QUASISYMMETRIC DISTORTION SPECTRUM

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joint work with

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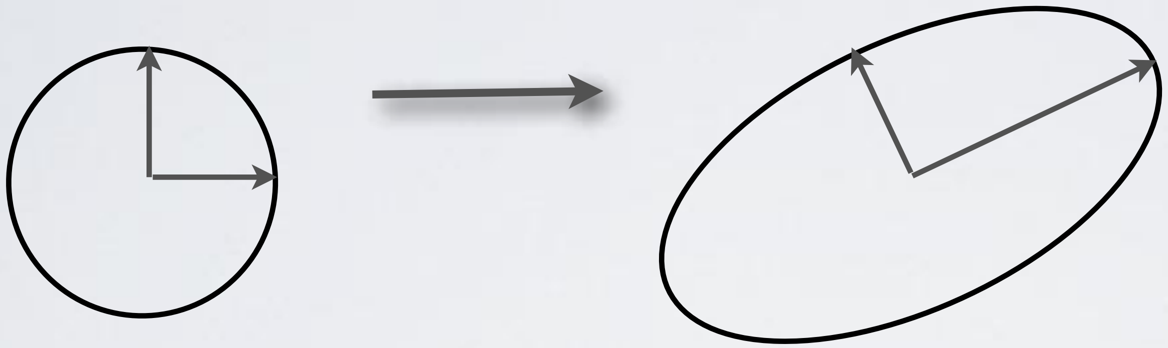
QUASICONFORMAL MAPS

$\varphi\colon \Omega \to \Omega'$ $W^{1,2}_{loc}$-homeomorphism $\qquad \|\mu\|_\infty \le k < 1$

$$\bar{\partial}\varphi(z) = \mu(z)\partial\varphi(z) \quad \text{a.e. } z \in \Omega$$

or equivalently



eccentricity ≤

$$K = \frac{1+k}{1-k}$$

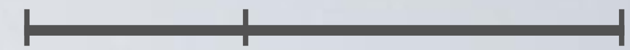
measurable Riemann mapping theorem:
(Morrey, Bojarski, Ahlfors-Bers,...)

unique (up to conformal change) solution exists

analytic dependence on μ

QUASISYMMETRIC MAPS

$g\colon \mathbf{R} \to \mathbf{R}$

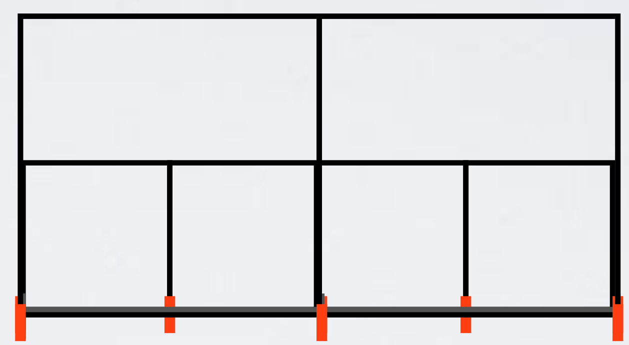


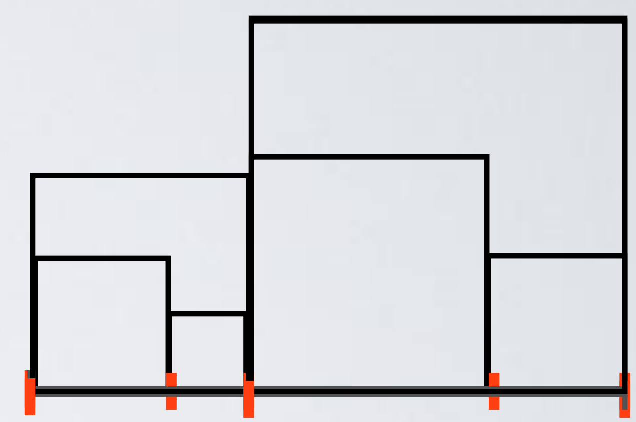
$$\frac{1}{\rho} \le \frac{|g(I)|}{|g(I')|} \le \rho$$

doubling measure $\quad g(x) = \int_0^x d\nu \qquad \nu = g^{-1}(m)$

can be singular wrt

Lebesgue measure *m*





$$\dim \nu = -(p \log_2 p + (1-p)\log_2(1-p)) =: I_p < 1 \qquad (\to 0 \text{ as } p \to 0)$$

$$\dim \nu \stackrel{\text{def}}{=} \inf\{\dim E : \nu(E^c) = 0\} = \inf\{\alpha : \nu \perp \mathcal{H}_\alpha\}$$

