb{R}^n$ and set of strictly positive real numbers $\rho_1, \ldots, \rho_n$ such that

$$Ar_j = \rho_j r_j, \text{ and } r_j \cdot r_k = \delta_{jk}.$$

Note that vector $r_j = (r_j^1, \ldots, r_j^n) \in \mathbb{R}^n$. Define vectors

$$w_j := r_j^l v_l \in T_x M.$$

Then it holds that

$$g(w_j, w_k) = g(r_j^l v_l, r_k^l v_l) = r_j^l r_k^\ell g(v_l, v_\ell) = \sum_{l=1}^n r_j^l r_k^l = r_j \cdot r_k = \delta_{jk}$$

and

$$\widetilde{g}(w_j, w_k) = r_j^l r_k^\ell \widetilde{g}(v_l, v_\ell) = a_{l\ell} r_j^l r_k^\ell = r_k \cdot (Ar_j) = r_k \cdot \rho_j r_j = \rho_j \delta_{jk}.$$

These calculations prove that $(w_j)_{j=1}^n$ is a basis of $T_x M$ and in this basis we have

$$g = \mathrm{diag}(1, \ldots, 1) \text{ and } \widetilde{g} = \mathrm{diag}(\rho_1, \ldots, \rho_n).$$

We say that σ_p is the unique elementary symmetric polynomial of degree p of n variables, if $p \leq n$ and

$$\sigma_p(X_1, \ldots, X_n) = \sum_{1 \leq j_1 < j_2 < \ldots < j_p} \prod_{k=1}^p X_{j_k}.$$

Let

$$\phi_i := \frac{1}{\rho_i} (\prod_{k=1}^n \rho_k)^{1/(n+1)}.$$

Then

$$\prod_{k=1}^n \rho_k = \det(G_x) = \det(g^{-1}(x) \widetilde{g}(x)) > 0,$$

and the numbers $\phi_i \in \mathbb{R}$ satisfy$\phi_1 \leq \phi_2 \leq \ldots \leq \phi_n$.

Let σ_p be the elementary symmetric polynomial of degree p of variables $\phi_1, \ldots, \phi_n$ and $\sigma_p(\widehat{\phi_i})$ the elementary symmetric polynomial of degree $p-1$ of variables $\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_n$.

Lemma 5.2. *The matrices of mappings S_k are given by*

$$S_k = (-1)^{n-k} diag(\sigma_{n-k-1}(\widehat{\phi_1}), \sigma_{n-k-1}(\widehat{\phi_2}), \ldots, \sigma_{n-k-1}(\widehat{\phi_n})).$$

Proof. Recall that the coefficients c_k of (53) are the same as the ones in (52). We will start by showing that the coefficients c_k can be calculated as

$$c_k = (-1)^{n-k} \frac{\sigma_{n-k}}{(\phi_1 \phi_2 \cdots \phi_n)^{k+1}}.$$

Let $k \in \{0, \dots, n\}$. First we calculate

$$(\phi_1\phi_2\cdots\phi_n)^{k+1} = (\prod_{j=1}^{n}\rho_j)^{-(k+1)}(\prod_{j=1}^{n}\rho_j)^{\frac{(k+1)n}{n+1}} = (\prod_{j=1}^{n}\rho_j)^{\frac{(k+1)n}{n+1}-(k+1)}.$$

If $k = n$ the calculation above and (52) prove that

$$c_n = \det G = \prod_{j=1}^{n}\rho_j = (\phi_1\phi_2\dots\phi_n)^{-(n+1)}.$$

Next we consider the case $k \in \{0, \dots, n-1\}$. By definition

$$\sigma_{n-k} = \sum_{1\le j_1<j_2<\cdots<j_{n-k}} \prod_{k=1}^{n-k}\phi_{j_k} = (\prod_{j=1}^{n}\rho_j)^{\frac{n-k}{n+1}} \sum_{1\le j_1<j_2<\cdots<j_{n-k}} \prod_{k=1}^{n-k}\frac{1}{\rho_{j_k}}.$$

Now it holds

$$\frac{\sigma_{n-k}}{(\phi_1\phi_2\cdots\phi_n)^{k+1}} = (\prod_{j=1}^{n}\rho_j)^{\frac{n-k}{n+1}-\frac{(k+1)n}{n+1}-(k+1)} \sum_{1\le j_1<j_2<\cdots<j_{n-k}} \prod_{k=1}^{n-k}\frac{1}{\rho_{j_k}}$$

$$= (\prod_{j=1}^{n}\rho_j) \sum_{1\le j_1<j_2<\cdots<j_{n-k}} \prod_{k=1}^{n-k}\frac{1}{\rho_{j_k}} = \sum_{1\le j_1<j_2<\cdots<j_{n-k}} \frac{\rho_1\rho_2\cdots\rho_n}{\prod_{k=1}^{n-k}\rho_{j_k}} = (-1)^{n-k}c_k.$$

To verify the claim of this Lemma, we use induction on k. For $k = 0$ we see that

$$S_0 = \Big(\frac{1}{\det G}\Big)^{\frac{2}{n+1}} c_0 G = (-1)^n(\phi_1\phi_2\dots\phi_n)^2 \mathrm{diag}(\rho_1, \rho_2, \dots \rho_n)$$

$$= (-1)^n(\phi_1\phi_2\dots\phi_n)^2 \mathrm{diag}\Big(\frac{1}{\phi_1(\phi_1\phi_2\dots\phi_n)}, \frac{1}{\phi_2(\phi_1\phi_2\dots\phi_n)}, \dots, \frac{1}{\phi_n(\phi_1\phi_2\dots\phi_n)}\Big)$$

$$= (-1)^n \mathrm{diag}(\phi_2\phi_3\dots\phi_n, \phi_1\phi_3\dots\phi_n, \dots, \phi_1\phi_2\dots\phi_{n-1})$$

$$= (-1)^n \mathrm{diag}(\sigma_{n-1}(\widehat{\phi_1}), \sigma_{n-1}(\widehat{\phi_2}), \dots, \sigma_{n-1}(\widehat{\phi_n})).$$

Next we assume that matrix S_{k-1} has form

$$S_{k-1} = (-1)^{n-(k-1)} \mathrm{diag}(\sigma_{n-(k-1)-1}(\widehat{\phi_1}), \sigma_{n-(k-1)-1}(\widehat{\phi_2}), \dots, \sigma_{n-(k-1)-1}(\widehat{\phi_n})).$$

Now it holds that

$$S_k = \Big(\frac{1}{\det G}\Big)^{\frac{1}{n+1}} G\Big(S_{k-1} + \Big(\frac{1}{\det G}\Big)^{\frac{k+1}{n+1}} c_k \,\mathrm{Id}\Big)$$

$$= (\phi_1\phi_2\dots\phi_n)\mathrm{diag}\ (\rho_1, \dots, \rho_n)((-1)^{n-(k-1)}\mathrm{diag}(\sigma_{n-k}(\widehat{\phi_1}), \sigma_{n-k}(\widehat{\phi_2}), \dots, \sigma_{n-k}(\widehat{\phi_n}))$$

$$+(\phi_1\phi_2\dots\phi_n)^{k+1}(-1)^{n-k}\frac{\sigma_{n-k}}{(\phi_1\phi_2\dots\phi_n)^{k+1}}\ \mathrm{Id})$$

$$= \mathrm{diag}\ (\phi_1^{-1}, \phi_2^{-1}, \dots, \phi_n^{-1})(-1)^{n-k}$$

$$* \mathrm{diag}\ (\sigma_{n-k} - \sigma_{n-k}(\widehat{\phi_1}), \sigma_{n-k} - \sigma_{n-k}(\widehat{\phi_2}), \dots, \sigma_{n-k} - \sigma_{n-k}(\widehat{\phi_n}))$$

$$= (-1)^{n-k}\mathrm{diag}\ \Big(\frac{\sigma_{n-k} - \sigma_{n-k}(\widehat{\phi_1})}{\phi_1}, \dots, \frac{\sigma_{n-k} - \sigma_{n-k}(\widehat{\phi_n})}{\phi_n}\Big).$$

The claims follows, since for every l polynomial $\sigma_l - \sigma_l(\widehat{\phi_i})$ is precisely sum of those l-products of ϕ_j's which all contain ϕ_i. Clearly same argument holds for polynomial $\phi_i \sigma_{l-1}(\widehat{\phi_i})$. □

Next we define a function $F : \mathbb{R} \times TM \to \mathbb{R}$ by formula

$$F_t(x,\xi) = t^{n-1} I_{n-1}(x,\xi) + \ldots + I_0(x,\xi).$$

For a fixed point (x,ξ) in tangent bundle of M, function $F_t(x,\xi)$ is a polynomial of t of degree $n-1$. Let the complex roots of polynomial $F_t(x,\xi)$ be $t_1(x,\xi), \ldots, t_{n-1}(x,\xi)$.

