PROBABILISTIC LIMIT SHAPES AND HARMONIC FUNCTIONS

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joint work with

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©Avid seminar - October 2021

TWO SIDES

Gradient variational problems in $\mathbb{R}^2$
arXiv:2006.01219, 2020

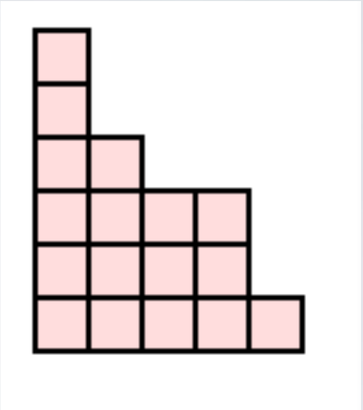
The genus-zero five-vertex model
arXiv:2101.04195, 2021

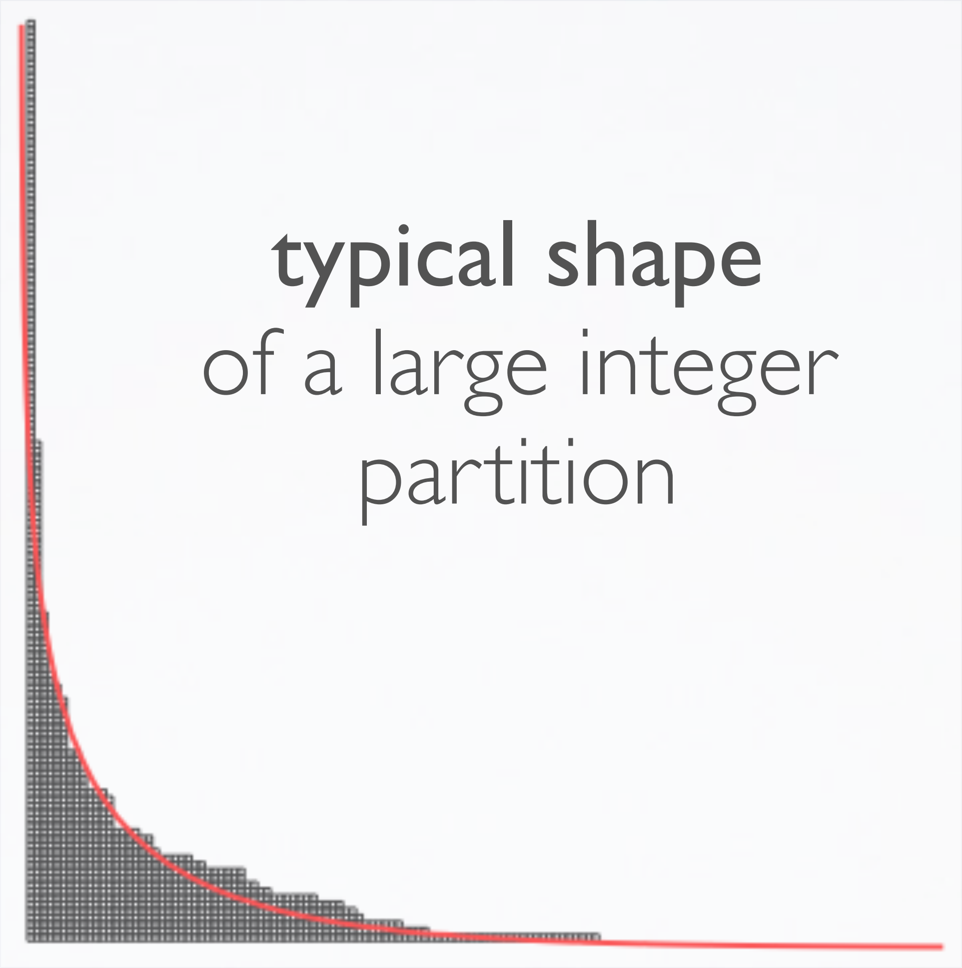
LIMIT SHAPE OF 2D YOUNG DIAGRAMS

Vershik

integer partitions

17=5+4+4+2+1+1

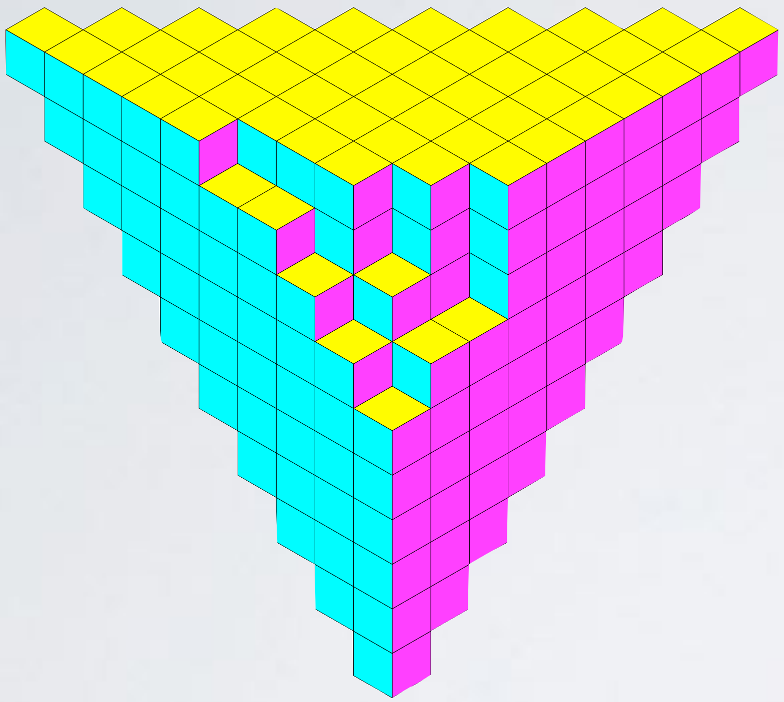


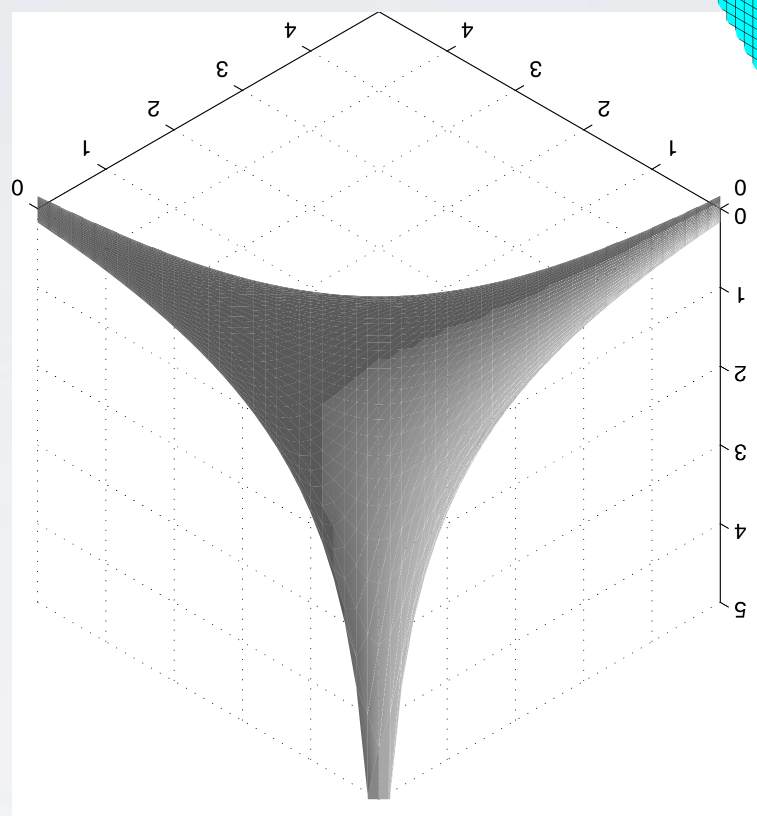


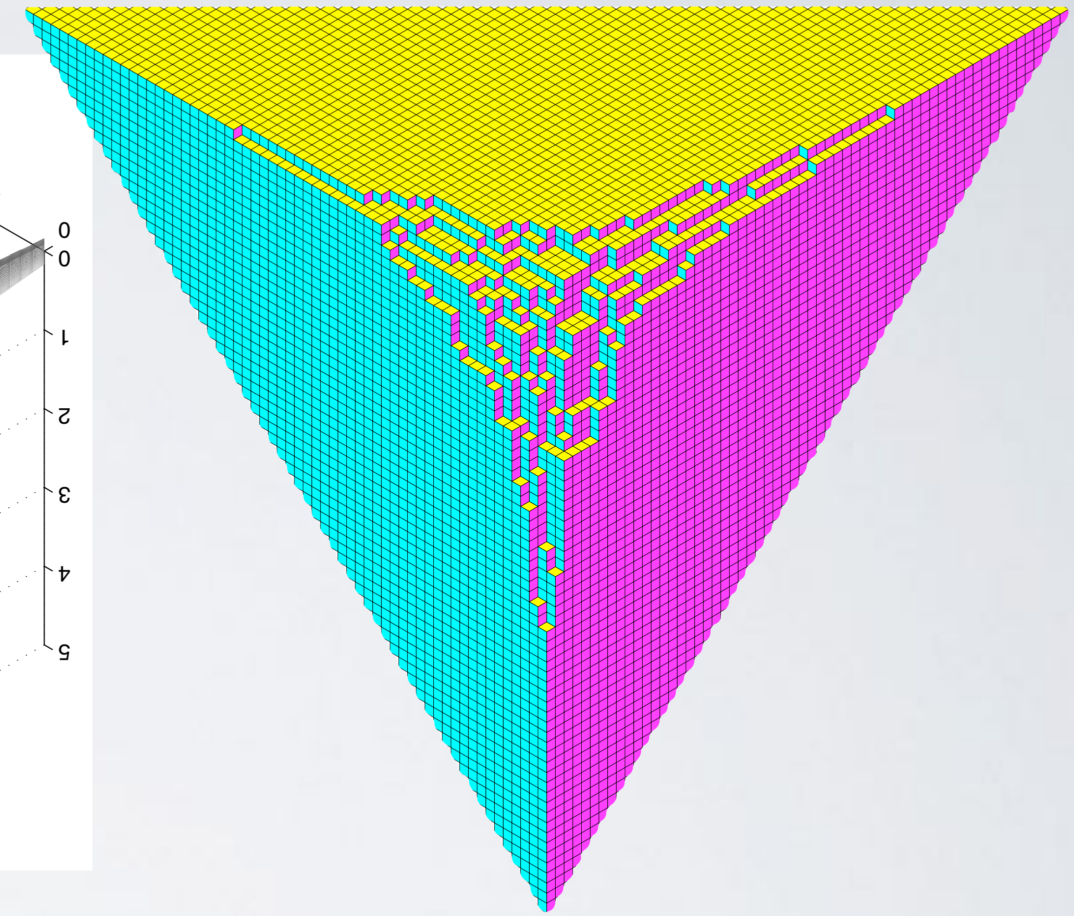