Heurteaux: for any ρ-qs map $\quad \dim \nu \ge I_p, \quad p = 1/(1+\rho)$

SINGULARITY SPECTRUM

Beurling-Ahlfors: as boundary values of qc maps

$\varphi\colon \mathbf{H}_+ \to \mathbf{H}_+$ K-quasiconformal

$g\colon \mathbf{R} \to \mathbf{R}$ boundary correspondence, call K-quasisym

Q: How singular g can be? (Estimates in terms of K)

$$\dim g(m) \geq ?$$

multifractal spectrum, in general

E.g. Higher Integrability of Astala yields

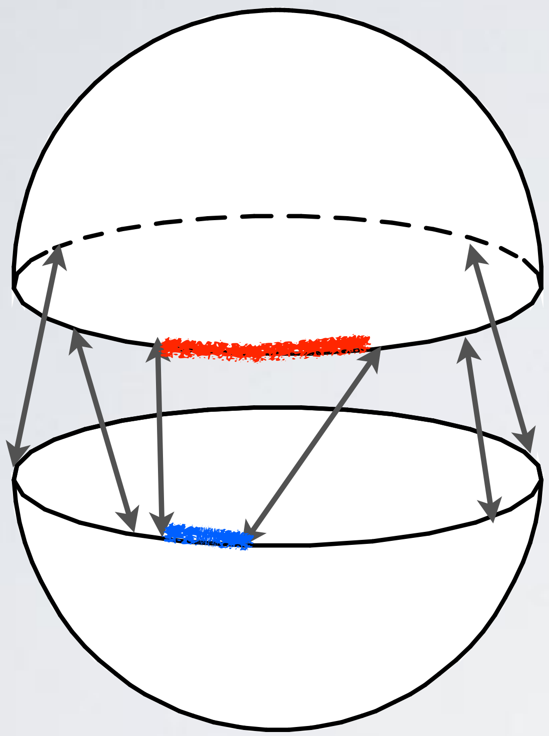
$\varphi \in W^{1,p}_{loc} \quad p < \dfrac{2K}{K-1}$ $\qquad \dim g(m) \geq 1 - k$

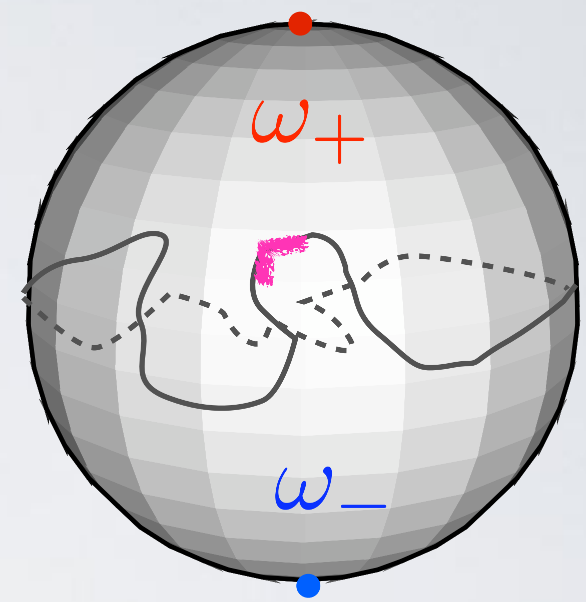
Possible to improve. Optimal estimates?