Lemma 5.3. *Let $x \in M$. Then for every $i \in \{1, 2, \ldots, n-1\}$ the following statements are true: For any $\xi \in T_xM$ the roots $t_1(x,\xi), \ldots, t_{n-1}(x,\xi)$ are real and*

$$\phi_i(x) \leq t_i(x,\xi) \leq \phi_{i+1}(x).$$

Proof. Fix a point $(x,\xi) \in TM$. For simplicity we write $t_i := t_i(x,\xi)$ and $\phi_i := \phi_i(x)$. Choose such a basis in T_xM that

$$g = \text{diag}\ (1, \ldots, 1) \text{ and } G = \text{diag}\ (\rho_1, \ldots, \rho_n).$$

Let P_i be the polynomial

$$P_i(t) := \prod_{k=1,\, k\neq i}^{n} (t - \phi_k) = \sum_{\alpha=0}^{n-1} (-1)^{n-\alpha-1}\ t^\alpha\ \sigma_{n-\alpha-1}(\widehat{\phi_i}).$$

Recall that we have defined the mapping $I_k(\xi) := g_x(S_k\xi, \xi)$. By the Lemma 5.2 we have that

$$I_k(\xi) = (-1)^{n-k} \sum_{i=1}^{n} \sigma_{n-k-1}(\widehat{\phi_i}) \xi_i^2.$$

Thus we can write polynomial $F_t(x,\xi)$ in form

$$\begin{aligned} F_t(x,\xi) &= \sum_{k=0}^{n-1} (-1)^{n-k}\ t^k \sum_{i=1}^{n} \sigma_{n-k-1}(\widehat{\phi_i}) \xi_i^2 \\ &= -\sum_{i=1}^{n} \xi_i^2 \sum_{k=0}^{n-1} (-1)^{n-k-1} t^k \sigma_{n-k-1}(\widehat{\phi_i}) \\ &= -(P_1(t)\xi_1^2 + P_2(t)\xi_2^2 + \ldots + P_n(t)\xi_n^2). \end{aligned}$$

Recall that $\phi_i \leq \phi_{i+1}$. We will split the rest of the proof to three different cases.

Suppose first that $\phi_i < \phi_{i+1}$ and $\xi_i \neq 0$ for every $i \in \{1, \ldots, n-1\}$. By the definition of polynomial P_j we see immediately that $P_j(\phi_i) = 0$ if $i \neq j$. Therefore it holds that

$$F_{\phi_i} = -P_i(\phi_i)\xi_i^2.$$

According to the definition of polynomials P_i and P_{i+1} it holds that

$$P_i(\phi_i) = \underbrace{(\phi_i - \phi_1)(\phi_i - \phi_2)\dots(\phi_i - \phi_{i-1})}_{>0}(\phi_i - \phi_{i+1})(\phi_i - \phi_{i+2})\dots(\phi_i - \phi_n)$$

and

$$P_{i+1}(\phi_{i+1}) = \underbrace{(\phi_{i+1} - \phi_1)(\phi_{i+1} - \phi_2)\dots(\phi_{i+1} - \phi_{i-1})(\phi_{i+1} - \phi_i)}_{>0}$$

$$\cdot(\phi_{i+1} - \phi_{i+2})\dots(\phi_{i+1} - \phi_n).$$

Therefore numbers F_{ϕ_i} and $F_{\phi_{i+1}}$ have different signs. Since F is real valued and continuous, it has atleast one root in interval $]\phi_i, \phi_{i+1}[$. Since degree of F is $n-1$ and intervals $]\phi_i, \phi_{i+1}[$ are disjoint we see that F has exactly one root t_i in interval $]\phi_i, \phi_{i+1}[$.

If $\phi_i \leq \phi_{i+1}$ and $\xi_k = 0$ for some $k \in 1, \dots, n-1$, we have that $F_{\phi_k} = 0$ and $\phi_k = t_k$.

If $\phi_k = \phi_{k+1}$ for some $k \in 1, \dots, n-1$, then $P_k(\phi_k) = 0$ and therefore $F_{\phi_k} = 0$.

Therefore for all the cases we have proved the claim of this Lemma. □

5.2. **Part II, Hamiltonian systems on regular level sets.** Let (M^{2n}, ω, H) and $(\widetilde{M}^{2n}, \widetilde{\omega}, \widetilde{H})$ be Hamiltonian systems with Hamiltonian vector fields V and, $\widetilde{V}$ respectively (For concepts, not explained here, see [38] chapter 18.). Suppose that h and $\widetilde{h}$ are regular values of H and, $\widetilde{H}$ respectively. We define regular level sets

$$Q = \{x \in M^{2n} : H(x) = h\}$$

and

$$\widetilde{Q} = \{x \in \widetilde{M}^{2n} : \widetilde{H}(x) = \widetilde{h}\}.$$

Lemma 5.4. *Let $x \in Q$. Then $T_xQ = \{X \in T_xM : XH = 0\}$.*

Proof. Let $i : Q \hookrightarrow M$ be the inclusion mapping. We first prove that

$$T_xQ = \mathrm{Ker} H_*, \; H_* : T_xM^{2n} \to T_h\mathbb{R}.$$

Since Q is a smooth submanifold of M^{2n} we will identify T_xQ with $i_*(T_xQ) \subset T_xM^{2n}$. Since Q is a level set of H we have that $H \circ i$ is constant. Therefore

$$(H \circ i)_* X = X(H \circ i) = 0, \text{ for every } X \in T_xQ.$$

But this means that mapping $(H \circ i)_* : T_xQ \to T_h\mathbb{R}$ is a zero mapping. Since $(H \circ i)_* = H_* \circ i_*$, we deduce $\mathrm{Im}\, i_* \subset \mathrm{Ker} H_*$. Since h is a regular value, we know that H_* is surjective. By rank-nullity law and since $\dim Q = \dim M^{2n} - \dim \mathbb{R}$ we have

$$\dim \mathrm{Ker} H_* = \dim T_xM^{2n} - \dim \mathbb{R} = \dim T_xQ = \dim \mathrm{Im}\, i_*.$$

This proves the claim $T_xQ = \mathrm{Ker} H_*$. Let $f \in C^\infty(\mathbb{R})$ and $X \in T_xM^{2n}$. Then we have

$$(H_*X)f = X(f \circ H) = (\dot{f} \circ H)XH.$$

According to what we proved earlier, we know $X \in T_xQ$ if and only if $H_*X = 0$. By preceding formula this is true if and only if $XH = 0$. □

Let $U \subset M^{2n}$ and $\widetilde{U} \subset \widetilde{M}^{2n}$ be neighborhoods of Q and $\widetilde{Q}$, respectively.

Definition 5.5. *Diffeomorphism $\Phi : U \to \widetilde{U}$ is orbital on Q if $\Phi(Q) = \widetilde{Q}$ and $\Phi|_Q$ maps orbits of V to the orbits of $\widetilde{V}$ and vice versa. In other words this means that for every orbit $\gamma : [a, b] \to Q$ of V there exists a diffeomorphisim $\alpha : [c, d] \to [a, b]$ such that*

$$\frac{d}{dt}(\Phi \circ \gamma \circ \alpha)|_{t=t_0} = \widetilde{V}_{(\Phi\circ\gamma\circ\alpha)(t_0)}.$$

Since Q is a regular level set of smooth mapping $H : M^{2n} \to \mathbb{R}$ on manifold M^{2n} it has dimension $2n - 1$. Let ϕ be the restriction $\Phi|_Q$.