$$e^{-cx} + e^{-cy} = 1$$

3D YOUNG DIAGRAM LIMIT SHAPE

Cerf-Kenyon, Okounkov-Reshetikhin



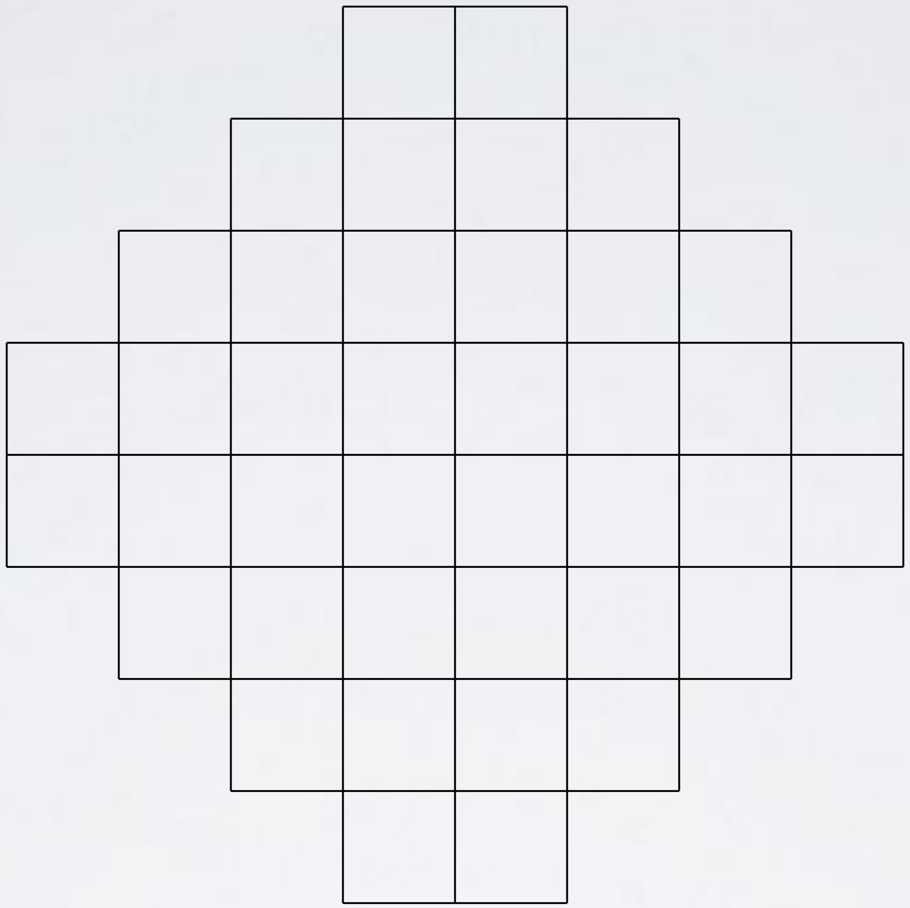




"*melting*" equilibrium crystal

planar projection: lozenge tilings, algebraic geometry

THE AZTEC DIAMOND



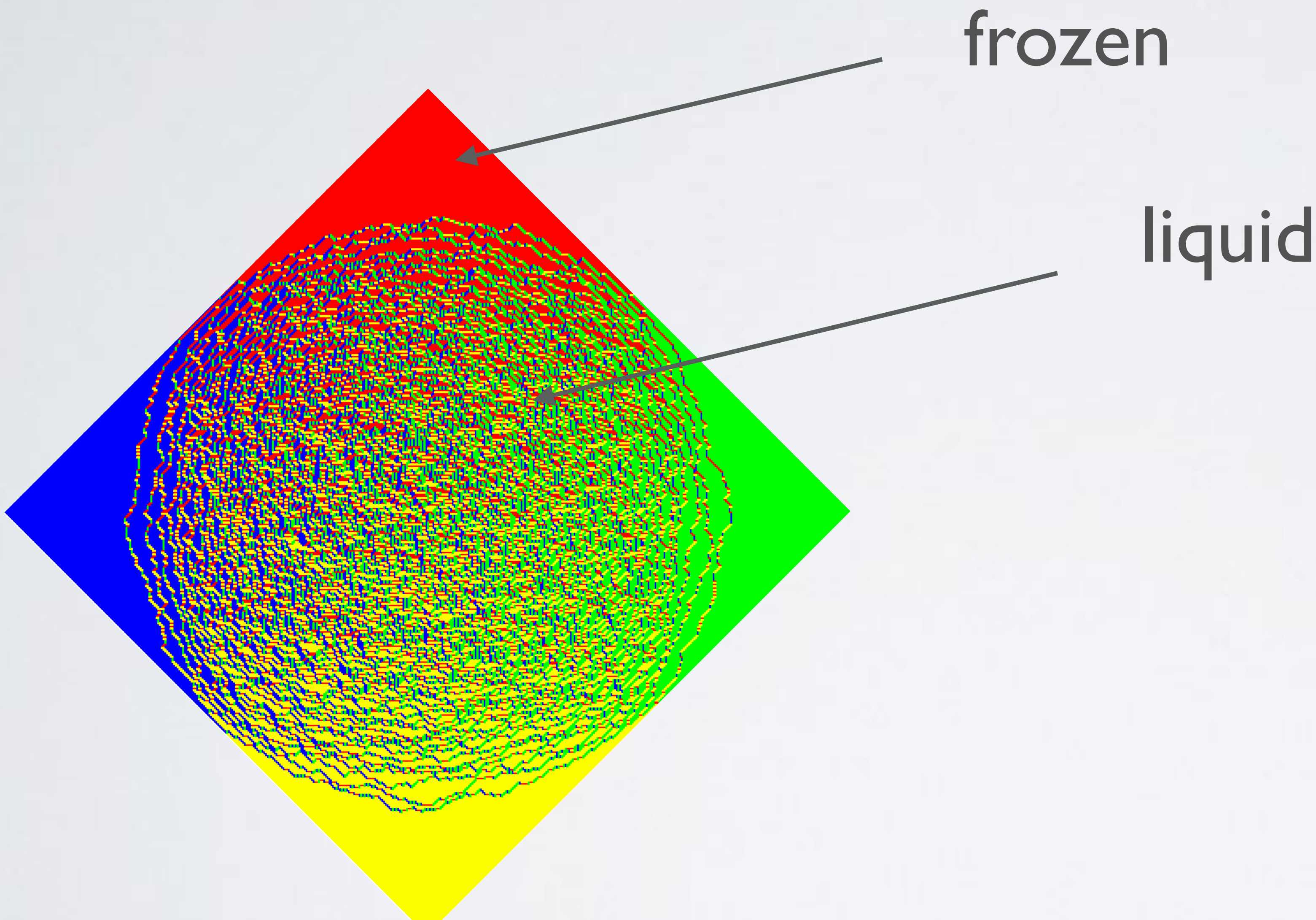
consider a uniformly random tiling by dominos (2x1)

How does it look like?

THE ARCTIC CIRCLE

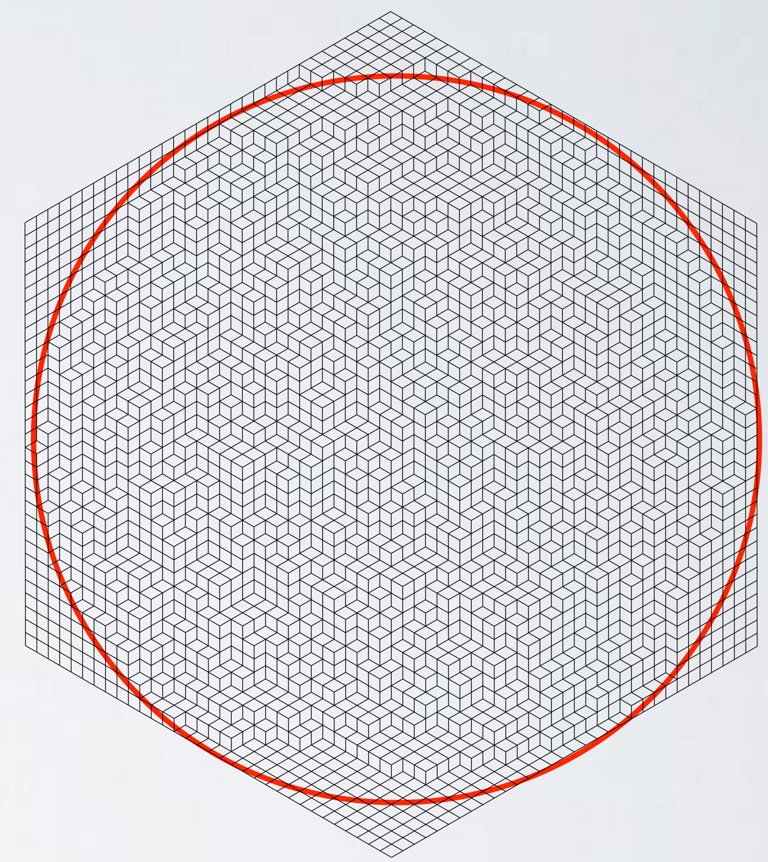
Jockusch-Propp-Shor

dominos



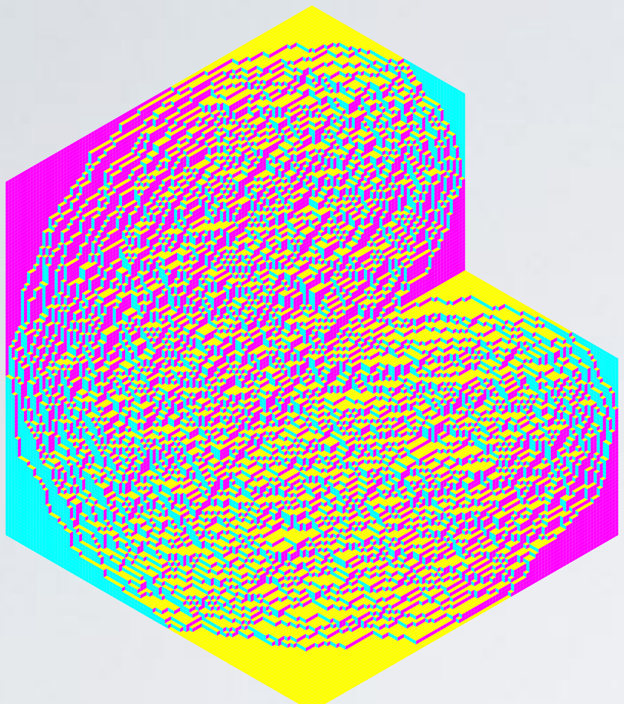
Cohn-Larsen-Propp

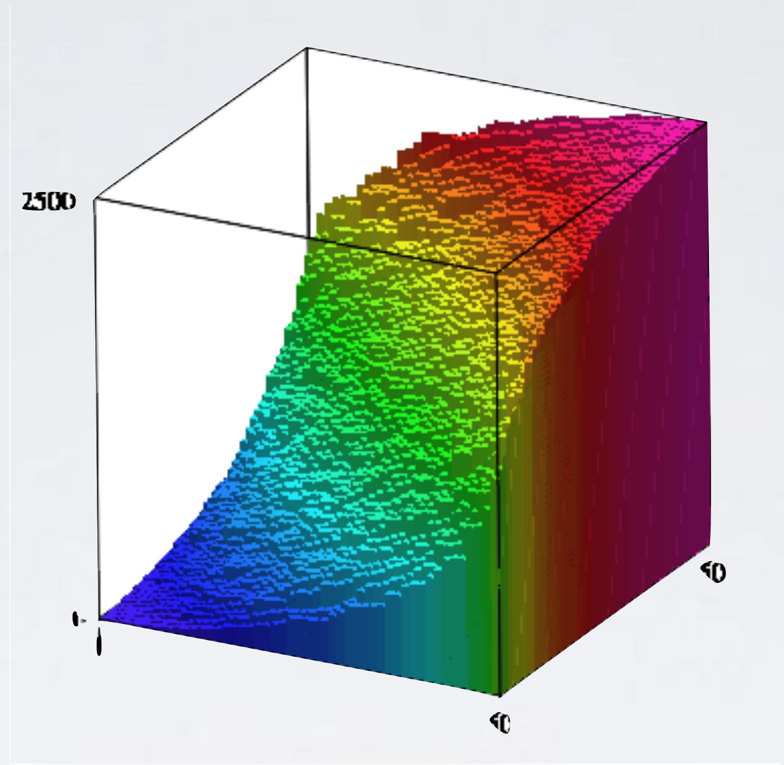
lozenges

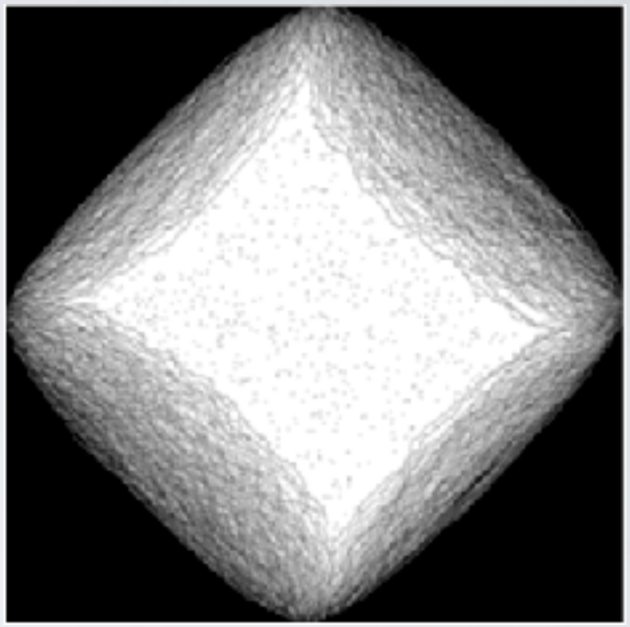


harmonic functions ?