Motivation:

CONFORMAL WELDING

K^2-quasisymmetric welding ⟷ K-quasicircle





singularity of the welding

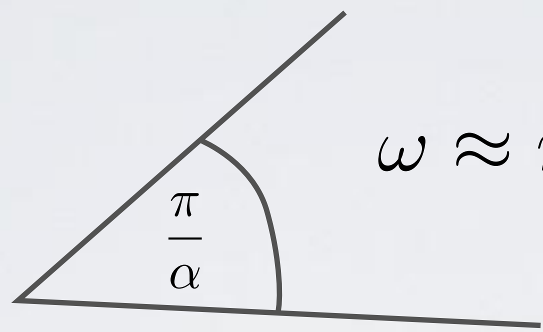
multifractality of harmonic measure

(compression/expansion of conformal maps)

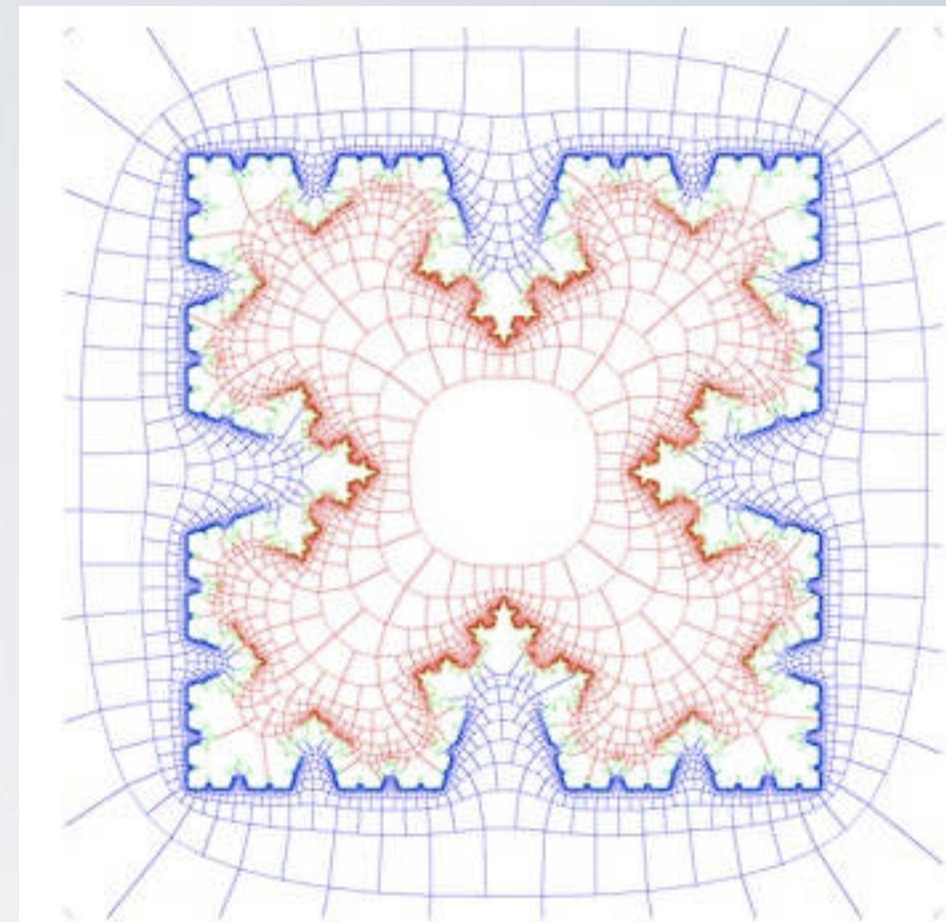
MULTIFRACTALITY OF ω

'fjords and spikes'

$\mathcal{F}_\alpha$ **scaling:** $\omega(B(x,r)) \approx r^\alpha$



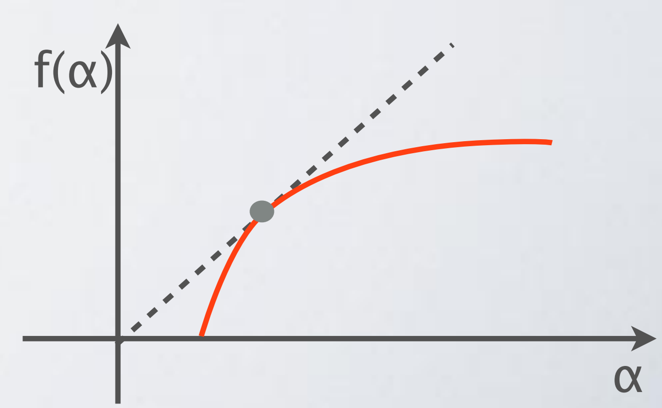
$\omega \approx r^\alpha$



Courtesy of D. Marshall

multifractal spectrum:

