If the conductivity is radial ($\sigma(z) = \sigma(|z|)$), then the DN map has trigonometric eigenfunctions

Claim: $\Lambda_\sigma \varphi_n = \lambda_n \varphi_n$ with real-valued eigenvalues λ_n. Moreover, $\lambda_{-n} = \lambda_n$ for all $n \in \mathbb{Z}$.

Define $\widetilde{u} := \rho_\varphi u$, where ρ_φ is the operator defined by $(\rho_\varphi u)(z) = u(e^{i\varphi} z)$. Consider the solutions

$$\boxed{\begin{aligned} \nabla \cdot \sigma \nabla u &= 0 \text{ in } \Omega \\ u|_{\partial\Omega} &= f \text{ on } \partial\Omega \end{aligned}} \qquad \boxed{\begin{aligned} \nabla \cdot \sigma \nabla \widetilde{u} &= 0 \text{ in } \Omega \\ \widetilde{u}|_{\partial\Omega} &= \rho_\varphi f \text{ on } \partial\Omega \end{aligned}}$$

Therefore $\Lambda_\sigma(\rho_\varphi f) = \sigma \frac{\partial \widetilde{u}}{\partial r}|_{r=1}$, and we can conclude that $\rho_\varphi \Lambda_\sigma = \Lambda_\sigma \rho_\varphi$. Consequently $\partial_\theta(\Lambda_\sigma f) = \Lambda_\sigma(\partial_\theta f)$. Now we can solve a differential equation to find a unique C for $f = \varphi_n$ such that

$$\partial_\theta(\Lambda_\sigma \varphi_n) = in \Lambda_\sigma \varphi_n \quad \Longrightarrow \quad \Lambda_\sigma \varphi_n = C e^{in\theta} = (C\sqrt{2\pi})\varphi_n.$$

Having defined the eigenvalues, we now need to show that $\lambda_{-n} = \lambda_n \in \mathbb{R}$. First show $\lambda_{-n} = \overline{\lambda_n}$.

Conductivity-type potentials always have $\lambda_0 = 0$.

Use the real-valuedness of q to see that

$$(-\Delta + q)\overline{u_n} = 0 \text{ in } \Omega, \qquad \overline{u_n}|_{\partial\Omega} = \varphi_{-n},$$

implying

$$\lambda_{-n}\varphi_{-n} = \Lambda_q \varphi_{-n} = \frac{\partial \overline{u_n}}{\partial \nu}|_{\partial\Omega} = \overline{\frac{\partial u_n}{\partial \nu}|_{\partial\Omega}} = \overline{\Lambda_q \varphi_n} = \overline{\lambda_n \varphi_n} = \overline{\lambda_n}\varphi_{-n},$$

so we get

$$\lambda_{-n} = \overline{\lambda_n}.$$

Finally, we prove that $\lambda_n = \overline{\lambda_n}$.

Note that $u_n = \overline{u_{-n}}$. Use the real-valuedness of q to compute

$$\begin{aligned}
\lambda_n &= \lambda_n \langle \varphi_{-n}, \varphi_n \rangle \\
&= \langle \varphi_{-n}, \lambda_n \varphi_n \rangle \\
&= \langle \varphi_{-n}, \Lambda_q \varphi_n \rangle \\
&= \int_\Omega (\nabla u_{-n} \cdot \nabla u_n + q u_{-n} u_n) dz \\
&= \int_\Omega \overline{(\nabla u_n \cdot \nabla u_{-n} + q u_n u_{-n})} dz \\
&= \overline{\int_\Omega (\nabla u_{-n} \cdot \nabla u_n + q u_{-n} u_n) dz} \\
&= \overline{\langle \varphi_{-n}, \Lambda_q \varphi_n \rangle} \\
&= \overline{\langle \varphi_{-n}, \lambda_n \varphi_n \rangle} \\
&= \overline{\lambda_n} \langle \varphi_{-n}, \varphi_n \rangle \\
&= \overline{\lambda_n}.
\end{aligned}$$