Lemma 5.6. *There exists functions $a_1 : Q \to \mathbb{R}$ and $a_2 : \widetilde{Q} \to \mathbb{R}$ such that for every $p \in Q$ holds:*

$$\Phi_*(V_p) = a_1(p)\widetilde{V}_{\Phi(p)} \text{ and } a_2(\Phi(p))\Phi_*(dH)_p = (d\widetilde{H})_{\Phi(p)}.$$

Proof. Since V is a Hamiltonian vector field on M^{2n} and Q is a regular level set of H, we know that $V|_Q$ is also a smooth vector field on Q ([38] 18.22). Let $p \in Q$. Since V is the Hamiltonian vector field on Q, there exists an orbit γ of V on Q such that $\gamma_0 = p$ and $\dot{\gamma}_0 = V_p$. Since Φ is orbital we have

$$\begin{aligned}
\Phi_*V_p &= \Phi_*\dot{\gamma}_0 \\
&= \Phi_*(\frac{d}{dt}(\gamma \circ \alpha)|_{t=\alpha^{-1}(0)}\frac{d}{dt}\alpha^{-1}|_{t=0}) \\
&= \frac{d}{dt}\alpha^{-1}|_{t=0}\Phi_*(\frac{d}{dt}(\gamma \circ \alpha)|_{t=\alpha^{-1}(0)}) \\
&= \frac{d}{dt}\alpha^{-1}|_{t=0}\frac{d}{dt}(\Phi \circ \gamma \circ \alpha)|_{t=\alpha^{-1}(0)} \\
&= \frac{d}{dt}\alpha^{-1}|_{t=0}\widetilde{V}_{\Phi(p)}.
\end{aligned}$$

Now we define that $a_1(p) = \frac{d}{dt}\alpha^{-1}|_{t=0}$. Since vector fields Φ_*V and $\widetilde{V}$ are smooth and the smoothness of vector fields is equivalent to the smoothness of its coefficient functions we deduce that a_1 is smooth.

Let $p \in M^{2n}$ and $X \in T_pQ$. To show that $a_2\Phi_*dH = d\widetilde{H}$, for some smooth function a_2, it suffices by Lemma 5.4 to prove that $\Phi^*d\widetilde{H}(X) =$

0. Since Φ maps Q onto $\widetilde{Q}$ we have

$$0 = X(\widetilde{H} \circ \Phi) = \Phi_* X(\widetilde{H}) = \Phi^* d\widetilde{H}(X).$$

□

Let $\sigma : TQ \to T^*Q$ be the restriction $\omega|_Q$ and $\widetilde{\sigma} = \widetilde{\omega}|_{\widetilde{Q}}$, respectively. Since ϕ takes Q onto $\widetilde{Q}$, we can consider $\phi^*\widetilde{\sigma}$ as a 2-form on Q.

Lemma 5.7. *Let θ be the flow of V. Then θ preserves the form $\phi^*\widetilde{\sigma}$.*

Proof. According to Theorem 18.16 of [38], it suffices to show that the Lie derivative $\mathcal{L}_V(\phi^*\widetilde{\sigma}) = 0$. Let us verify this using Cartan's formula

$$\mathcal{L}_V(\phi^*\widetilde{\sigma}) = d(\iota_V(\phi^*\widetilde{\sigma})) + \iota_V(d(\phi^*\widetilde{\sigma})). \tag{54}$$

Let $X \in TQ$. By the Lemma 5.6 we have that

$$\iota_V(\phi^*\widetilde{\sigma})(X) = \widetilde{\sigma}(\phi_*V, \phi_*X) = \widetilde{\sigma}(a_1\widetilde{V}, \phi_*X)$$
$$= a_1 d\widetilde{H}(\phi_*X) = a\phi_*X(H) = aX(H \circ \phi) = 0.$$

Hence the first term of (54) vanishes. We defined that $\widetilde{\sigma} = \widetilde{\omega}|_{\widetilde{Q}}$. Since $\widetilde{Q}$ is a submanifold of $\widetilde{M}^{2n}$, the inclusion mapping $i : \widetilde{Q} \hookrightarrow \widetilde{M}^{2n}$ is a smooth embedding. Therefore for all $X, Y \in \mathcal{T}(\widetilde{Q})$ it holds that

$$i^*\widetilde{\omega}(X, Y) = \widetilde{\omega}(i_*X, i_*Y) = \widetilde{\omega}(X, Y) = \widetilde{\sigma}(X, Y),$$

i.e., $\widetilde{\sigma} = i^*\widetilde{\omega}$. Since form $\widetilde{\omega}$ is closed on $\widetilde{M}^{2n}$, we know that $\widetilde{\sigma}$ is closed on $\widetilde{Q}$ by Lemma 12.16 of [38]. Now we also have that the second term of (54) vanishes, since $d(\phi^*\widetilde{\sigma}) = \phi^*(d\widetilde{\sigma}) = 0$ by equation (12.18) of [38]. □

We can consider 2-form ω on M^{2n} as a mapping

$\omega : TM^{2n} \to T^*M^{2n}, \omega(v)(w) = \omega(v, w)$ for all $x \in M^{2n}, v, w \in T_xM^{2n}$.

Since form ω is non-degenerate, we know that mapping ω is a linear isomorphism on fibers T_xM^{2n}. Let $\sigma := \omega|_Q : TQ \to T^*Q$. We want to show that

$$\text{Ker } \sigma|_{TxQ} = \text{span}(V(x)).$$

We start with observing that $V(x) \in T_xQ$ if $x \in Q$. This holds since $V(H) = \omega(V, V) = 0$ and therefore V satisfies the conditions of Lemma 5.4.

Next we show that $\omega(V, X) = 0$ for any $X \in T_xQ$. Let γ be such a smooth path on Q that at point x it has a velocity X. Since Q is a level set, we have

$$0 = \frac{d}{dt}(H \circ \gamma) = VH = \omega(V, X).$$

We denote by A^ω the sympletic complement of set A with respect to sympletic form ω. Suppose that for $X \in T_xQ$ it holds $\sigma(X, \cdot) \equiv 0$. Therefore dim span$(X, V)^\omega = 2n-1$, since for every $W \in T_xQ$ we have

$$\omega(aV + bX, W) = a\omega(V, W) + b\omega(X, W) = 0.$$

On the other hand it holds that

$$\dim \operatorname{span}(V, X)^{\omega} + \dim \operatorname{span}(V, X) = 2n$$

$$\Rightarrow \dim \operatorname{span}(X, V) = 1.$$

But this means precisely that $X \in \operatorname{span}(V)$. Denote $\operatorname{span}(V) = \langle V \rangle$.

By considerations made above we know that $\langle V \rangle = \operatorname{Ker} \sigma$. The kernel of form $\phi^* \widetilde{\sigma}$ is also $\langle V \rangle$, since by Lemma 5.6 we have for every $X \in T_x Q$ that

$$\phi^* \widetilde{\sigma}(V, \widetilde{X}) = \widetilde{\sigma}(\phi_* V, \phi_* X) = \widetilde{\sigma}(a_1 \widetilde{V}, \phi_* X) = a_1 \phi_* X(\widetilde{H}) = a_1 X(\widetilde{H} \circ \phi) = 0,$$

since $\phi(Q) = \widetilde{Q}$.

Let $TQ/\langle V \rangle$ be the quotient bundle of TQ i.e. the fiber of $TQ/\langle V \rangle$ is the vector space $T_x Q/\langle V_x \rangle$. Next we consider two induced tensor fields σ and $\phi^* \widetilde{\sigma}$ on quotient bundle $TQ/\langle V \rangle$, i.e., we define for $[X], [Y] \in TQ/\langle V \rangle$

$$\sigma([X], [Y]) = \sigma(X, Y) \text{ and } \phi^* \widetilde{\sigma}([X], [Y]) = \phi^* \widetilde{\sigma}(X, Y).$$

These forms are well defined, since both forms σ and $\phi^* \widetilde{\sigma}$ have the same kernel $\langle V \rangle$. These induced forms are both nondegenerate since

$$\sigma([X], \cdot) \equiv 0 \text{ if and only if } X \in \langle V \rangle .$$

Therefore we can define an operator $\sigma^{-1}(\phi^* \widetilde{\sigma})$ on the quotient bundle $TQ/\langle V \rangle$, if we again consider forms as a fiber vice linear mappings from $TQ/\langle V \rangle$ to $(TQ/\langle V \rangle)^*$.