ZOO OF LIMIT SHAPES







general boundary conditions & a variety of models

variational approach

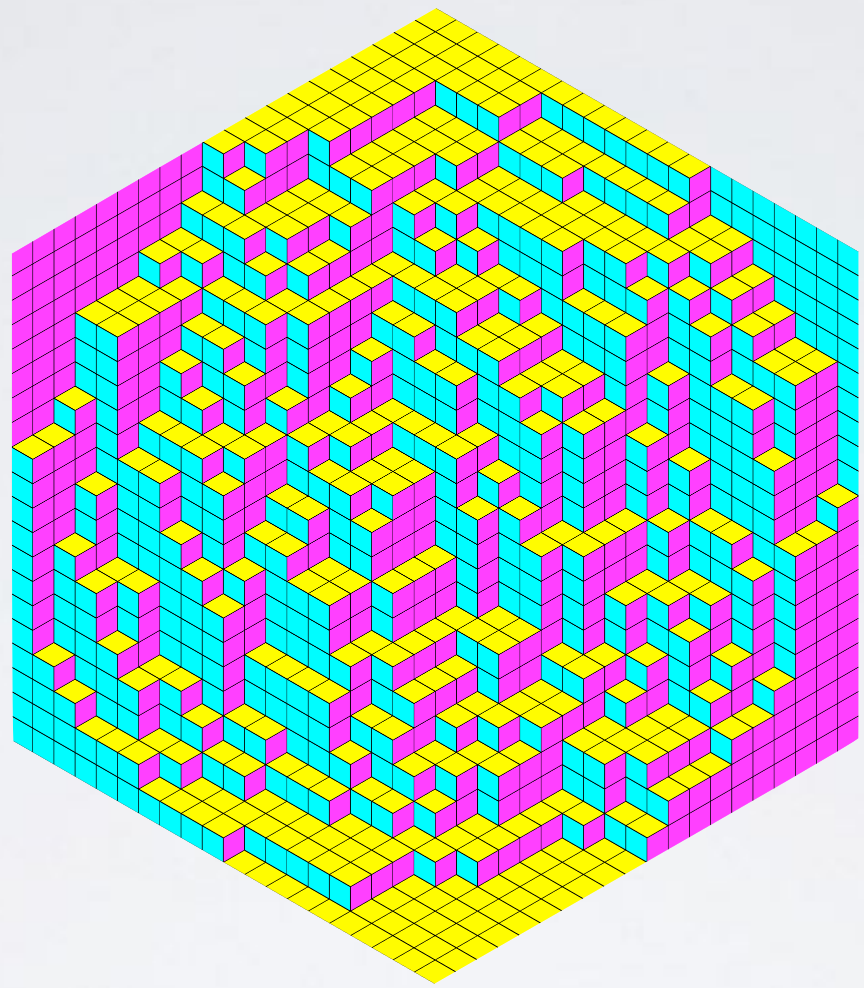
dimer model
(domino/lozenge tilings
etc)

random
Young tableaux

five-vertex model

HEIGHT FUNCTION

Thurston



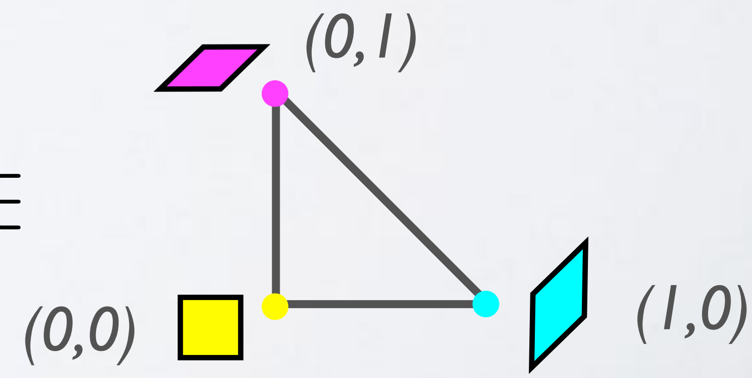
3d surface
(simulations
by A. & M. Borodin)

random stepped surface

$\mathcal{N}$ = *unit triangle*

height function

$\nabla h \in$

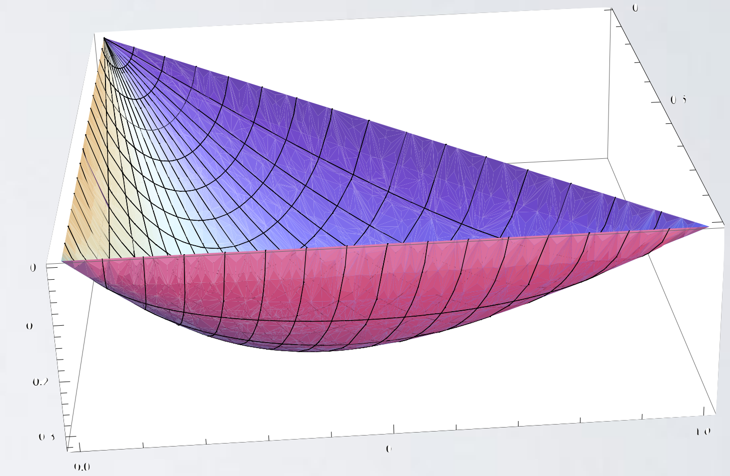


LIMIT SHAPE THEOREM

Cohn-Kenyon-Propp

The random surfaces as mesh size →0 concentrate around a deterministic surface, called limit shape

The limit shape is a minimizer of a variational problem



'minimal surface' spanning a wire-frame

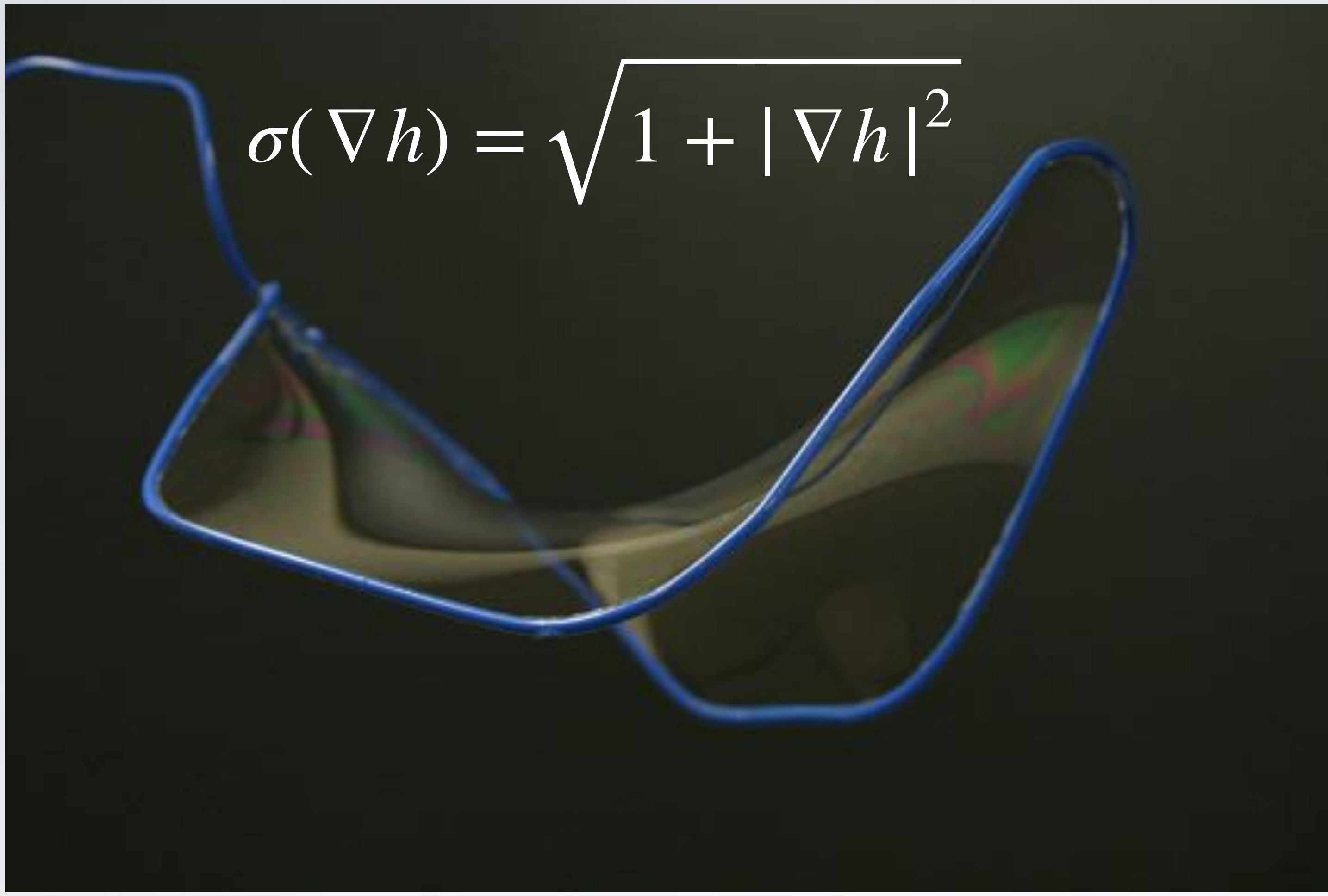
$h : \Omega \to \mathbb{R}$ Lipschitz

$$\min_h \int_\Omega \sigma(\nabla h), \quad \begin{array}{c} \nabla h \in \mathcal{N} \\ h_{|\partial\Omega} = h_0 \end{array}$$

analytic, strictly convex surface tension in the interior

singular and *degenerates* on the boundary

LIMIT SHAPE THEOREM

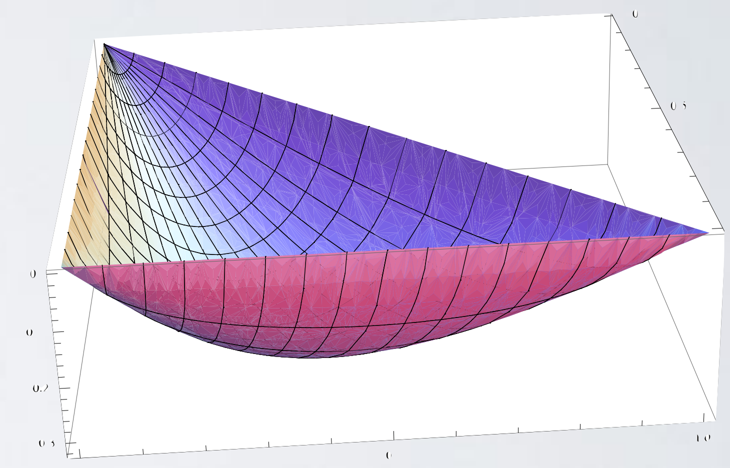


PP

h size $\to 0$

istic surface,

er of a



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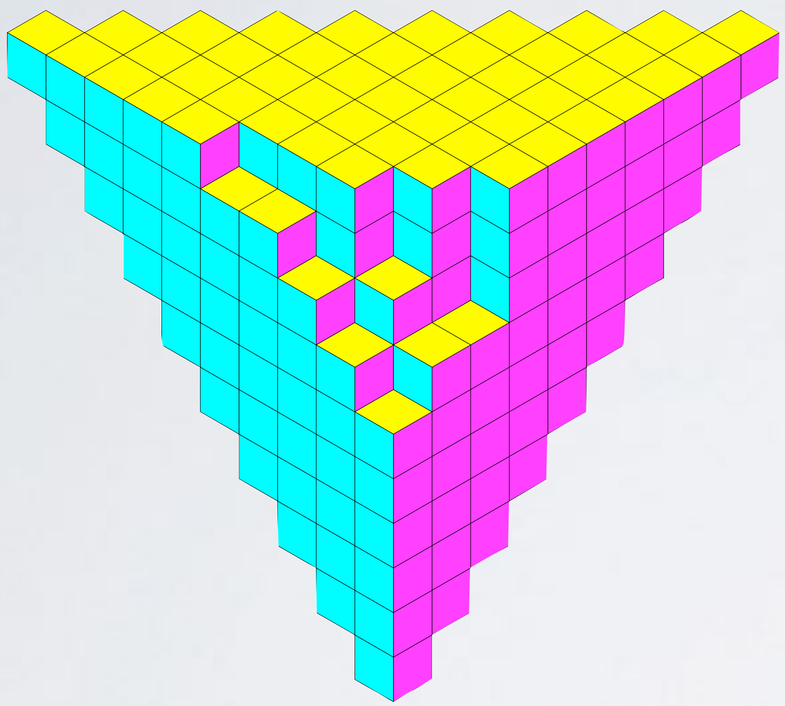
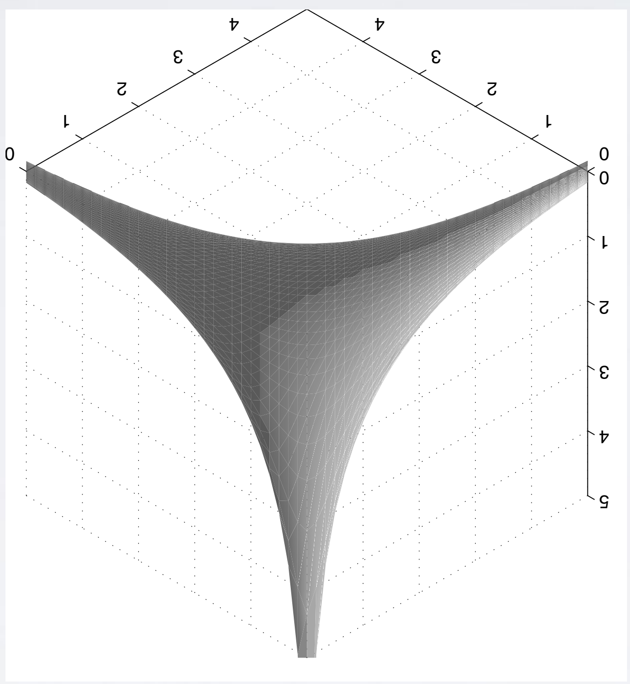
WULFF SHAPE