$$f(\alpha) = \dim\{x : \omega B(x,r) \approx r^\alpha\}$$



Problem: **Universal bounds???**

QUASISYMMETRIC DISTORTION

Smirnov: dim(K-quasicircle) ≤ 1+k²

Theorem: $g : \mathbf{R} \to \mathbf{R}$ *K*-quasisymmetric

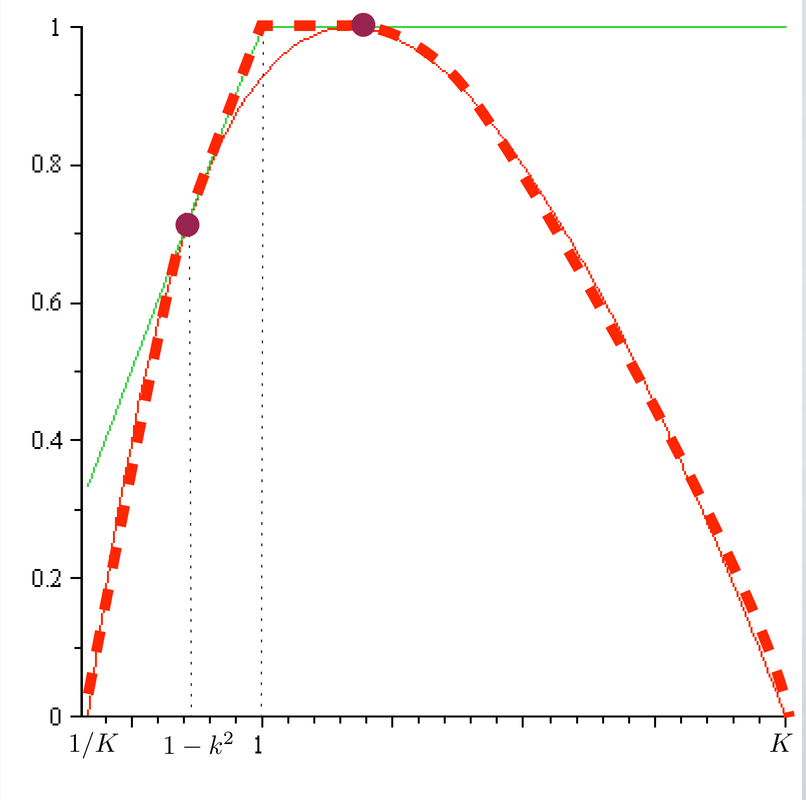
$$E \subset \mathbf{R}, \quad \dim E = t \quad \Longrightarrow \quad \dim g(E) \geq \frac{t(1-k^2)}{(1+k\sqrt{1-t})^2} =: t(k)$$

and $\dim g(E) \leq t(-\min\{k, \sqrt{1-t}\})$

Remark: t=1 case: $\dim g(m) \geq 1-k^2$

$$f_{K\text{-}qs}(\alpha) \leq -\frac{4K}{(K-1)^2}(\sqrt{\alpha}-\sqrt{K})(\sqrt{\alpha}-1/\sqrt{K})$$

$$\frac{1}{K} \leq \alpha \leq 1-k^2$$



PROOF:

- **builds on Astala's higher integrability, "complex interpolation"**
  use holomorphic dependence to upgrade apriori bounds

- **underlying holomorphic dependece**
  holomorphic motions

- **exploits extra symmetry in the motion**
  quasisymmetric maps, "quadratic" improvement