Let $p \in Q$. Since p is a regular point for H, we can choose the Darboux coordinates $x_1, y_1, \ldots, x_n, y_n$ for p such that $y_1 = H$ and $V = \partial x_1 := \frac{\partial}{\partial x_1}$. By Lemma 5.4 it holds that vector fields $\partial x_1, \partial x_2, \partial y_2, \ldots, \partial x_n, \partial y_n$ are tangential to Q at p since by Darboux criterion

(55)
$$0 = \{y_1, y_k\}_{\omega} = \{H, y_k\}_{\omega} = \partial x_k H \text{ and } -\delta_{1k} = \{y_1, x_k\}_{\omega} = -\partial y_k H.$$

According to Lemma 5.6 there exists functions a_1 and a_2 such that

$$\Phi_*(V_p) = a_1(p) \widetilde{V}_{\Phi(p)} \text{ and } a_2(\Phi(p)) \Phi_*(dH)_p = (d\widetilde{H})_{\Phi(p)}.$$

We denote $a := a_1 a_2$. Therefore

$$\Phi^* \widetilde{\omega}(\partial x_1, \partial y_1) = \widetilde{\omega}(\Phi_* \partial x_1, \Phi_* \partial y_1) = \widetilde{\omega}(\Phi_* \partial x_1, \Phi_* \partial y_1) = a_1 \widetilde{\omega}(\widetilde{V}, \Phi_* \partial y_1)$$

$$= a_1 d\widetilde{H}(\Phi_* \partial y_1) = a_1 a_2 (\Phi_* dH)(\Phi_* \partial y_1)) = a dH(\partial y_1) = a \frac{\partial y_1}{\partial y_1} = a.$$

With similar computations one gets

$$\Phi^* \widetilde{\omega}(\partial x_1, \partial x_k) = 0 \text{ and also } \Phi^* \widetilde{\omega}(\partial x_1, \partial y_j) = 0,\ j \neq 1.$$

This means that the matrices of ω^{-1} and $\Phi^*\widetilde{\omega}$ are of the form

$$\omega^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots 0 \\ -1 & 0 & 0 & 0 & \dots 0 \\ 0 & 0 & & & \\ 0 & 0 & & & \\ \vdots & \vdots & & A^{-1} & \\ 0 & 0 & & & \end{pmatrix} \tag{56}$$

and

$$\Phi^*\widetilde{\omega} = \begin{pmatrix} 0 & a & 0 & 0 & \dots 0 \\ -a & 0 & c_1 & c_2 & \dots c_{2n-2} \\ 0 & -c_1 & & & \\ 0 & -c_2 & & & \\ \vdots & \vdots & & B & \\ 0 & -c_{2n-2} & & & \end{pmatrix}. \tag{57}$$

Therefore we get that

$$\omega^{-1}\Phi^*\widetilde{\omega} = \begin{pmatrix} -a & 0 & c_1 & c_2 & \dots & c_{2n-2} \\ 0 & -a & 0 & 0 & \dots & 0 \\ 0 & d_1 & & & & \\ 0 & d_2 & & & & \\ \vdots & \vdots & & A^{-1}B & & \\ 0 & d_{2n-2} & & & & \end{pmatrix}. \tag{58}$$

This proves that

$$\det(\omega^{-1}\Phi^*\widetilde{\omega}) = a^2 \det(A^{-1}B). \tag{59}$$

Matrix $A^{-1}B$ is now the matrix of mapping $\sigma^{-1}\phi^*\widetilde{\sigma}$, since by formula (55) we get $\sigma^{-1}\phi^*\widetilde{\sigma}$ by removing the first and second rows and columns of matrix (58).

Lemma 5.8. *For every $t \in \mathbb{R}$ the characteristic polynomial $\chi_{\sigma^{-1}(\phi^*\widetilde{\sigma})}(t)$ is preserved by the flow θ of V in Q.*

Proof. Note first that by Lemma 5.6 it holds

$$\mathcal{L}_V(\Phi^*\widetilde{\omega}) = a\mathcal{L}_{\frac{1}{a}V}(\Phi^*\widetilde{\omega}) = a\mathcal{L}_{\Phi^*\widetilde{V}}(\Phi^*\widetilde{\omega}) = 0,$$

since Hamiltonian vector fields are sympletic. For all invertible tensors fields T and vector fields X the following are equivalent

$$\mathcal{L}_X(T) = 0 \text{ and } \mathcal{L}_X(T^{-1}) = 0.$$

We will also use two following facts

$$0 = \mathcal{L}_V(\omega^{-1}\Phi^*\widetilde{\omega}) = \mathcal{L}_V(\omega^{-1})\Phi^*\widetilde{\omega} + \omega^{-1}\mathcal{L}_V(\Phi^*\widetilde{\omega})$$

and

$$\text{if } \mathcal{L}_V(\omega^{-1}\Phi^*\widetilde{\omega}) = 0 \text{ then } \mathcal{L}_V(\det(\omega^{-1}\Phi^*\widetilde{\omega})) = 0.$$

The first one is the product rule of Lie derivatives and the latter one is the Jacobi formula for the Lie derivative of a determinant.

Take the Lie derivative from both sides of equation (59) to conclude

$$\det(\sigma^{-1}\phi^*\widetilde{\sigma})\mathcal{L}_V(a^2) + a^2\mathcal{L}_V(\det(\sigma^{-1}\phi^*\widetilde{\sigma})) = 0. \tag{60}$$

By formula (58) we can write

$$\Phi^*\widetilde{\omega} = adx^1 \wedge dy^1 + \phi^*\widetilde{\sigma}. \tag{61}$$

Take the Lie derivatives from both sides of equation (61) and recall Lemma 5.7. Then we get

$$0 = (\mathcal{L}_V a)dx^1 \wedge dy^1 + a\mathcal{L}_V(dx^1 \wedge dy^1).$$

By Corollary 18.11 of [38], it holds

$$\mathcal{L}_V(dx^1) = d(Vx^1) = d(\frac{\partial x^1}{\partial x_1}) = d(1) = 0 \text{ and } \mathcal{L}_V(dy^1) = d(\frac{\partial y^1}{\partial x_1}) = 0.$$

Thus we must have that $\mathcal{L}_V a = 0$. By formula (60) and the Leibnitz rule we must have that

$$\mathcal{L}_V(\det(\sigma^{-1}\phi^*\widetilde{\sigma})) = 0.$$

□

Next we will point out few important properties of skew symmetric matrices. Recall that a square matrix A with real entries is skew symmetric, if $A^T = -A$.

Lemma 5.9. *For a real skew symmetric matrix A the following statements are true:*

1. *If A has an odd number of columns and rows, then $detA = 0$.*
2. *If A is invertible, then A^{-1} is skew symmetric.*
3. *If A and B are skew symmetric n-matrices and n is even, then for every $t \in \mathbb{R}$, $det(A^{-1}B - tId) = p^2(t)$ for some polynomial p.*
4. *Kernel of A has an even dimension, if number of rows is even. Kernel of A has an odd dimension, if number of rows is odd.*

Proof. (1) If A has an odd number of rows and columns, then it holds that

$$\det(A) = \det(A^T) = \det(-A) = (-1)^{2n+1}\det(A) = -\det(A).$$

Therefore $\det A = 0$.

(2) If A is an invertible skew symmetric matrix, it follows that

$$(A^{-1}A)^T = A^T(A^{-1})^T = -A(A^{-1})^T = \text{Id}.$$

Thus $(A^{-1})^T = -A^{-1}$.

(3) According to [37], $\det A = q^2$, where q is a polynimial of degree n with variables of entries of A. Since the sum of skew symmetric matrices is skew skymmetric, we have for skew symmetric matrices A and B

$$\det(A^{-1}B - t\,\mathrm{Id}) = \det(A^{-1})\det(B - tA) = s^2 q^2(t),$$

for some polynomial q of degree n and real number s. Especially this means that the characteristic polynomial $\chi_{A^{-1}B}(t)$ is a square of a polynomial $p(t) := sq(t)$.

(4) By Spectral theory every symmetric matrix $B \in M(2n, \mathbb{C})$ has $2n$ real eigenvalues. It holds that iA is a symmetric matrix since

$$(iA)^* = -iA^* = -iA^T = iA.$$

Suppose that λ is an eigenvalue of A with eigenvector v. Then it holds that

$$iAv = i\lambda v,$$

i.e., $i\lambda$ is an eigenvaluea of iA. Since all the eigenvalues of iA are real, it must be that $i\lambda$ is real. But then we must have that λ is purely imaginary. Since the characteristic polynomial of A has only real coefficients and since conjugation of complex numbers commutates with sum and product, we see that also $\overline{\lambda}$ is an eigenvalue of A.