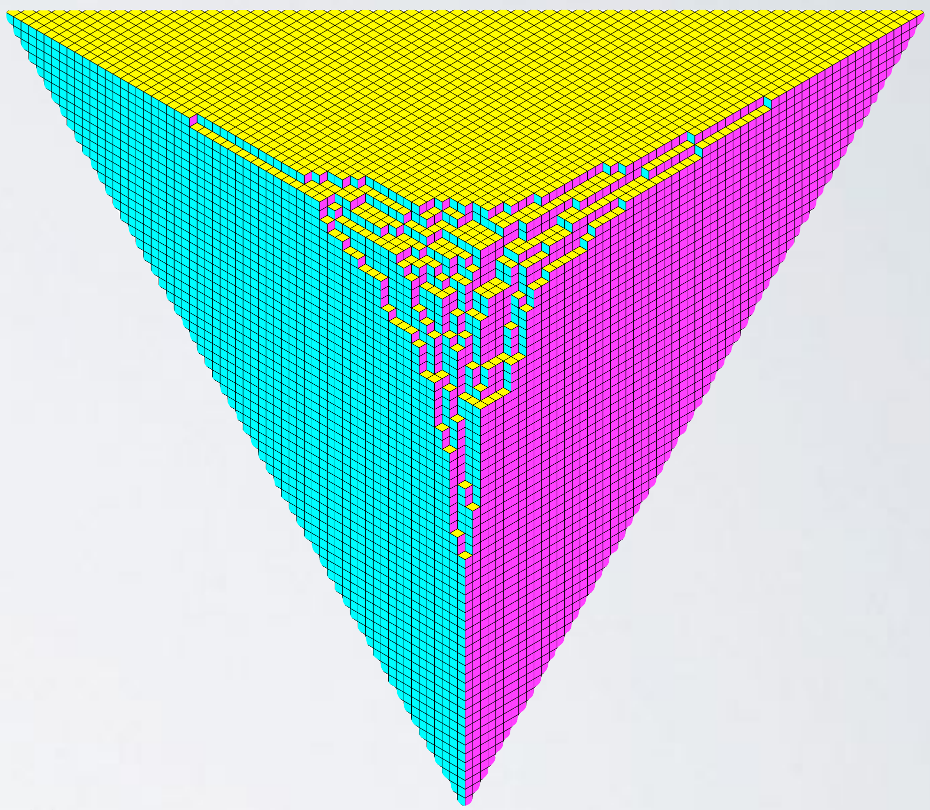
Wulff shape - Legendre dual of surface tension

itself a limit shape

(lozenge tilings ↔ 3D Young diagram)







"fundamental solution"

(facets, facet-rough transition, phases, algebraic boundary etc)

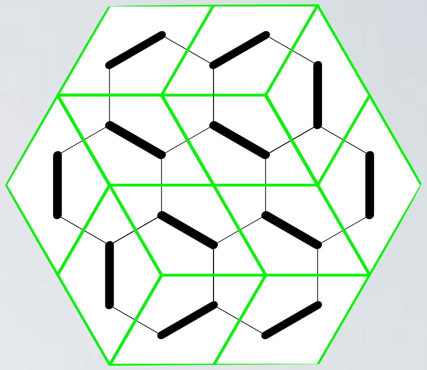
Kenyon-Okounkov-Sheffield

$\det D^2\sigma \equiv \pi^2$ for the dimer model (determinantal)

DETERMINANTAL MODEL

Kasteleyn

The number of dimer covers for G (a subgraph of hexagonal lattice) is $|\det(K)|$, where



K is the (bi)adjacency matrix of the bipartite graph G.

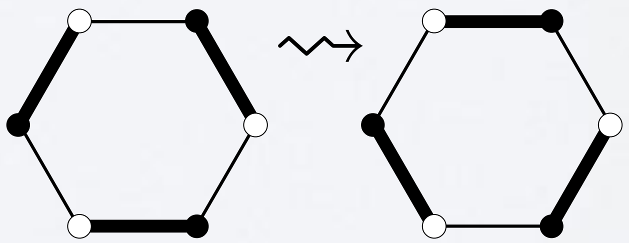
$$\det(K) = \sum_{\sigma \in S_n} \left(\mathrm{sgn}(\sigma) \prod_{i=1}^{n} K(b_i, w_{\sigma(i)}) \right)$$

determinant counts *perfect matchings* with signs

key point: signs are consistent

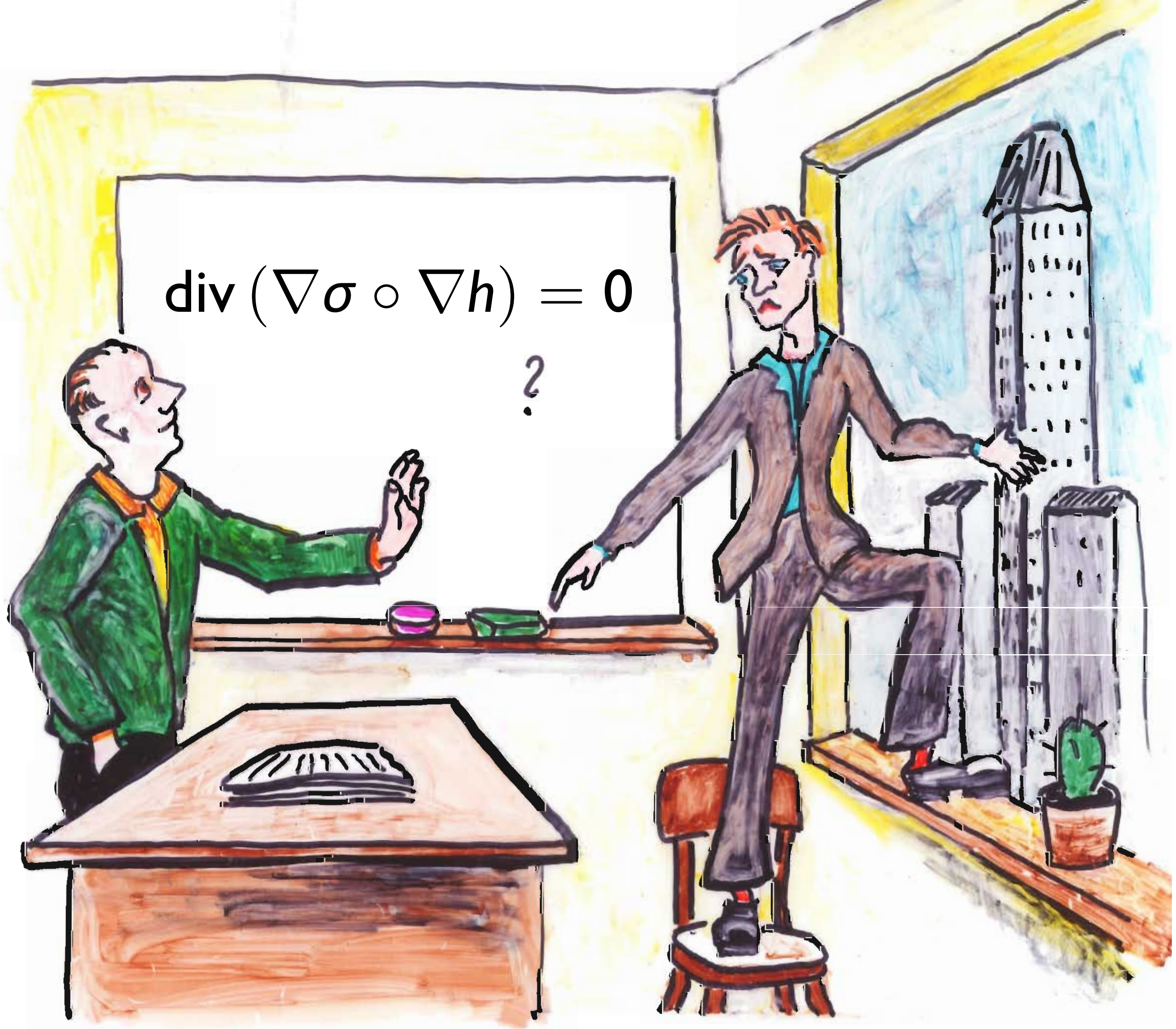
a hexagon flip move changes σ by a 3-cycle (even permutation)

adds/removes cubes



INTEGRABLE PDE ?

$$\begin{tikzcd} \text{limit shape} \arrow[d] & \text{variational principle} \arrow[l] & \text{Wulff shape} \arrow[l] & \text{free energy in magnetic field} \arrow[l] \\ \text{PDE} \arrow[rrr, leftrightarrow, "?"] & & & \text{integrability of the model} \arrow[u] \end{tikzcd}$$



Don't jump, *"complexify to simplify"* !

a not-so-"hidden" complex variable

(fluctuations, integrability, isothermal)

ISOTHERMAL COORDINATE

Gauss

Riemannian metric associated to the Wulff shape

let z be the isothermal coordinate

$$\sigma_{ss} ds^2 + 2\sigma_{st} ds dt + \sigma_{tt} dt^2 = \rho \, |dz|^2$$

$$(s,t) \in \mathcal{N} \leftrightarrow z \in \mathbb{C}$$

$$(x,y) \in \mathscr{L} \mapsto \nabla h(x,y) \mapsto z(x,y)$$

liquid

this is the conformal coordinate of the model

“Write everything in terms of z”

κ-HARMONIC ENVELOPE

Kenyon-Prause

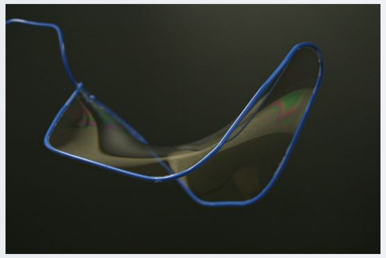
$$\kappa(z) = \sqrt{\det D^2\sigma} \quad \text{as a function of } z$$

$$\nabla \cdot \kappa \nabla u = 0$$

Thm: s, t and h-(sx+ty) are all κ-harmonic(z) in the liquid region

(multi-valued in z)

Corollary: Limit shapes in the dimer model are *envelopes* of harmonically moving planes in R^3

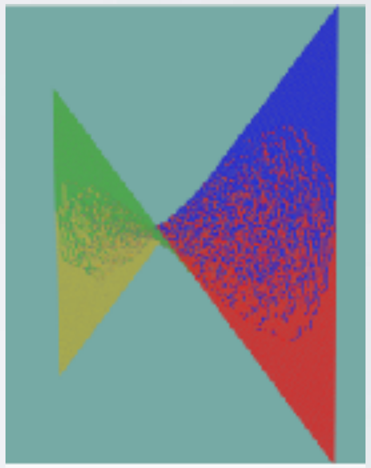


minimal surfaces

$$(x, y, h(x, y))$$

$$\mu_z = 0$$

$$\longleftrightarrow$$



dimer limit shapes

$$(h_x, h_y, h - \nabla h \cdot (x, y))$$

$$\mu_{\bar{z}} = 0$$

κ-HARMONIC ENVELOPE

Kenyon-Prause

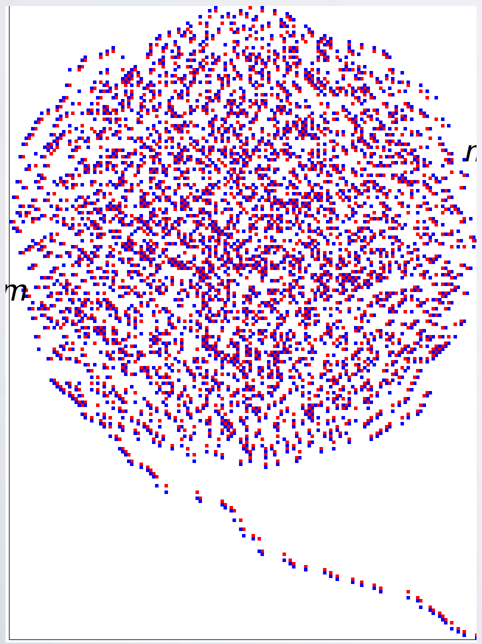
$$\kappa(z) = \sqrt{\det D^2\sigma} \quad \text{as a function of } z$$

$$\nabla \cdot \kappa \nabla u = 0$$

Thm: s, t and $h\text{-}(sx+ty)$ are all κ-harmonic(z) in the liquid region

(multi-valued in z)

