If the conductivity is radial ($\sigma(z) = \sigma(|z|)$), then the scattering transform is radial and real-valued

Remember the definition $q = \sigma^{-1/2}\Delta\sigma^{1/2}$. Clearly, the assumption $\sigma(z) = \sigma(|z|)$ implies $q(z) = q(|z|)$. We will prove two claims:

Claim 1: We have $\mathbf{t}(k) = \mathbf{t}(k_\varphi)$, where $k_\varphi := e^{i\varphi}k$.

Claim 2: We have $\overline{\mathbf{t}(-\overline{k})} = \mathbf{t}(k)$.

The proofs of Claims 1 and 2 are based on the uniqueness of the solution $(-\Delta + q)\psi(z,k) = 0$ satisfying $\psi(z,k) \sim e^{ikz}$.

Note that for each $k \in \mathbb{C}$ there is an angle $\varphi(k)$ with the property $-\overline{k} = k_{\varphi(k)}$. Then Claims 1 and 2 imply

$$\mathbf{t}(k) = \overline{\mathbf{t}(-\overline{k})} = \overline{\mathbf{t}(k_{\varphi(k)})} = \overline{\mathbf{t}(k)}.$$

Proof of Claim 1: $\mathbf{t}(k) = \mathbf{t}(k_\varphi)$

The solution $\psi(z_\varphi, k_{-\varphi})$ satisfies equation

$$\begin{aligned} 0 &= (-\Delta + q(z_\varphi))\psi(z_\varphi, k_{-\varphi}) \\ &= (-\Delta + q(z))\psi(z_\varphi, k_{-\varphi}) \end{aligned}$$

with asymptotics $\psi(z_\varphi, k_{-\varphi}) \sim e^{ik_{-\varphi}z_\varphi} = e^{ie^{-i\varphi}ke^{i\varphi}z} = e^{ikz}$.
The uniqueness of the solution $(-\Delta + q)\psi(z,k) = 0$ satisfying $\psi(z,k) \sim e^{ikz}$ implies that $\psi(z_\varphi, k_{-\varphi}) = \psi(z,k)$. Now

$$\begin{aligned} \mathbf{t}(k_\varphi) &= \int_{\mathbb{R}^2} e^{i\overline{k_\varphi}\bar{z}} q(z)\psi(z,k_\varphi)dxdy \\ &= \int_{\mathbb{R}^2} e^{i\bar{k}\overline{z_\varphi}} q(z)\psi(z_\varphi,k)dxdy \\ &= \int_{\mathbb{R}^2} e^{i\bar{k}\bar{w}} q(w)\psi(w,k)dw_1dw_2 \;=\; \mathbf{t}(k). \end{aligned}$$

Proof of Claim 2: $\overline{\mathbf{t}(-\overline{k})} = \mathbf{t}(k)$

Consider $(-\Delta + q(\overline{z}))\psi(\overline{z}, -\overline{k}) = 0$ with $\psi(\overline{z}, -\overline{k}) \sim e^{i(-\overline{k})\overline{z}}$.
Complex conjugating and using $\overline{q} = q$ and $q(\overline{z}) = q(z)$ gives

$$(-\Delta + q(z))\overline{\psi(\overline{z}, -\overline{k})} = 0 \quad \text{with} \quad \overline{\psi(\overline{z}, -\overline{k})} \sim e^{ikz}.$$

The uniqueness of the solution $(-\Delta + q)\psi(z, k) = 0$ satisfying $\psi(z, k) \sim e^{ikz}$ implies that $\overline{\psi(\overline{z}, -\overline{k})} = \psi(z, k)$. Now

$$\begin{aligned}
\overline{\mathbf{t}(-\overline{k})} &= \overline{\int_{\mathbb{R}^2} e^{i\,\overline{(-\overline{k})}\,\overline{z}} q(z)\psi(z, -\overline{k})dxdy} \\
&= \int_{\mathbb{R}^2} e^{i\overline{k}z} q(z)\overline{\psi(z, -\overline{k})}dxdy \\
&= \int_{\mathbb{R}^2} e^{i\overline{k}\overline{w}} q(\overline{w})\overline{\psi(\overline{w}, -\overline{k})}dw_1 dw_2 \\
&= \int_{\mathbb{R}^2} e^{i\overline{k}\overline{w}} q(w)\psi(w, k)dw_1 dw_2 \;=\; \mathbf{t}(k).
\end{aligned}$$