By Fundamental theorem of Algebra we have that the characteristic polynomial of A has as many roots as A has rows. Now it must be that 0 is a root of even/odd multiplicity for χ_A, if A has even/odd number of rows, since every non-zero root comes in pairs λ and $\overline{\lambda}$. This proves the claim, thus eigenspace of 0 is precisely the kernel of A.

□

We define a Pfaffian of skew symmetric matrix A as

$$\mathrm{Pf}(A) = \delta, \text{ if } \det A = \delta^2.$$

Let us return back to the our Hamiltionian setting. Let $p(t) = \chi_{\sigma^{-1}(\phi^*\widetilde{\sigma})}(t)$. Since given in the basis $x_1, H, x_2, y_2, \dots, x_n, y_n$ the both forms σ and $\phi^*\widetilde{\sigma}$ are skew symmetric, we see by the considerations made above of skew symmetric matrices that $p(t) = (\delta(t))^2$ where $\deg(\delta) = n-1$. By Lemma 5.8 polynomial p is preserved by flow of V, therefore also δ is preserved.

Since we have that

$$p(t) = \det(\sigma^{-1}(\phi^*\widetilde{\sigma}) - t\mathrm{Id}) = \frac{\det(\phi^*\widetilde{\sigma} - t\sigma)}{\det\sigma},$$

it follows that

$$\delta(t) = \frac{\mathrm{Pf}(\phi^*\widetilde{\sigma} - t\sigma)}{\mathrm{Pf}(\sigma)}.$$

Theorem 5.10. *Let diffeomorphism $\Phi : U(Q) \to U(\widetilde{Q})$ be orbital on Q. Then for each $t \in \mathbb{R}$ the polynomial*

$$\mathcal{P}^{n-1}(t) := \frac{Pf(\Phi^*\widetilde{\omega} - t\omega)}{Pf(\omega)(a-t)}$$

is an integral of Hamiltonian flow of V. In particular all the coefficients of $\mathcal{P}$ are integrals of V. Here $a = a(x) = a_1(x)a_2(x)$, where a_1 and a_2 are as in Lemma 5.6.

Proof. Choose a point $x \in Q$ and let $a_1 = a_1(x), a_2 = a_2(\Phi(x))$. First we note that $V_x \neq 0$ since ω is non-degenerate and $\omega(V_x, X) = XH = 0$ if and only if $X \in T_xQ$ by Lemma 5.4. We consider a two form $\Phi^*\widetilde{\omega} - a\omega$ on T_xM^{2n}. Let $u \in T_xM^{2n}$. By Lemma 5.6 we have that

$$\iota_V(\Phi^*\widetilde{\omega} - a\omega)(u) = \Phi^*\widetilde{\omega}(V, u) - a\omega(V, u) = a_1\widetilde{\omega}(\widetilde{V}, \Phi_*u) - adH(u)$$

$$= a_1 d\widetilde{H}(\Phi_*u) - adH(u) = a_1\Phi^*(d\widetilde{H})(u) - adH(u)$$

and

$$a_1\Phi^*(a_2\Phi_*dH)(u) - adH(u) = a_1(a_2 \circ \Phi)dH - adH = 0.$$

Therefore we have proved that $\langle V_x \rangle \subset \mathrm{Ker}(\Phi^*\widetilde{\omega} - a\omega)$, if we consider again $\Phi^*\widetilde{\omega} - a\omega$ as a linear mapping from T_xM^{2n} to $T_x^*M^{2n}$.

By Lemma 5.9 we know that the kernel of $\Phi^*\widetilde{\omega} - a\omega$ has an even dimension since $\dim M^{2n} = 2n$. By Lemma 5.9 we also know that the kernel restriction of $\Phi^*\widetilde{\omega} - a\omega$ into T_xQ has dimension of odd multiplicity. Therefore the set $(T_xM^{2n} \setminus T_xQ) \cap \mathrm{Ker}\,(\Phi^*\widetilde{\omega} - a\omega)$ is not empty. Let $A \in (T_xM^{2n} \setminus T_xQ) \cap \mathrm{Ker}\,(\Phi^*\widetilde{\omega} - a\omega)$. Now it must be that $A \in \mathrm{Ker}\,(\Phi^*\widetilde{\omega} - a\omega)$ and

$$\omega(V, A) = dH(A) = AH \neq 0,$$

by Lemma 5.4. Since ω is bilinear, one can choose vector A such that $\omega(V, A) = 1$. Since $V \in T_xQ$ we can choose vectors $\{e_1, \ldots, e_{2n-2}\} \subset T_xQ$ such that $(V, e_1, \ldots, e_{2n-2})$ is a basis of T_xQ. Now it holds that $(A, V, e_1, \ldots, e_{2n-2})$ is a basis of T_xM^{2n}.

Let $(A^*, V^*, e_1^*, \ldots, e_{2n-2}^*)$ be a basis in T_x^*M that is a dual basis for $(A, V, e_1, \ldots, e_{2n-2})$. Since $\omega(V, \cdot) \in T_x^*M$, we can write it as

$$\omega(V, \cdot) = b_1A^* + b_2V^* + \sum_{i=1}^{2n-2} b_{i+2}e_i^*.$$

Since we assumed that $\omega(V, A) = 1$, it follows that $b_1 = 1$. Since ω is a 2-from it holds that $0 = \omega(V, V) = b_2$. For the other base vectors we have that

$$b_{i+2} = \omega(V, e_i) = e_iH = 0.$$

Now we have proved that $(t\omega(V, \cdot))_i = t\delta_1^i$. Since we know that $V \in \ker(\Phi^*\widetilde{\omega} - a\omega)$ it follows that $(\Phi^*\widetilde{\omega} - t\omega)(V, \cdot)_i = (a-t)\delta_1^i$.

Since $\Phi^*\widetilde{\omega} - t\omega$ is a skew symmetric matrix, we can easily calculate its determinant as we have obtained a nice representations for matrix $(\Phi^*\widetilde{\omega} - t\omega)$,

$$\det(\Phi^*\widetilde{\omega} - t\omega) = \begin{vmatrix} 0 & a-t & (*) \\ -(a-t) & 0 & 0\dots0 \\ -(*) & 0 & (\Phi^*\widetilde{\omega} - t\omega)|_{T_xQ} \end{vmatrix}$$

$$= (a-t)^2\det((\Phi^*\widetilde{\omega} - t\omega)|_{T_xQ}) = (a-t)^2\det(\phi^*\widetilde{\sigma} - t\sigma).$$

From this equation we see that

$$\frac{\mathrm{Pf}(\Phi^*\widetilde{\omega} - t\omega)}{a-t} = \mathrm{Pf}(\phi^*\widetilde{\sigma} - t\sigma).$$

We also see that

$$\det(\omega) = \begin{vmatrix} 0 & 1 & (*) \\ -1 & 0 & 0\dots0 \\ -(*) & 0 & \omega|_{T_xQ} \end{vmatrix} = \det(\omega|_{T_xQ}) = \det(\sigma).$$

Finally we have proved that

$$\mathcal{P}^{n-1}(t) = \frac{\mathrm{Pf}(\Phi^*\widetilde{\omega} - t\omega)}{\mathrm{Pf}(\omega)(a-t)} = \frac{\mathrm{Pf}(\phi^*\widetilde{\sigma} - t\sigma)}{\mathrm{Pf}(\sigma)} = \delta(t).$$

Now our claim follows from reasoning made after Lemma 5.9. □

5.3. **Part III, Proof of main theorem of** [41]**.** Let $M, n > 1$ be a smooth manifold with geodesically equivalent metric tensor fields g and $\widetilde{g}$. In this part we will provide a proof to the main theorem of [41]. We will use tools introduced in [41]. The main idea is to use Theorem 5.10. Let $r > 0$. Define sets

$$U_g^r M := \{(p,\xi) \in TM : \|\xi\|_g(p) = r\}$$

and

$$U_{\widetilde{g}}^r M := \{(p,\xi) \in TM : \|\xi\|_{\widetilde{g}}(p) = r\}$$

respectively.

Choose $p \in M$ and let (U, x) be any smooth coordinates near p. Let $(TU, (x, v))$ be coordinates in TM related to (U, x) as

$$TU = \pi^{-1}U, \text{ and } (x,v)(p,\xi) = (x(p), v(\xi)) = (x(p), (\xi^i)_{i=1}^n),$$

where $\pi : TM \to M$ is the projection to the base point and $\xi = \xi^i \partial x_i|_p$.