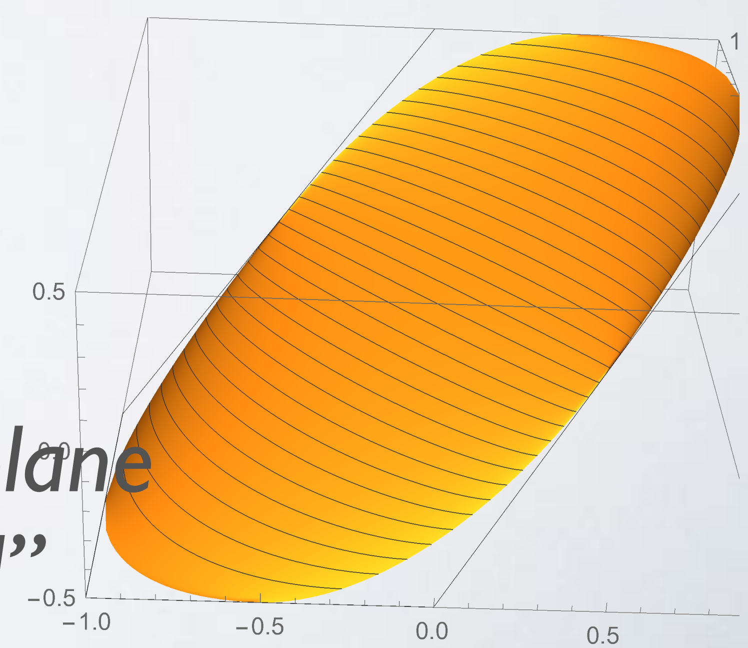
Corollary: Limit shapes in the dimer model are *envelopes* of harmonically moving planes in R^3



limit shape algorithm

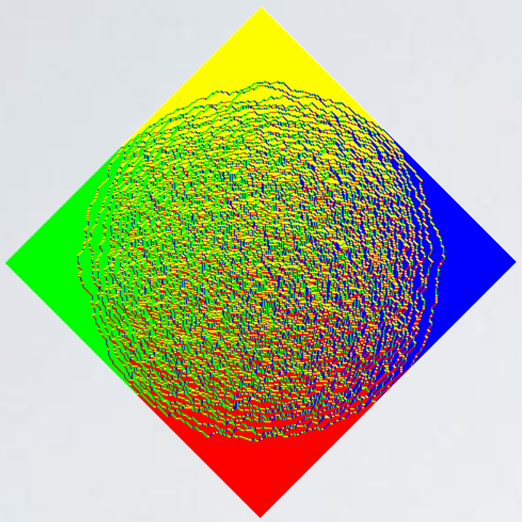
tangent method
Colomo-Sportiello

"tangent plane method"