- **main tool**
  variational principle from thermodynamical formalism

HOLOMORPHIC MOTION

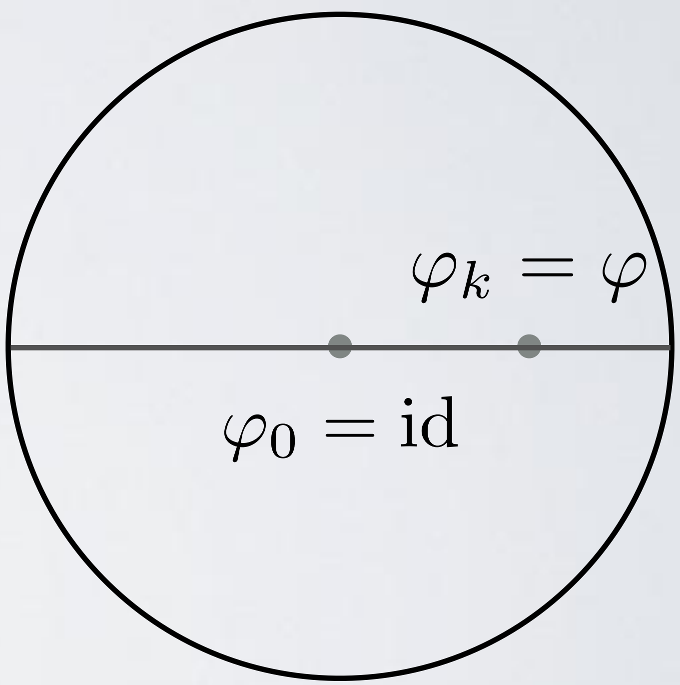
Extension by reflection $\varphi(z) = \overline{\varphi(\bar{z})}$ $\varphi_{|\mathbf{R}} = g$

Symmetric Beltrami $\|\mu\|_\infty = 1$ $\bar{\partial}\varphi = k\mu\,\partial\varphi$

$\mu(z) = \overline{\mu(\bar{z})}$

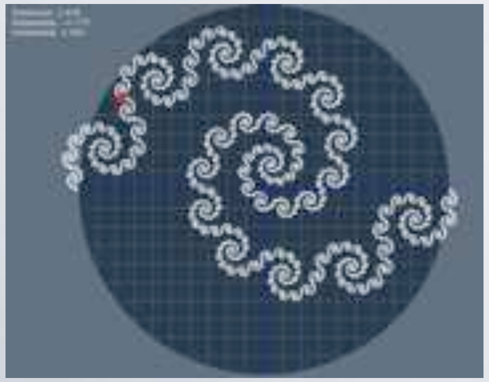
Solve for $\lambda \in \mathbb{D}$ the Beltrami equation