Let (TM, ω_g, H_g) be the Hamiltonian system with $\omega_g = d[g_{ij}v^j dx^i]$ and $H_g(\xi) = \frac{1}{2}g_{ij}\xi^i\xi^j$ in local coordinates. Then the Hamiltonian vector field of this system is same as the geodesic vectorfield

$$X_H := v^k\frac{\partial}{\partial x_k} - \Gamma_{ij}^k v^i v^j \frac{\partial}{\partial v_k}.$$

Let Z be the zero section of M. We define a mapping $\Phi : TM \to TM$ by formula

$$\begin{cases} \Phi(q,\xi) = \left(q, \xi \frac{\|\xi\|_g}{\|\xi\|_{\widetilde{g}}}\right), & \xi \notin Z \\ \Phi(q,\xi) = (q,0), & \xi \in Z \end{cases}.$$

By this formula Φ is continuous, since at each q norms $\|\cdot\|_{g(q)}$ and $\|\cdot\|_{\widetilde{g}(q)}$ are equivalent. Therefore there exists $0 < c(q) \leq C(q)$ such that

$$\|\xi\|_{\widetilde{g}}\, c(q) \leq \|\xi\|_g \leq \|\xi\|_{\widetilde{g}}\, C(q) \text{ for all } \xi \in T_qM.$$

By continuity of norms it holds that $\Phi(q,\xi) \longrightarrow 0$ as $\xi \longrightarrow 0_q$. By this definition mapping Φ is clearly smooth in $TM \setminus Z$ and has a smooth inverse $\Psi : TM \setminus Z \to TM \setminus Z$

$$\Psi(q,\xi) = \left(q, \xi \frac{\|\xi\|_{\widetilde{g}}}{\|\xi\|_g}\right)$$

since

$$(\Phi \circ \Psi)(q,\xi) = \left(q, \xi \frac{\|\xi\|_g}{\|\xi\|_{\widetilde{g}}} \frac{\|\xi \frac{\|\xi\|_g}{\|\xi\|_{\widetilde{g}}}\|_{\widetilde{g}}}{\|\xi \frac{\|\xi\|_g}{\|\xi\|_{\widetilde{g}}}\|_g}\right) = (x,\xi).$$

Lemma 5.11. *Let θ_g be the geodesic flow of g and $\theta_{\widetilde{g}}$ the geodesic flow of $\widetilde{g}$ respectively. Let $r > 0$. Function Φ maps U_g^rM onto $U_{\widetilde{g}}^rM$, takes orbits of θ_g to the orbits of $\theta_{\widetilde{g}}$ and is orbital on U_g^rM.*

Proof. Let $(q,\xi) \in U_g^rM$. Then it holds that

$$\Phi(q,\xi) = \left(q, \xi \frac{r}{\|\xi\|_{\widetilde{g}}}\right)$$

and therefore

$$\|\xi \frac{r}{\|\xi\|_{\widetilde{g}}}\|_{\widetilde{g}} = r.$$

Thus by symmetry of Ψ we have shown that $\Phi(U_g^rM) = U_{\widetilde{g}}^rM$.

Since metrics g and $\widetilde{g}$ are geodesically equivalent, it follows that the geodesics of both metrics have same images on manifold M. Let $\gamma : (-\epsilon,\epsilon) \to M$ be a geodesic of metric g parameterized with arclength. Then it holds that $t \mapsto (\gamma_t, \dot{\gamma}_t)$ is an orbit of θ_g and

$$\Phi(\gamma_t,\dot{\gamma}_t) = \left(\gamma_t, \frac{1}{\|\dot{\gamma}_t\|_{\widetilde{g}}}\dot{\gamma}_t\right), \text{ where } \left\|\frac{1}{\|\dot{\gamma}_t\|_{\widetilde{g}}}\dot{\gamma}_t\right\|_{\widetilde{g}} = 1.$$

Here we use notation $\gamma(t) = \gamma_t$. Give an unit speed parametrization for γ with respect to metric $\widetilde{g}$ and it follows that $\Phi(\gamma_t,\dot{\gamma}_t)$ is an orbit of $\theta_{\widetilde{g}}$.

Since $TM \backslash Z$ is open, $U_g^rM, U_{\widetilde{g}}^rM \subset TM\backslash Z$ and $\Phi : TM\backslash Z \to TM\backslash Z$ is a diffeomorphisim, we have proven that Φ is orbital on U_g^rM.

$\square$

By definition of sets $U^r_g M$ and $U^r_{\widetilde{g}} M$ it is obvious that

$$U^r_g M = H^{-1}\Big\{\frac{r}{2}\Big\} \text{ and } U^r_{\widetilde{g}} M = \widetilde{H}^{-1}\Big\{\frac{r}{2}\Big\}.$$

Recall that a standard way to embed T_qM in $T_{(q,\xi)}TM$, $\xi \in T_qM$ is the mapping $w \mapsto w^i \frac{\partial}{\partial x^i}$. Let $(q,\xi) \in TM\backslash Z$ and embed $T_qM \hookrightarrow T_{(q,\xi)}TM$. Define $c(t) = \xi t \in T_qM$. Calculate

$$H_*\xi = \frac{d}{dt}(H \circ c)(t)|_{t=1} = \frac{d}{dt}\frac{1}{2}\|\xi t\|_g|_{t=1} = \frac{1}{2}\|\xi\|_g \neq 0.$$

Since $\mathbb{R}$ is one dimensional, we have shown that, for any $(q,\xi) \in TM\backslash Z$ push forward mapping H_* is onto. Thus sets $U^r_g M$ and $U^r_{\widetilde{g}} M$ are regular level sets and therefore sets $U^r_g M$ and $U^r_{\widetilde{g}} M$ are smooth sub manifolds of M with codimension 1.

Clearly it holds that $H_g = H_{\widetilde{g}} \circ \Phi$ and $H_{\widetilde{g}} = H_g \circ \Psi$. By [38], 6.12. it also holds that $\Phi^* dH_g = d(H_{\widetilde{g}} \circ \Phi)$. Therefore it holds that

$$dH_g = d(H_{\widetilde{g}} \circ \Phi) = \Phi^*(dH_{\widetilde{g}}) = \Psi_*(dH_{\widetilde{g}}) = d(H_{\widetilde{g}} \circ \Psi) = dH_{\widetilde{g}}.$$

Let γ be a geodesic of g with respect to initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$. Let α be an unit speed parametrization of γ with respect to metric $\widetilde{g}$ such that $\alpha(0) = 0$. Consider now a curve $\widetilde{\gamma} : \mathbb{R} \to M$ defined by

$$\widetilde{\gamma}(t) = \gamma(\|\xi\|_g \alpha(t)).$$

Since γ is a geodesic of g, it follows that $\widetilde{\gamma}$ is a geodesic of $\widetilde{g}$ with condition

$$\frac{d}{dt}\widetilde{\gamma}(t)\Big|_{t=0} = \dot{\gamma}(\alpha(t))\|\xi\|_g \dot{\alpha}(t)\Big|_{t=0} = \frac{\|\xi\|_g}{\|\xi\|_{\widetilde{g}}}\xi.$$

Then the geodesic vector fields X_{H_g} and $X_{H_{\widetilde{g}}}$ satisfy

$$X_{H_g}(p,\xi) = (\dot{\gamma}(0), \ddot{\gamma}(0))$$

and

$$X_{H_{\widetilde{g}}}(\Phi(p,\xi)) = X_{H_{\widetilde{g}}}(p, \frac{\|\xi\|_g}{\|\xi\|_{\widetilde{g}}}\xi) = (\dot{\widetilde{\gamma}}(t), \ddot{\widetilde{\gamma}}(t))|_{t=0},$$

since $\widetilde{\gamma}$ is an geodesic of $\widetilde{g}$ with initial values $(p, \frac{\|\xi\|_g}{\|\xi\|_{\widetilde{g}}}\xi)$. Let $\beta(t) = \|\xi\|_g \alpha(t)$. According to the proof of Lemma 5.6 it holds that

$$X_{H_g}(p,\xi) = \frac{d}{dt}\beta^{-1}(t)\Big|_{t=0} X_{H_{\widetilde{g}}}(p, \frac{\|\xi\|_g}{\|\xi\|_{\widetilde{g}}}\xi) = \frac{\|\xi\|_{\widetilde{g}}}{\|\xi\|_g} X_{H_{\widetilde{g}}}(p, \frac{\|\xi\|_g}{\|\xi\|_{\widetilde{g}}}\xi).$$