BACK TO THE ARCTIC CIRCLE

envelope of planes $x_3 = sx + ty + c$



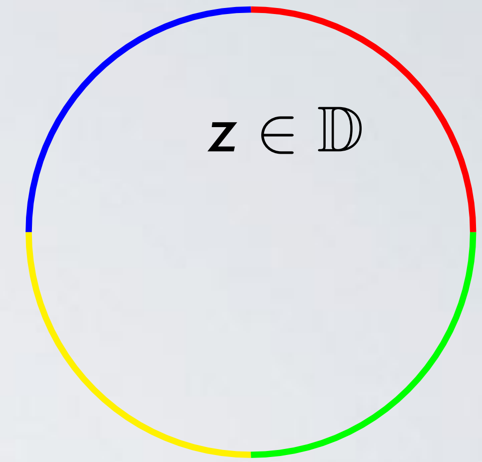
$(x, y) \in \mathcal{L}$

$$s_z x + t_z y + c_z = 0$$

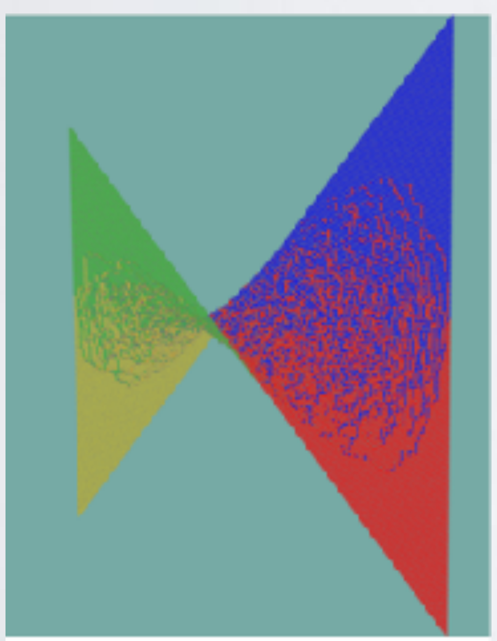
$$s = \omega - \omega$$

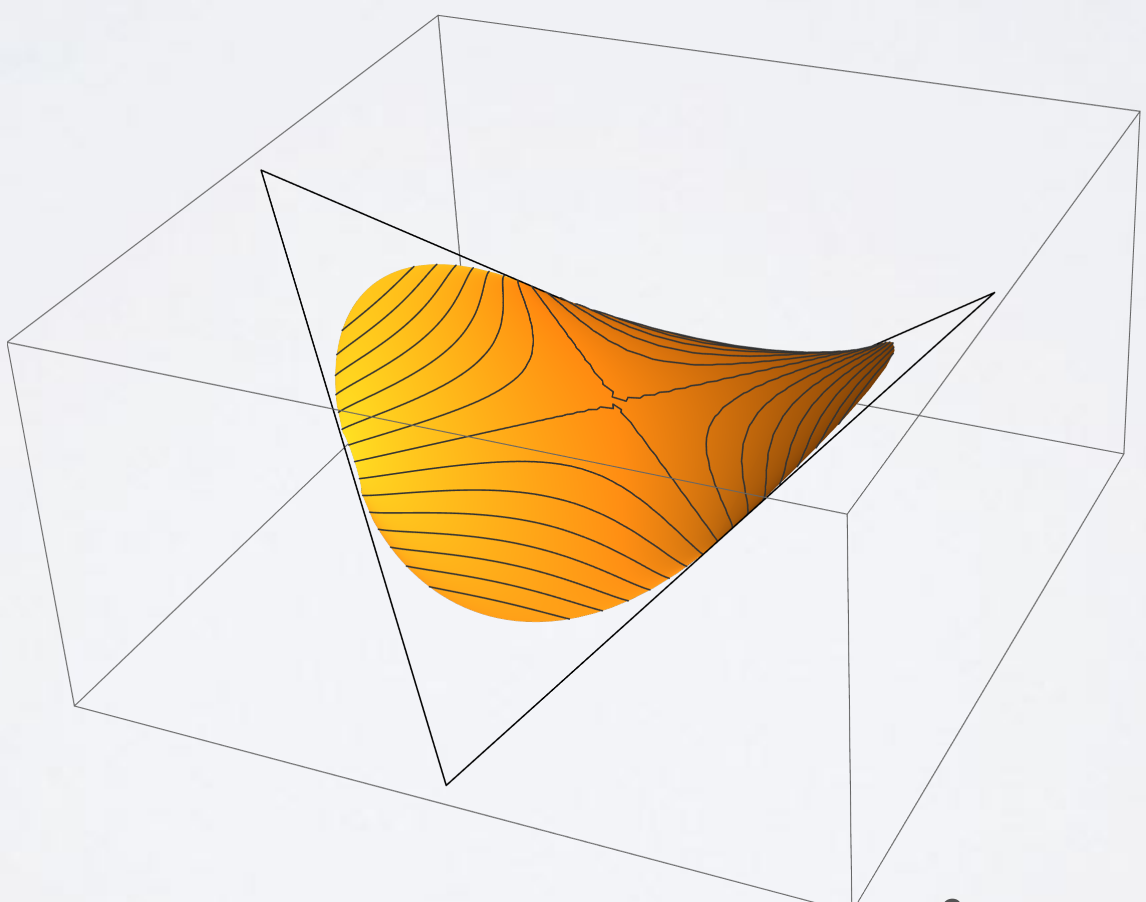
$$t = \omega - \omega$$

$$c = \omega - \omega + \omega - \omega$$

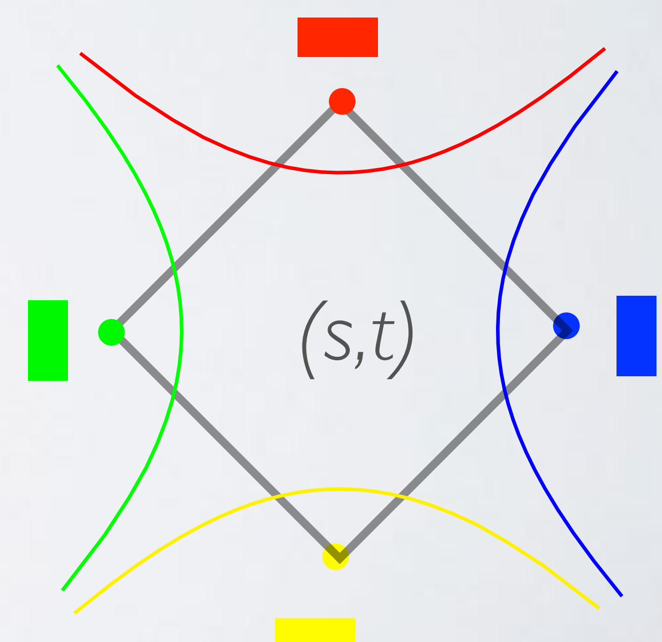


$\downarrow$ harmonic

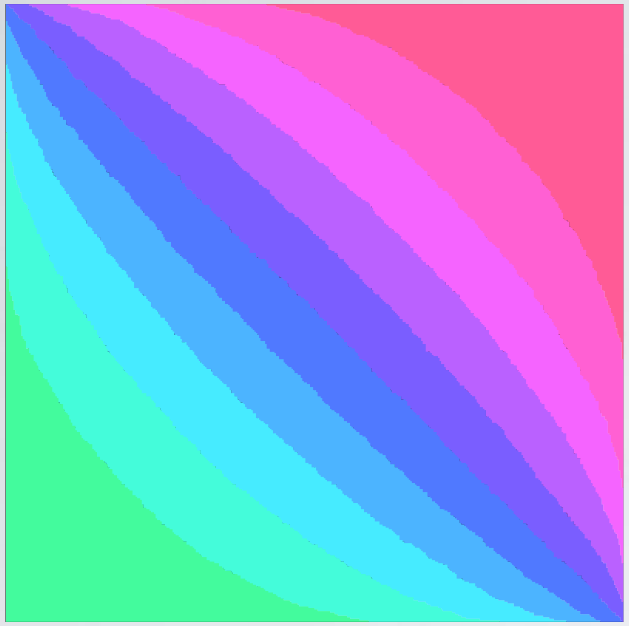




$z \mapsto (x(z), y(z), x_3(z)) \subset \mathbb{R}^3$



YOUNG TABLEAUX EXAMPLE

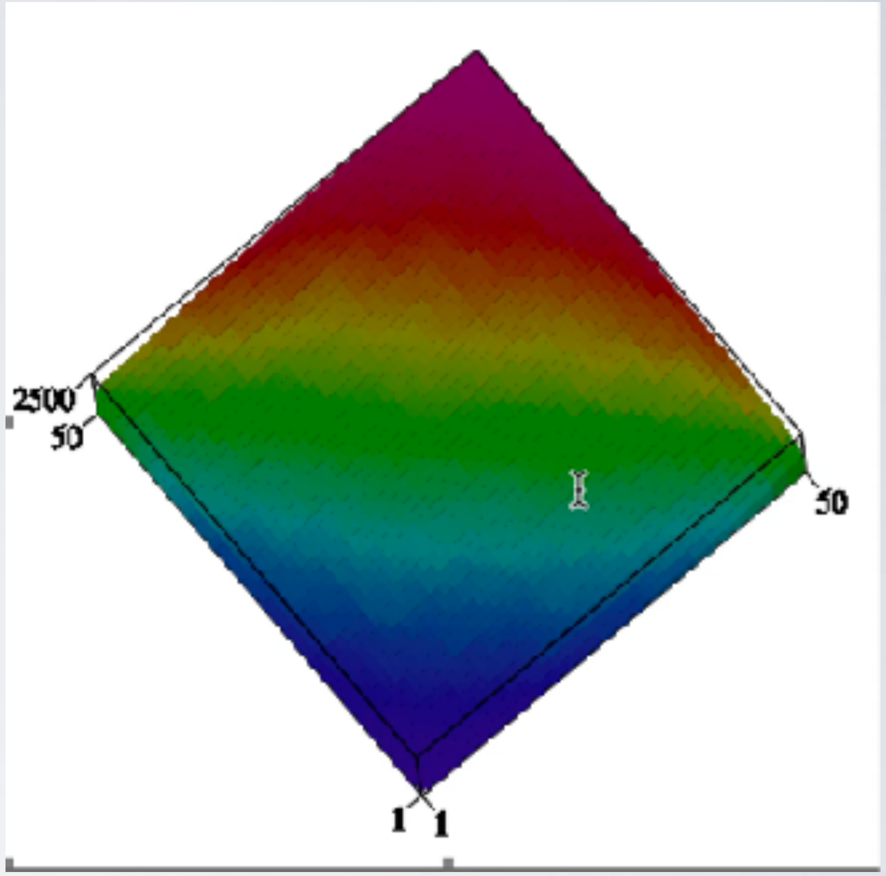


Dan Romik's
MacTableaux

square shape

Pittel-Romik

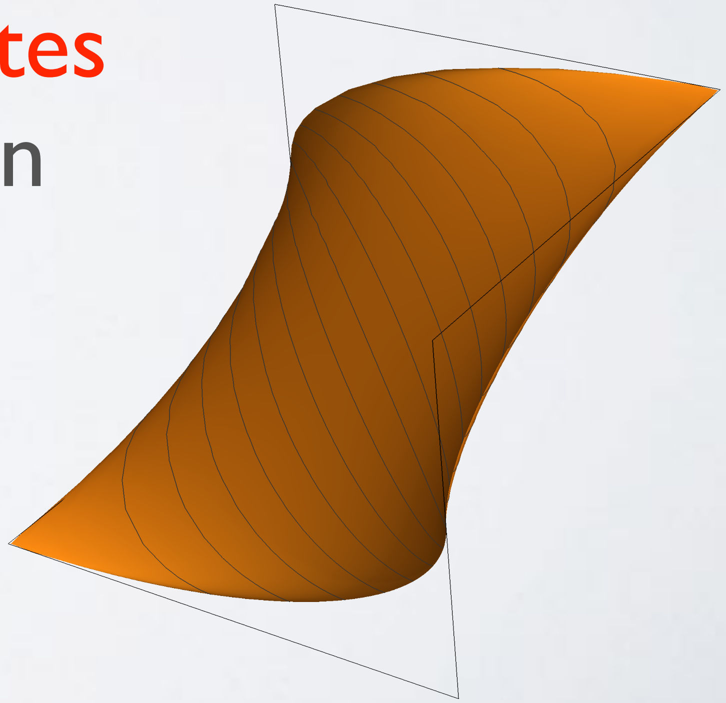
Biane



Cyril Banderier et al.'s
YoungPackage

W. Sun variational principle

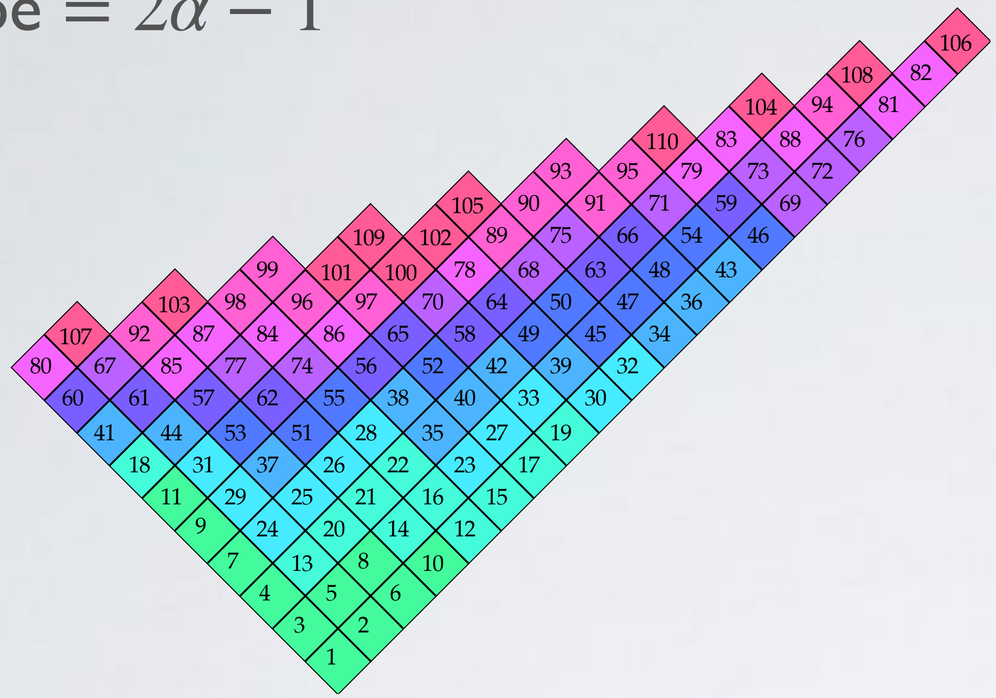
$\kappa \equiv 1 \longrightarrow$ harmonic coordinates
for surface tension



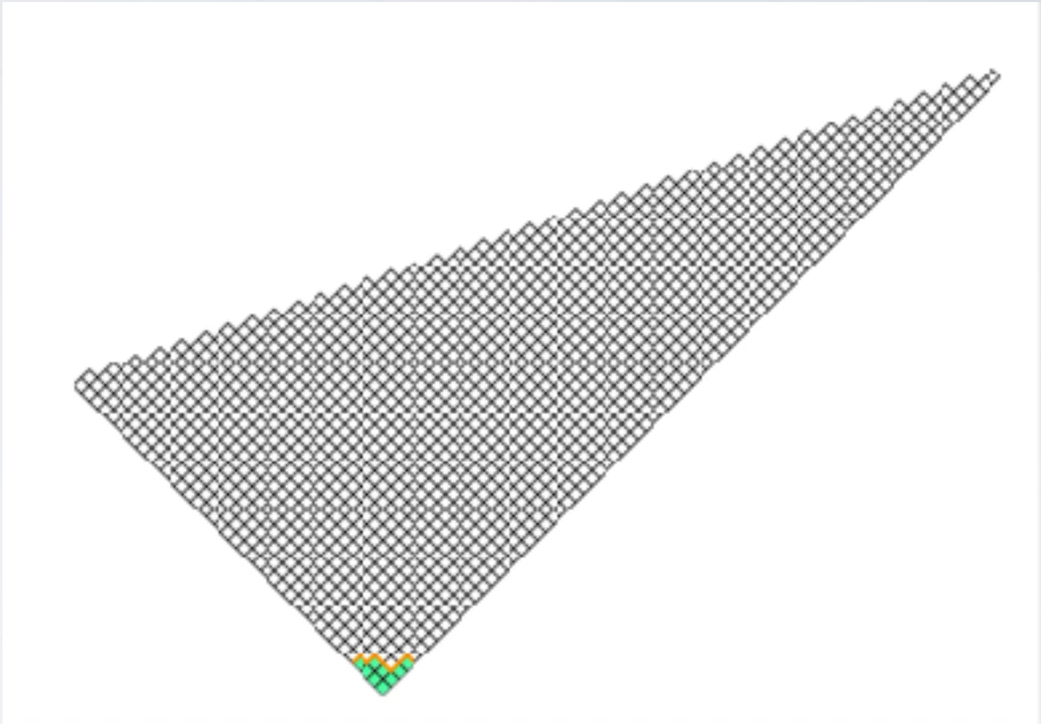
limit surface as harmonic envelope
(use harmonic extension of boundary facets)

SLANTED STAIRCASE

slope $= 2\alpha - 1$

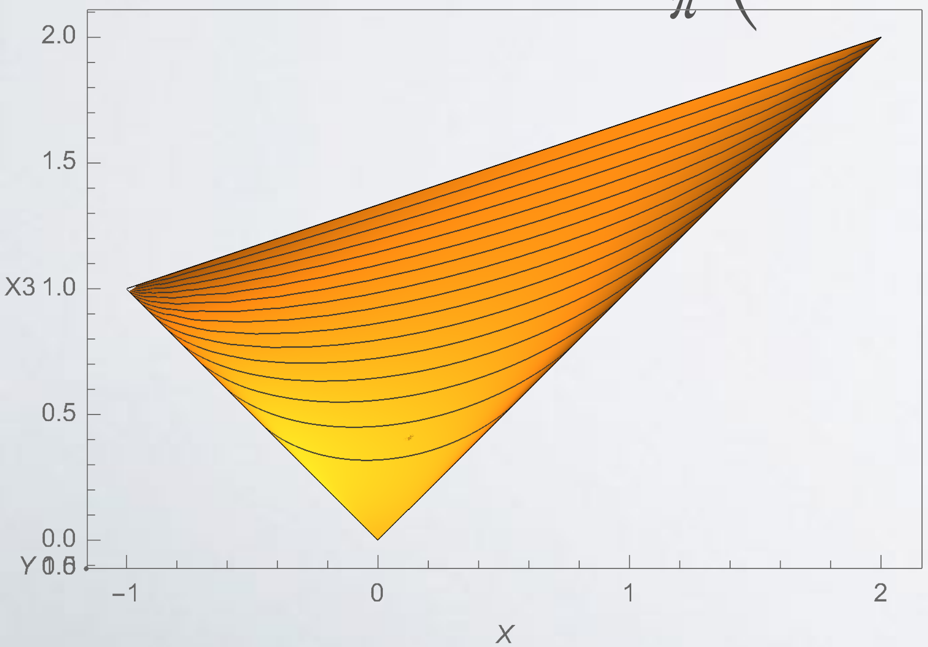


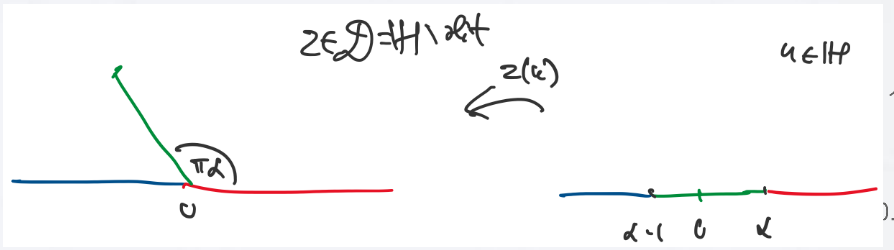
P



$$x(u) = -\frac{1}{1-\alpha}\frac{\mathrm{Im}\frac{u-2\alpha+1}{z(u)}}{\mathrm{Im}\frac{1}{z(u)}}, \quad y(u) = 1 - \frac{1}{1-\alpha}\frac{\mathrm{Im}(u-2\alpha+1)}{\mathrm{Im}z(u)} \qquad u \in \mathbb{H}$$

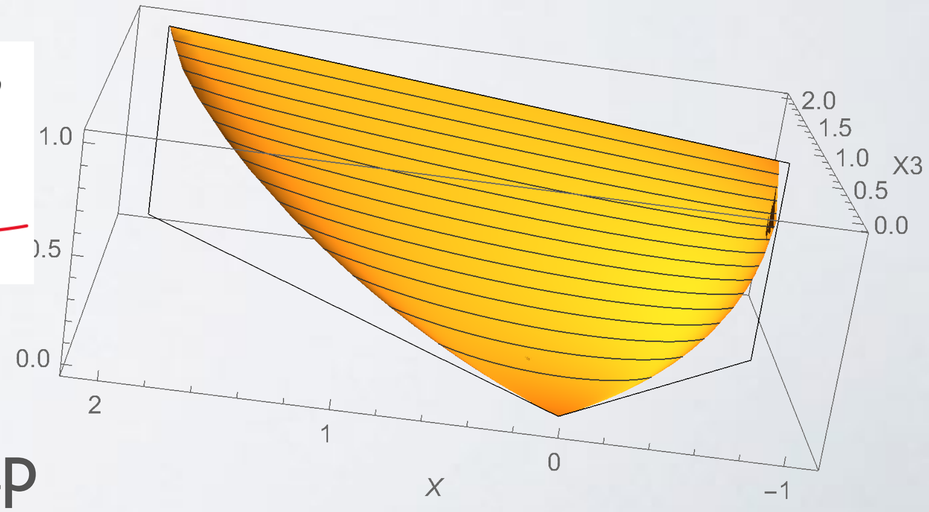
$$x_3(u) = \frac{2}{\pi}\left((\arg(z(u)) - 1)x(u) + \mathrm{Im}(z(u))(y(u)-1) + \frac{1}{1-\alpha}\mathrm{Im}\ u + \alpha \arg\frac{u-\alpha}{u-\alpha+1}\right)$$