$$\bar{\partial}\varphi_\lambda = \lambda\mu\,\partial\varphi_\lambda$$



symmetry $\varphi_\lambda(z) = \overline{\varphi_{\bar{\lambda}}(\bar{z})}$

$\varphi_\lambda(\mathbf{R}) = \mathbf{R}$ $\lambda \in \mathbf{R}$



java animation by Aleksi Vähäkangas

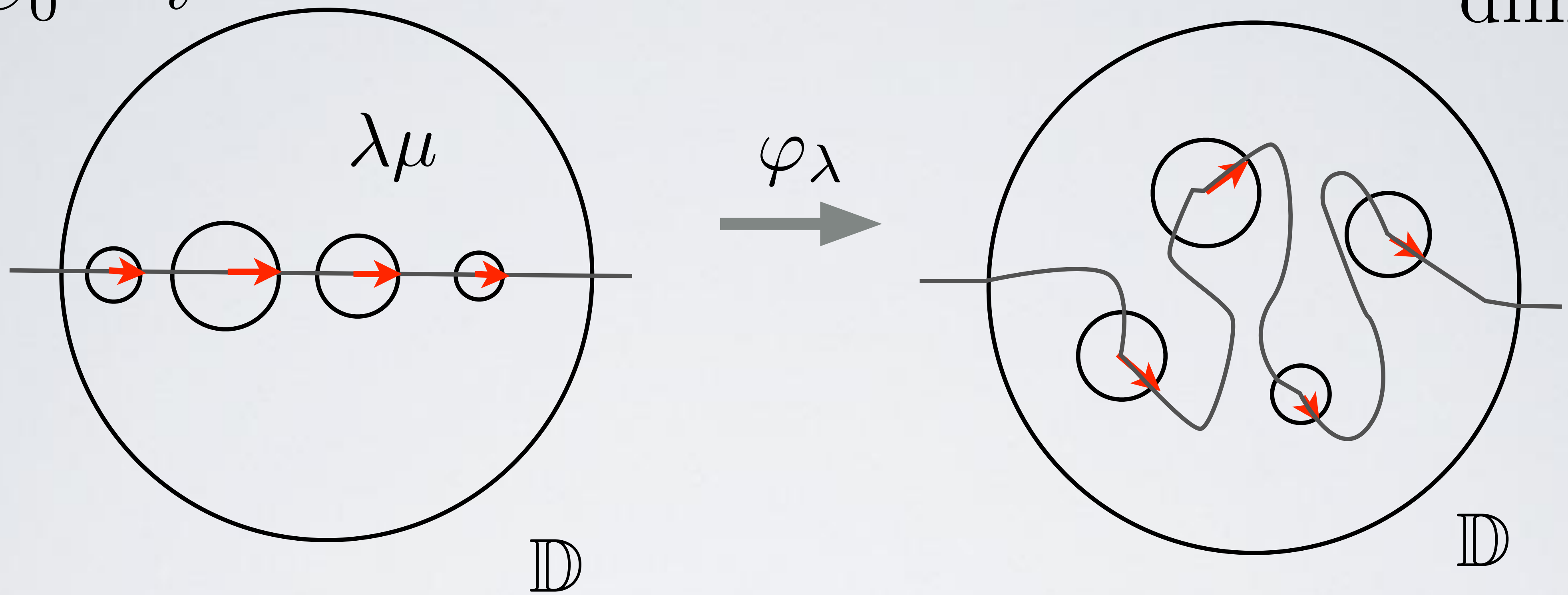
holomorphic motion of snowflake (Astala-Rohde-Schramm)

CANTOR SETS

'fractal approximation'

$E \approx C_0$ $\qquad$ $\varphi(E) \approx C_k$

$\dim C_0 = t$ $\qquad$ $\dim C_k = ?$



a packing of disks $\{B_\lambda\}$

"complex radius" $r_i(\lambda) := \varphi_\lambda(z_i + r_i) - \varphi_\lambda(z_i)$

Cantor sets C_λ

THERMODYNAMICS

variational principle (Ruelle, Bowen)

$$P_\lambda(t) := \log\left(\sum |r_i(\lambda)|^t\right) = \sup_{p\in\mathbf{Prob}} (\mathrm{I}_p - t\,\mathrm{Re}\,\Lambda_p(\lambda))$$

$$\mathrm{I}_p = \sum p_i \log\frac{1}{p_i}$$

entropy

$$\Lambda_p(\lambda) = \sum p_i \log\frac{1}{r_i(\lambda)}$$

(complex) Lyapunov exponent

$\dim C_\lambda$ =root of P_λ $= \sup_p \dfrac{\mathrm{I}_p}{\mathrm{Re}\,\Lambda_p(\lambda)} = \sup_p \dim p_{C_\lambda}$

$\lambda \mapsto \dfrac{\mathrm{I}_p}{\Lambda_p(\lambda)}$ holomorphic!

APRIORI BOUNDS

$$\Phi(\lambda) = 1 - \frac{\mathrm{I}_p}{\Lambda_p(\lambda)}$$ holomorphic

natural bounds: dim ≤2 ⟶ $|\Phi| < 1$ $\Phi : \mathbb{D} \to \mathbb{D}$

$\lambda \in \mathbf{R}$ $C_\lambda \subset \mathbf{R}$ $\dim C_\lambda \leq 1$ ⟶ $\Phi(\lambda) \geq 0$