In coordinates (x,v) of TU we have at p that

$$\omega_g = d[g_{ij}v^i dx^j] \text{ and } \omega_{\widetilde{g}} = d[\widetilde{g}_{ij}v^i dx^j].$$

Then it holds that

$$\omega_g = d[g_{ij}v^i dx^j] = d(g_{ij}v^i) \wedge dx^j = \partial_{x_k}(g_{ij}v^i)dx^k \wedge dx^j - \partial_{v_k}(g_{ij}v^i)dx^j \wedge dv^k$$
$$= \partial_{x_k}(g_{ij}v^i)dx^k \wedge dx^j - g_{kj}\delta_{kj}dx^j \wedge dv^k$$

and

$$\Phi^*\omega_{\widetilde{g}} = \Phi^*(d[\widetilde{g}_{ij}v^i dx^j]) = \Phi^*(d[\widetilde{g}_{ij}v^j] \wedge dx^i) = \Phi^*(d[\widetilde{g}_{ij}v^j]) \wedge \Phi^*(dx^i)$$
$$= d[((\widetilde{g}_{ij}v^i)\circ\Phi]\wedge d[x^i\circ\Phi] = d\Big[\frac{\|v(\cdot)\|_g}{\|v(\cdot)\|_{\widetilde{g}}}\widetilde{g}_{ij}v^j\Big]\wedge dx^i = d\Big[\frac{\|v(\cdot)\|_g}{\|v(\cdot)\|_{\widetilde{g}}}\widetilde{g}_{ij}v^j dx^i\Big].$$

Therefore it holds that

$$\begin{aligned}\Phi^*\omega_{\widetilde{g}} &= d\Big[\frac{\|v(\cdot)\|_g}{\|v(\cdot)\|_{\widetilde{g}}}\widetilde{g}_{ij}v^j\Big] \wedge dx^i \\ &= \frac{\partial}{\partial x_k}\Big[\frac{\|v(\cdot)\|_g}{\|v(\cdot)\|_{\widetilde{g}}}\widetilde{g}_{ij}v^j\Big]dx^k \wedge dx^i - \frac{\partial}{\partial v_k}\Big[\frac{\|v(\cdot)\|_g}{\|v(\cdot)\|_{\widetilde{g}}}\widetilde{g}_{ij}v^j\Big]dx^i \wedge dv^k. \qquad (62)\end{aligned}$$

For each $\xi \in T_pM \setminus \{0\}$ we define

$$A_{ik} = -\frac{\partial}{\partial v_k}\Big[\frac{\|v(\cdot)\|_g}{\|v(\cdot)\|_{\widetilde{g}}}\widetilde{g}_{ij}v^j\Big]\Big|_{\xi}.$$

From now on we identify vector ξ with mapping v. With this small abuse of notation we have

$$A_{ik} = -\frac{\partial}{\partial v_k}\Big[\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}\widetilde{g}_{ij}v^j\Big].$$

As in the first section of this appendix, we can choose such a basis for T_pM that matrices g and $\widetilde{g}$ are

$$g = \operatorname{diag}(1,\dots,1) \text{ and } \widetilde{g} = \operatorname{diag}(\rho_1,\dots,\rho_n),$$

here $\rho_1 \geq \dots \geq \rho_n > 0$. In this basis it holds that

$$A_{ik} := -\rho_i\frac{\partial}{\partial v_k}\Big(v_i\frac{\sqrt{\sum_{j=1}^n v_j^2}}{\sqrt{\sum_{j=1}^n \rho_j v_j^2}}\Big) = -\rho_i\delta_{ik}\frac{\|v\|_g}{\|v\|_{\widetilde{g}}} - \rho_i v_i\Big(\frac{\frac{\|v\|_{\widetilde{g}}}{\|v\|_g} - \rho_k\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}}{\|v\|_{\widetilde{g}}^2}v_k\Big).$$

Let us define

$$\mu_i := -\rho_i\frac{\|v\|_g}{\|v\|_{\widetilde{g}}},\ A_i = -\rho_i v_i \text{ and } B_i := \frac{\frac{\|v\|_{\widetilde{g}}}{\|v\|_g} - \rho_i\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}}{\|v\|_{\widetilde{g}}^2}v_i.$$

Also set $A = (A_1,\dots,A_n)$ and $B = (B_1,\dots,B_n)$. Then we can write

$$[A_{ij}] = \operatorname{diag}(\mu_1,\dots,\mu_n) - A \otimes B. \qquad (63)$$

Since in T_pM we have $g = \delta_{ij}$, it holds that

$$\omega_g(p) = \partial_{x_k}(g_{ij}v^i)dx^k \wedge dx^j - \delta_{jk}dx^j \wedge dv^k.$$

Due this, formula (62) and the considerations made above a $2n \times 2n$-square matrix $\Phi^*\omega_{\widetilde{g}} - t\omega_g$ at (p,v) looks like

$$\begin{pmatrix} * & [A_{ij}] + t\delta_{ij} \\ -([A_{ij}] + t\delta_{ij}) & 0 \end{pmatrix}.$$

Calculate

$$\det(\Phi^*\omega_{\widetilde{g}} - t\omega_g) = \begin{vmatrix} * & [A_{ij}] + t\delta_{ij} \\ -([A_{ij}] + t\delta_{ij}) & 0 \end{vmatrix} = \det([A_{ij}] + t\delta_{ij})^2$$

and

$$\det(\omega_g) = \begin{vmatrix} * & \delta_{ij} \\ -\delta_{ij} & 0 \end{vmatrix} = 1.$$

We define

$$\Delta^n(t) := \det([A_{ij}] + t\delta_{ij}) = \det(\mathrm{diag}\ (t+\mu_1,\dots,t+\mu_n) - A\otimes B).$$

Note that the last equality holds due equation (63). By the considerations we made about the Paffians of the skew symmetric matrices, it holds that

$$\Delta^n(t) = \frac{\mathrm{Pf}(\Phi^*\widetilde{\omega} - t\omega)}{\mathrm{Pf}(\omega)}.$$

Lemma 5.12. *It holds that*

$$\begin{aligned} \Delta^n(t) = \ & (t+\mu_1)(t+\mu_2)\cdots(t+\mu_n) \\ & -(A_1B_1)(t+\mu_2)(t+\mu_3)\cdots(t+\mu_n) \\ & \ldots - (t+\mu_1)(t+\mu_2)(t+\mu_3)\cdots(t+\mu_{n-1})(A_nB_n). \end{aligned} \tag{64}$$

Proof. We prove the claim by induction. We start with step $n = 2$.

$$\Delta^2(t) = \begin{vmatrix} t+\mu_1 - A_1B_1 & -A_2B_1 \\ -A_1B_2 & t+\mu_2 - A_2B_2 \end{vmatrix}$$

$$= (t+\mu_1)(t+\mu_2) - A_1B_1(t+\mu_2) - (t+\mu_1)A_2B_2 + A_1A_2B_1B_2 - A_1A_2B_1B_2$$

$$= (t+\mu_1)(t+\mu_2) - A_1B_1(t+\mu_2) - (t+\mu_2)A_2B_2.$$

Assume that $\Delta^{n-1}(t)$ and $\Delta^{n-2}(t)$ satisfy equation (64). Let us prove the claim in the case of $\Delta^n(t)$. By induction assumption we have

$$n\Delta^n(t) = \sum_{i=1}^n (t+\mu_i - A_iB_i)\det M_{ii} + \sum_{i,j=1\ j\neq i}^n (-1)^{i+j+1} A_iB_j \det M_{ij}$$

$$= \sum_{i=1}^n (t+\mu_i - A_iB_i)\det M_{ii} + \sum_{i,j=1\ j\neq i}^n (-1)^{i+j+1} A_iA_jB_iB_j \det \widetilde{M}_{ij}$$

$$= n \begin{pmatrix} (t+\mu_1)(t+\mu_2)\cdots(t+\mu_n) \\ -(A_1B_1)(t+\mu_2)(t+\mu_3)\cdots(t+\mu_n) \\ \ldots - (t+\mu_1)(t+\mu_2)(t+\mu_3)\cdots(t+\mu_{n-1})(A_nB_n) \end{pmatrix}$$