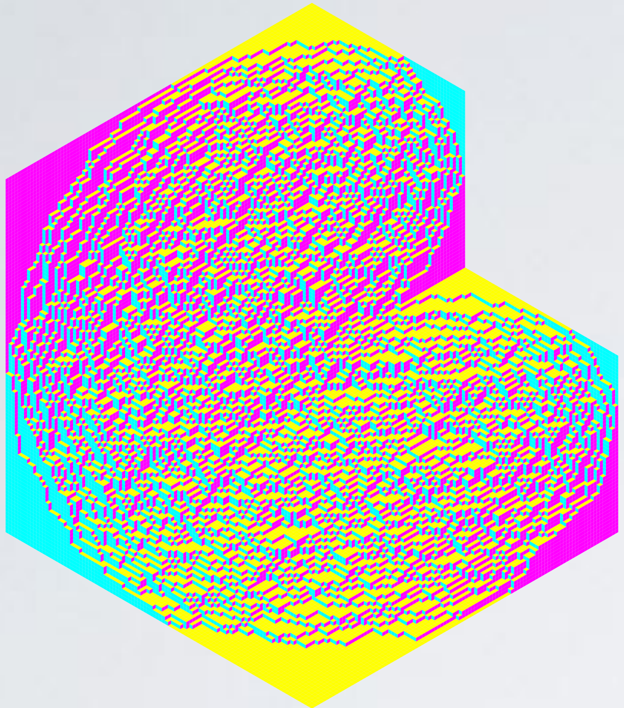


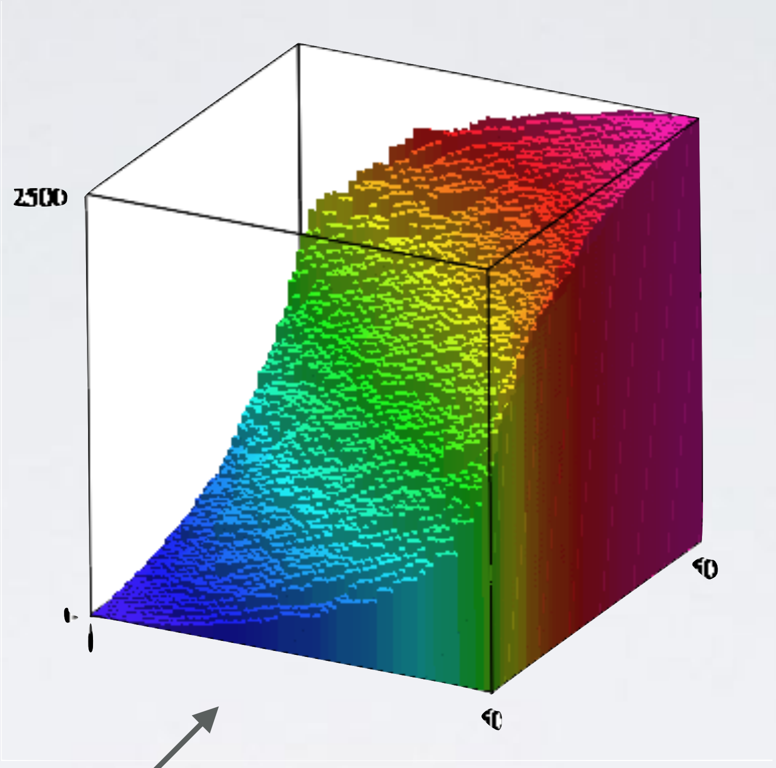
$u \mapsto z(u)$

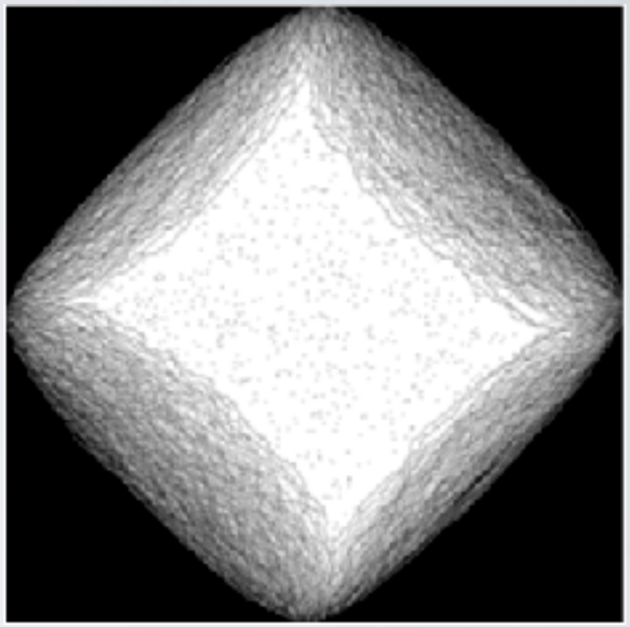
explicit (slit) conformal map



ZOO OF LIMIT SHAPES







non-determinantal

$\kappa = ?$

determinantal

$\kappa \equiv 1$

dimer model
(domino/lozenge tilings
etc)

random
Young tableaux

five-vertex model

TRIVIAL POTENTIAL

Kenyon-Prause

$$\nabla \cdot \kappa \nabla u = 0$$

reduction to Schrödinger equation

$$(-\Delta + q)(\kappa^{1/2} u) = 0 \qquad q = \frac{\Delta \kappa^{1/2}}{\kappa^{1/2}} \quad \text{potential}$$

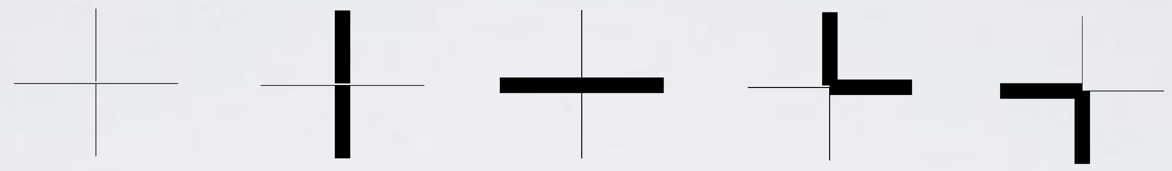
Def: a surface tension has *trivial potential* if $\sqrt[4]{\det D^2 \sigma}$ is a harmonic function of the intrinsic coordinate z

Then κ-harmonic: $$\frac{\text{harmonic}(z)}{\sqrt[4]{\det D^2 \sigma}} \qquad (q=0)$$

5-VERTEX MODEL

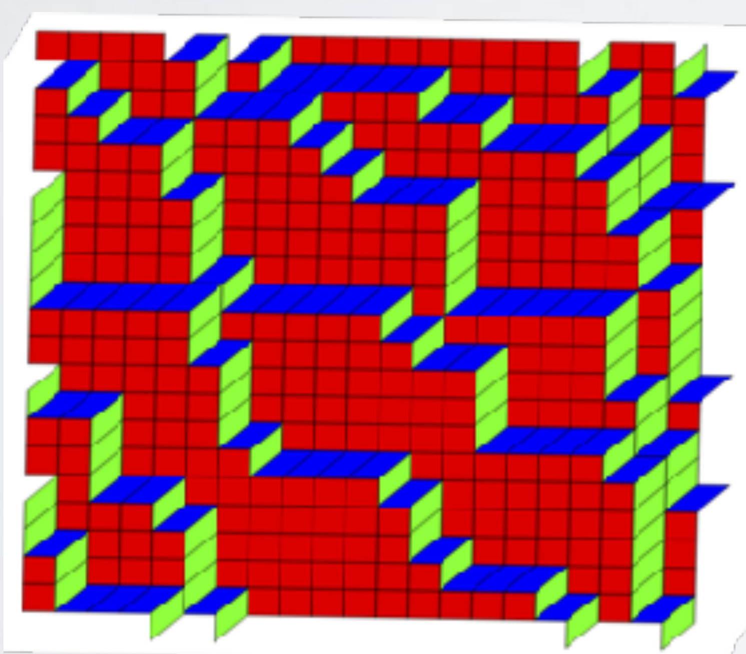
r≠1 (non-determinantal) *"interacting fermions"*

Bethe ansatz

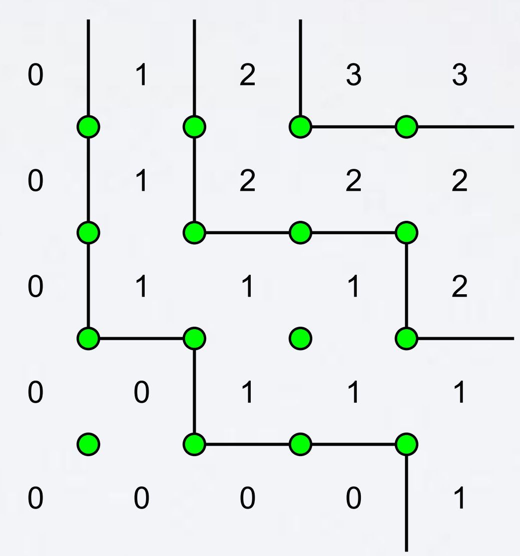


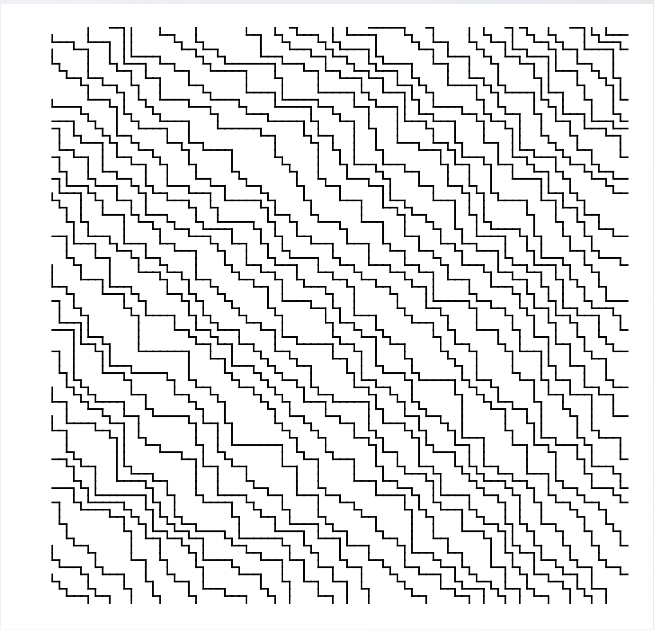
1 1 1 r r

$$P(\mathit{configuration}) \propto r^{\#\text{corners}}$$



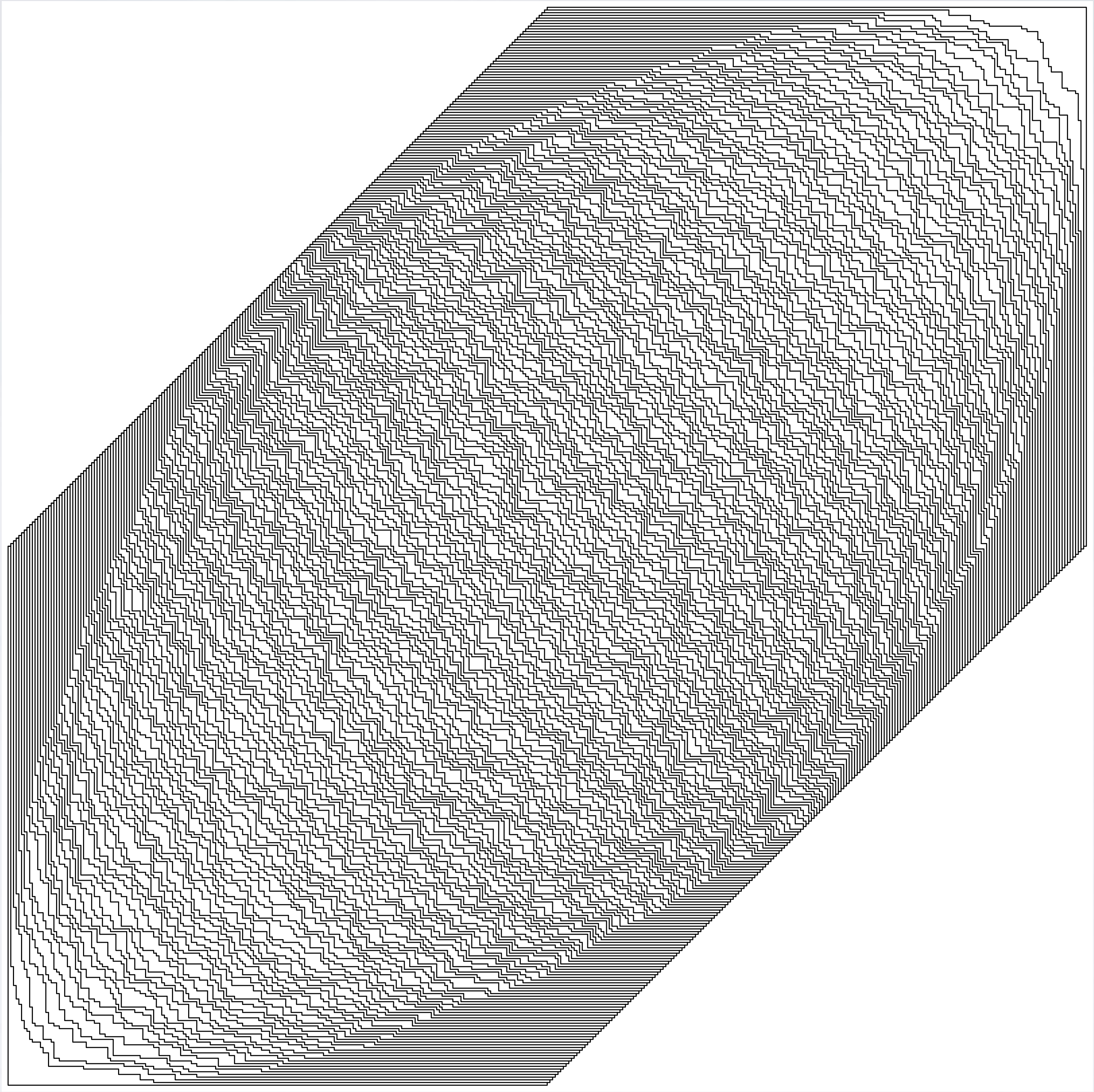
lozenge tilings with (blue-green) *interaction*