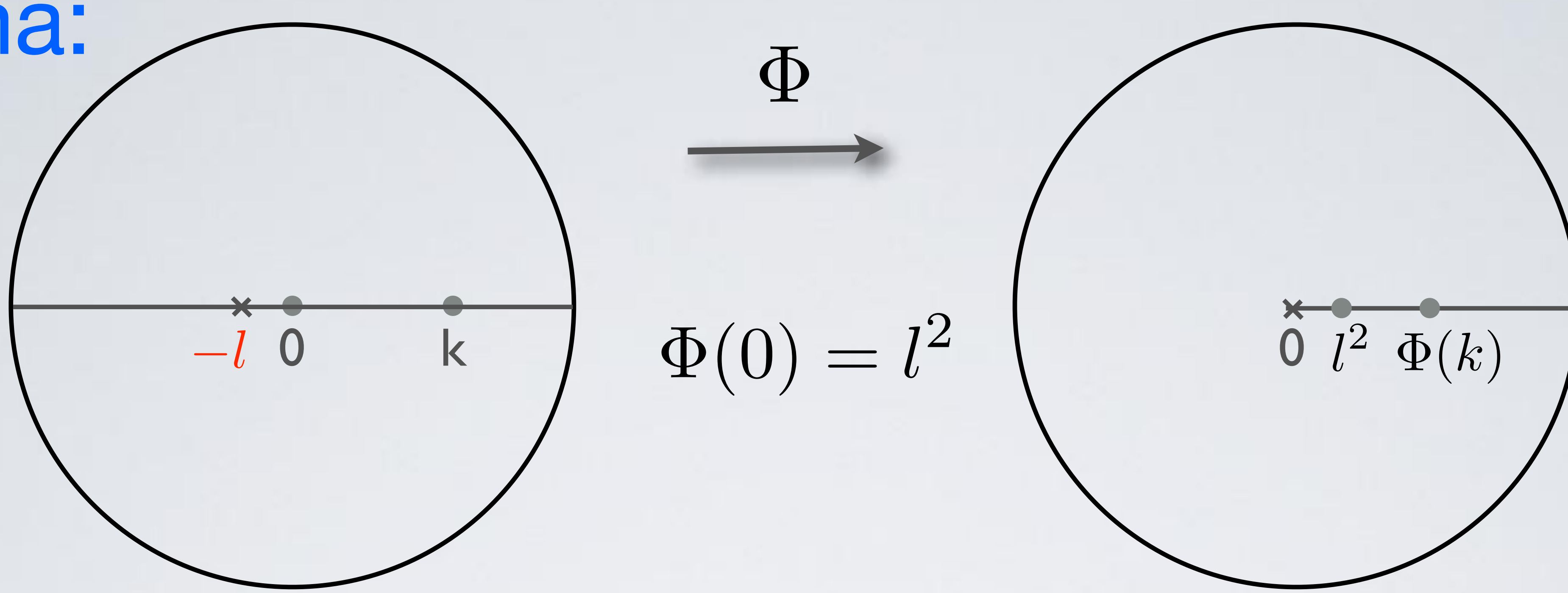
'freeze' p at 0: $\frac{\mathrm{I}_p}{\Lambda_p(0)} = \dim C_0 = t$ $\Phi(0) = 1 - t =: l^2$

$$\Phi(k) = ?$$

Remark: Schwarz-lemma ⟶ Higher Integrability

BLASCHKE PRODUCT

Lemma:



$$\Phi(0) = l^2$$

$$\Phi : \mathbb{D} \to \mathbb{D}$$

$$\Phi(\lambda) \geq 0$$

$$\lambda \in \mathbf{R}$$

$$\Rightarrow \quad \Phi(k) \leq \left(\frac{k+l}{1+kl}\right)^2 = B_{-l}(k)$$

$$\dim C_k \geq \frac{I_p}{\Lambda_p(k)} = 1 - \Phi(k) \geq 1 - B_{-\sqrt{1-t}}(k) = \frac{t(1-k^2)}{(1+k\sqrt{1-t})^2}$$

$\square$

3-POINT SCHWARZ LEMMA

Beardon-Minda

Theorem 3.1. *Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic but not a conformal automorphism of $\mathbb{D}$. Then for any z, w and v in $\mathbb{D}$,*

$$d\big(f^*(z,v), f^*(w,v)\big) \le d(z,w). \tag{3.1}$$

Further, equality holds in (3.1) *if and only if f is a Blaschke product of degree two.*

Proof:

$$[z,w] = \frac{z-w}{1-\bar{w}z} \qquad |[z,w]| = \tanh \frac{1}{2} d(z,w)$$

$$f^*(z,w) = \frac{[fz, fw]}{[z,w]} \qquad |f^*(z,w)| < 1 \quad \text{(Schwarz-Pick)}$$

Apply another Schwarz-lemma to $z \mapsto f^*(z,v)$ $\square$

Remark: For previous lemma, pick *v*=-*I* as auxiliary point.