(65)

$$+ \sum_{i,j=1\ i\neq j}^n \Big(A_iA_jB_iB_j \prod_{k=1,\,k\neq i,j}^n (t+\mu_k)\Big) + \sum_{i,j=1,\ j\neq i}^n (-1)^{i+j+1} A_iA_jB_iB_j \det \widetilde{M}_{ij}.$$

Here M_{ij} is the $n-1$-square matrix where we have deleted the i^{th} column and j^{th} row. Matrix $\widetilde{M}_{ij}$ is like M_{ij}, but B_i and A_j are removed from j^{th} row and i^{th} column. For instance

$$\widetilde{M}_{12} = \widetilde{M}_{21}$$

$$= \begin{pmatrix} -1 & -A_3 & -A_4 & \cdots & -A_n \\ -B_3 & t+\mu_3 - A_3B_3 & -A_4B_3 & \cdots & -A_nB_3 \\ -B_4 & -A_3B_4 & t+\mu_4 - A_4B_4 & \cdots & -A_nB_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B_n & -A_3B_n & -A_4B_n & \cdots & t+\mu_n - A_nB_n \end{pmatrix}.$$

If we can prove that (65) is zero, then we are done with the induction claim for $\Delta^n(t)$. To verify that (65) is zero, it suffices to prove that

$$\det\widetilde{M}_{ij} = (-1)^{i+j} \prod_{k=1, k\neq i,j}^{n} (t+\mu_k).$$

This follows by an induction in a following way. Calculate $\det\widetilde{M}_{ij}$ with respect to the i^{th} row. Then one of the sub determinants of $\widetilde{M}_{ij}$ is $(-1)^{i+j}\Delta^{n-2}(t)$ and all the others have a similar looking form as $\widetilde{M}_{ij}$. Therefore

$$\begin{aligned} &\det\widetilde{M}_{ij} = \\ &(-1)^{i+j}\Big(\prod_{k=1,\ k\neq i,j}^{n}(t+\mu_1) - \sum_{k=1,\ k\neq i,j}^{n}\Big(A_kB_k\prod_{\ell=1,\ \ell\neq i,j,k}^{n}(t+\mu_\ell)\Big)\Big) \\ &+ \sum_{k=1,\ k\neq j}^{n}\Big((-1)^{i+k}A_kB_k(-1)^{j+k}\prod_{\ell=1,\ \ell\neq i,j,k}^{n}(t+\mu_\ell)\Big) \\ &= (-1)^{i+j}\prod_{k=1, k\neq i,j}^{n}(t+\mu_k). \end{aligned}$$

□

Let $\delta^{n-1}(t)$ be such a polynomial of degree $n-1$ that

$$\Delta^n(t) = (a-t)\delta^{n-1}(t),$$

where $a = a(p,v) = \frac{\|v\|_{\tilde{g}}}{\|v\|_g}$. The polynomial δ^{n-1} exists since

$$(a+\mu_i) = \frac{\|v\|_{\tilde{g}}}{\|v\|_g} - \rho_i\frac{\|v\|_g}{\|v\|_{\tilde{g}}} = A_iB_i\frac{\|v\|^2_{\tilde{g}}}{\rho_i v_i^2}$$

and by equation (64) it holds that

$$\Delta^n(a) = \prod_{i=1}^{n}(a+\mu_i)\Big(1 - \frac{\sum_{j=1}^n \rho_j v_j^2}{\|v\|^2_{\tilde{g}}}\Big) = 0.$$

Write

$\Delta^n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ and $\delta^{n-1}(t) = t^{n-1} + b_{n-2}t^{n-2} + \cdots + b_0$ for some $(a_i)_{i=0}^{n-1}$ and $(b_i)_{i=0}^{b-2}$. Calculate

$$(t-a)\delta^{n-1}(t) = (t-a)(t^{n-1} + b_{n-2}t^{n-2} + \cdots + b_0)$$

$$= t^n + (b_{n-2}-a)t^{n-1} + \ldots + (b_{n-k-1} - ab_{n-k})t^{n-k} + \ldots + (b_0 - ab_1)t - ab_0.$$

Then it holds that

$$\begin{aligned}
&a_n = b_n = 1\\
&a_{n-1} = b_{n-2} - a\\
&\vdots\\
&a_{n-k} = b_{n-k-1} - ab_{n-k}\\
&\vdots\\
&a_0 = -ab_0
\end{aligned} \tag{66}$$

Note that the coefficient a_0 can be calculated from formula (64) in the following way

$$a_0 = \Delta^n(0) = \mu_1 \cdots \mu_n - A_1B_1\mu_2\mu_3 \cdots \mu_n - \ldots - \mu_1\mu_2\mu_3 \cdots \mu_{n-1}A_nB_n$$

$$= (-1)^n(\rho_1 \cdots \rho_n)\Big(\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}\Big)^n + (-1)^n\rho_1 \cdots \rho_n\Big(\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}\Big)^{n-1}\sum_{i=1}^n v_i^2\Big(\frac{\frac{\|v\|_{\widetilde{g}}}{\|v\|_g} - \rho_i\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}}{\|v\|^2_{\widetilde{g}}}\Big)$$

$$\begin{aligned}
&= (-1)^n(\rho_1 \cdots \rho_n)\Big(\Big(\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}\Big)^n + \Big(\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}\Big)^{n-1}\Big(\frac{\frac{\|v\|_{\widetilde{g}}}{\|v\|_g}\|v\|^2_g - \frac{\|v\|_g}{\|v\|_{\widetilde{g}}}\|v\|^2_{\widetilde{g}}}{\|v\|^2_{\widetilde{g}}}\Big)\Big)\\
&= (-1)^n(\rho_1 \cdots \rho_n)\Big(\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}\Big)^n.
\end{aligned}$$

By formula (66) it holds that

$$b_0 = -\frac{a_0}{a} = (-1)^{n+1}(\rho_1 \cdots \rho_n)\Big(\frac{\|v\|_g}{\|v\|_{\widetilde{g}}}\Big)^{n+1}. \tag{67}$$

In the first section of this appendix we defined that

$$S_k((p,v)) := \Big(\frac{\det g_p}{\det \widetilde{g}_p}\Big)^{\frac{k+2}{n+1}}\sum_{i=0}^k c_iG^{k-i+1}(v)$$

and

$$I_k((p,v)) = g_p(S_k((p,v)), v).$$

Thus it holds that

$$S_0((p,v)) = \Big(\frac{\det g_p}{\det \widetilde{g}_p}\Big)^{\frac{2}{n+1}}G(v)$$

and therefore we have that

$$\begin{aligned} I_0(x,v) &= \Big(\frac{\det g_p}{\det \widetilde{g}_p}\Big)^{\frac{2}{n+1}} g_x(G(v),v) \\ &= \Big(\frac{\det g_p}{\det \widetilde{g}_p}\Big)^{\frac{2}{n+1}} v^T(gg^{-1}\widetilde{g})v \\ &= \Big(\frac{\det g_p}{\det \widetilde{g}_p}\Big)^{\frac{2}{n+1}} \widetilde{g}_{ij}v^iv^j \\ &= (\rho_1\cdots\rho_n)^{-\frac{2}{n+1}}\widetilde{g}(v,v) \\ &= (\rho_1\cdots\rho_n)^{-\frac{2}{n+1}}\|v\|_g^2\Big(\frac{\|v\|_{\widetilde{g}}}{\|v\|_g}\Big)^2 \\ &= 2H_g(v)((-1)^{n+1}b_0)^{-\frac{2}{n+1}}. \end{aligned}$$

Note that

$$\delta^{n-1}(t) = \frac{\mathrm{Pf}(\Phi^*\widetilde{\omega} - t\omega)}{\mathrm{Pf}(\omega)(a-t)}$$

and by Theorem 5.10 each coefficient of polynomial δ^{n-1} is an integral of geodesic flow of g. Since H_g is an integral of geodesic flow of g, also I_0 is an integral of geodesic flow of g by calculations made above.

Since we do not need other functions I_k, $k \geq 1$ in our results, we skip the proofs to show that each I_k has a similar kind of connection to coefficient b_k as I_0 and b_0 have. This is done in [41]. Thus we have proved Theorem 5.1 that is by Matveev and Topalov and is the main result of [41].

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