SPECTRUM OF QUASIDISKS

$\varphi\colon \mathbb{D} \to \Omega$ conformal with K-quasiconformal extension

$$\beta_\varphi(t) = \inf\left\{\beta\colon \int |\varphi'(re^{i\theta})|^t d\theta = O(\frac{1}{1-r})^\beta\right\}$$ *integral means spectrum*

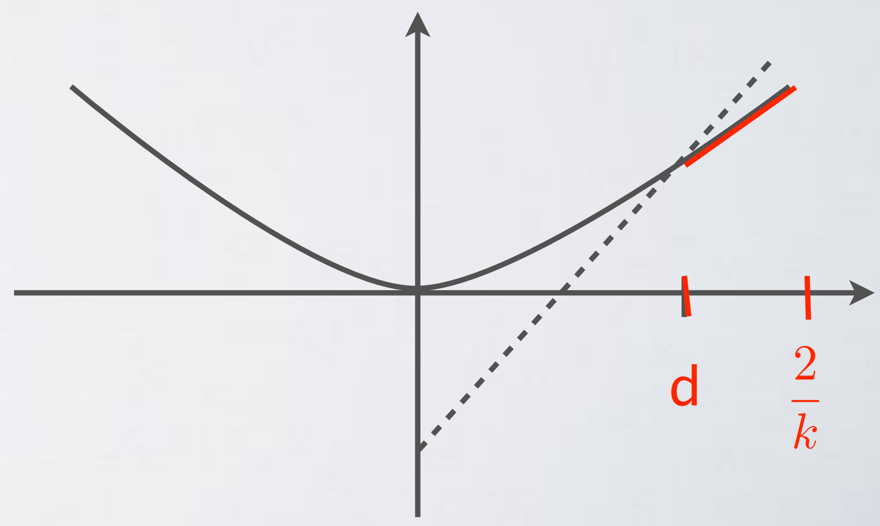
$$B_K(t) = \sup_\varphi \beta_\varphi(t)$$

Conjecture: (universal spectrum) $B_K(t) = \frac{k^2 t^2}{4}$ for $|t| \le \frac{2}{k}$, $k = \frac{K-1}{K+1}$.

Theorem: $B_K(t) \le \frac{k^2 t^2}{4}$ for $t \ge d$, $1 \le d \le 2$ $\frac{k^2 d^2}{4} = d - 1$.

$t = d$ $1 + k^2$-bound for quasicircles

$t = \frac{2}{k}$ higher integrability up to $p < \frac{2(K+1)}{K-1}$

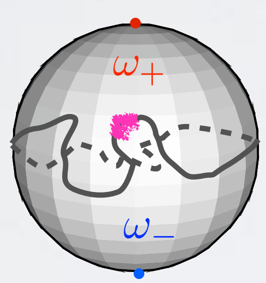


P-Tolsa-U-Tuero: $\varphi' \in \mathrm{weak} - L^{2/k}$

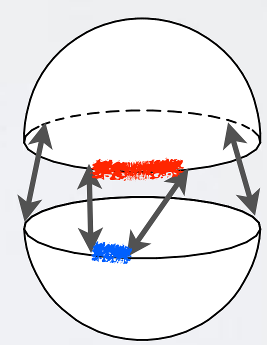
OPEN QUESTIONS

- **Sharpness of the estimates**
  existence of quasicircles with dimension $1+k^2$
  → *sharpness of quasisymmetric spectrum*
  → $t^2/4$ *lower bound for integral means spectrum*

- **Understand the precise relation**
  multifractality of ω ↔ *singularity of the welding*



$$|\Phi|^2 \leq \Phi_- \Phi_+ \ ??$$



- **Compression for general (non-symmetric) qc maps**
  Same multifractal spectrum? $\dim \varphi(m) \geq 1 - k^2$ *still true.*