monotone non-intersecting lattice paths with corners *penalized*

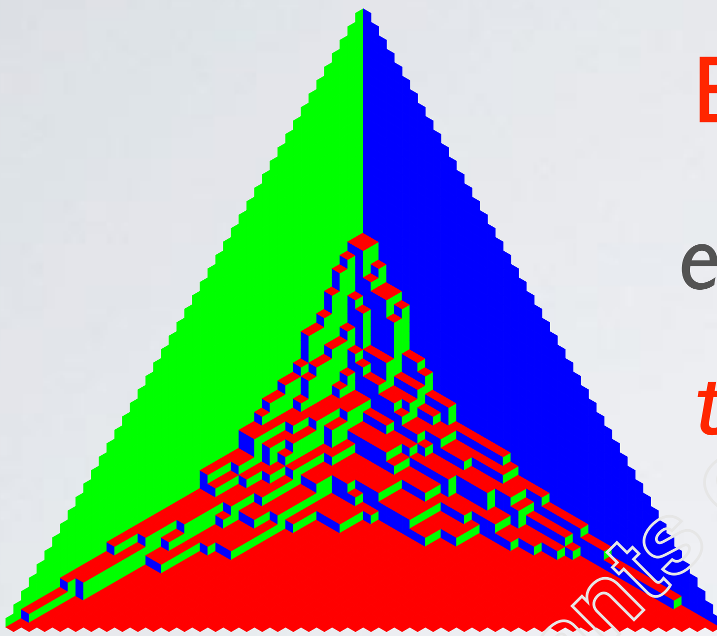
5-VERTEX BOXED PLANE PARTITION



r=0.6

5-VERTEX SURFACE TENSION

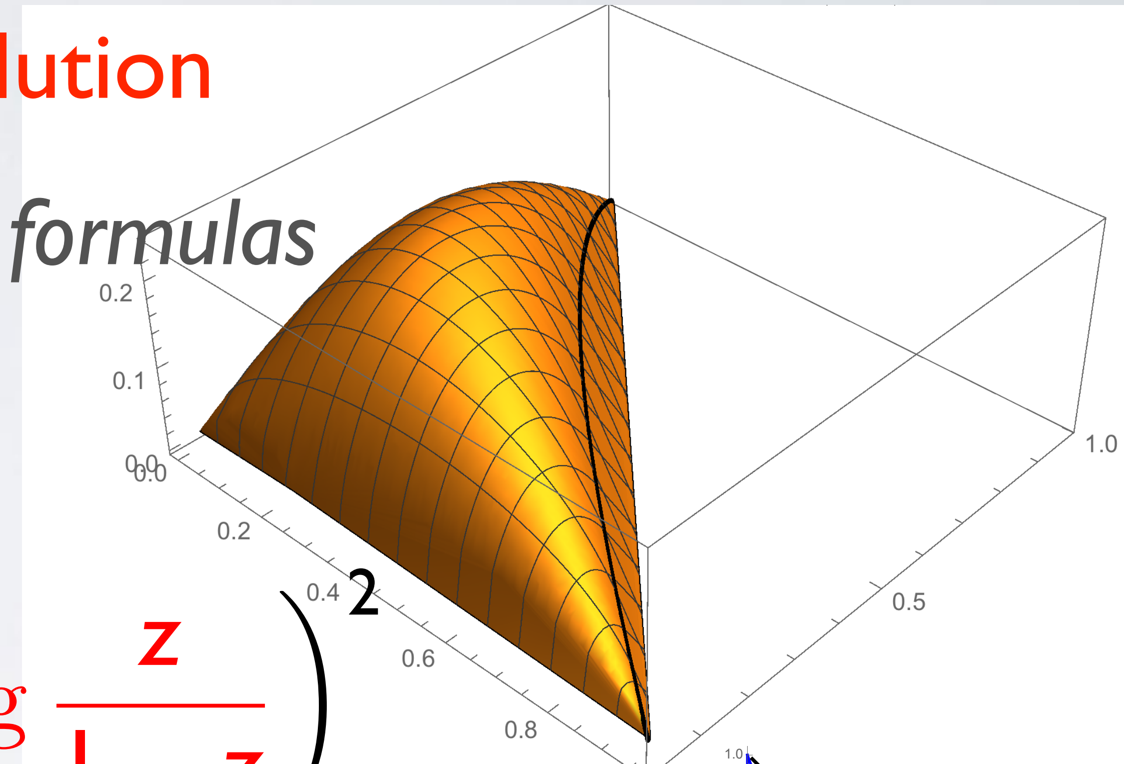
de Gier-Kenyon-Watson



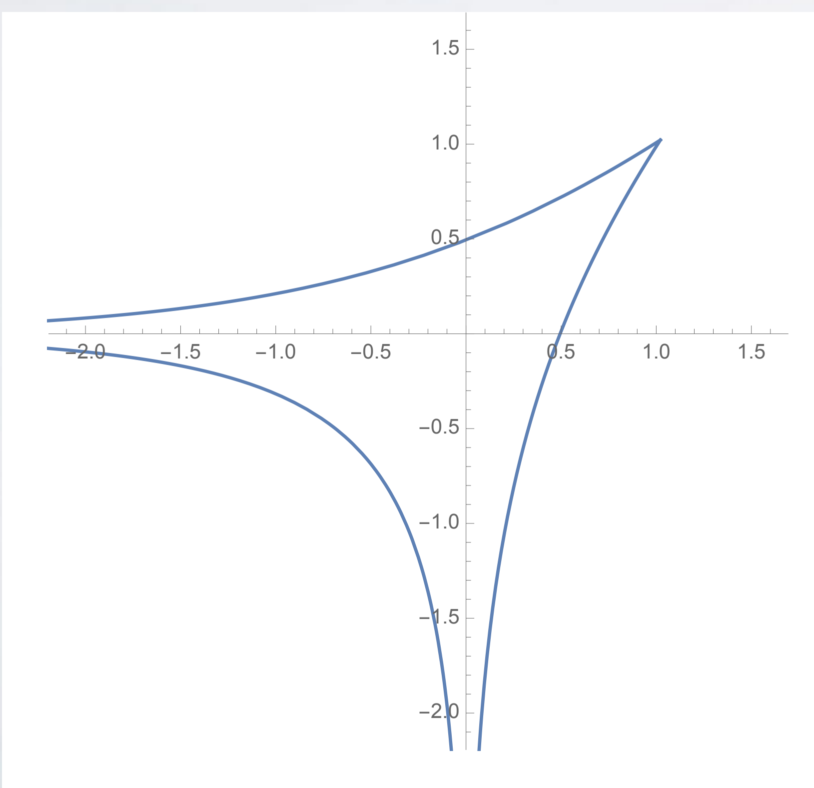
Bethe ansatz solution

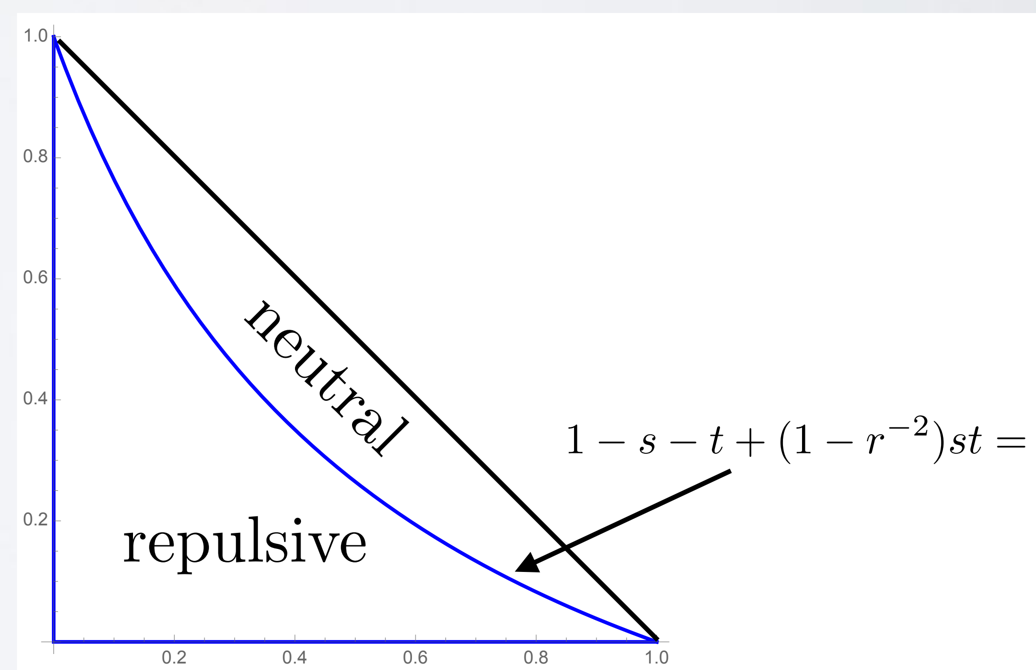
explicit (involved) formulas

trivial potential !



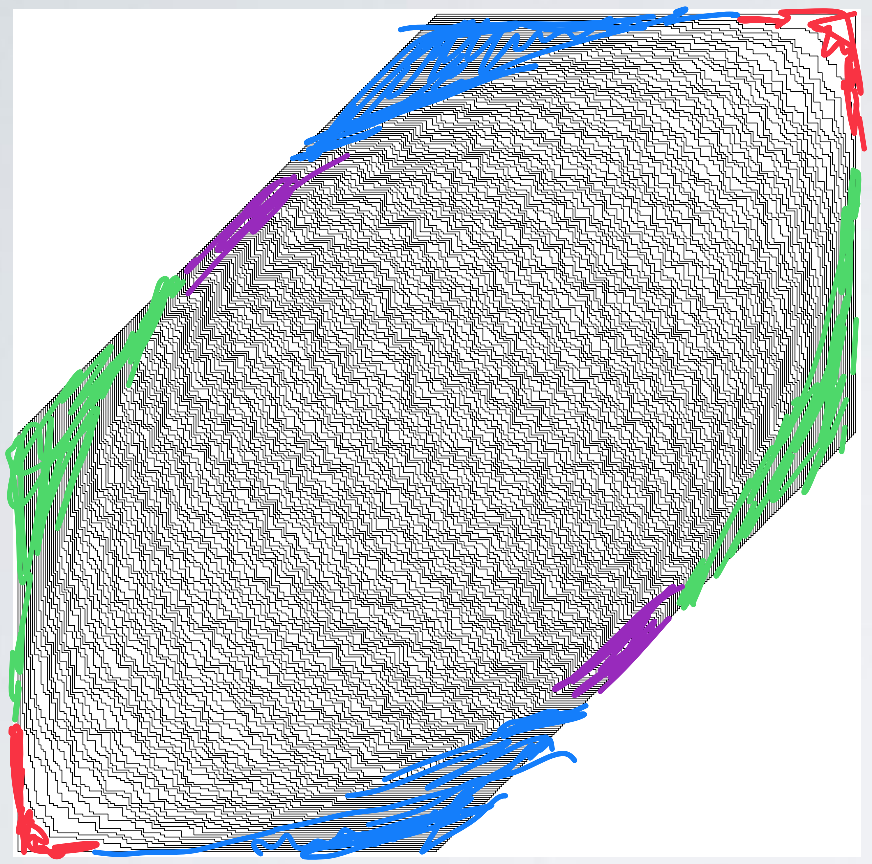
$$\pi\sqrt{\det D^2\sigma} = \left(\arg\frac{z}{1-z}\right)^2$$



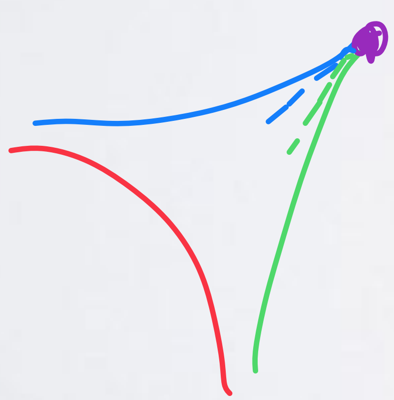


BPP EXAMPLE

small r

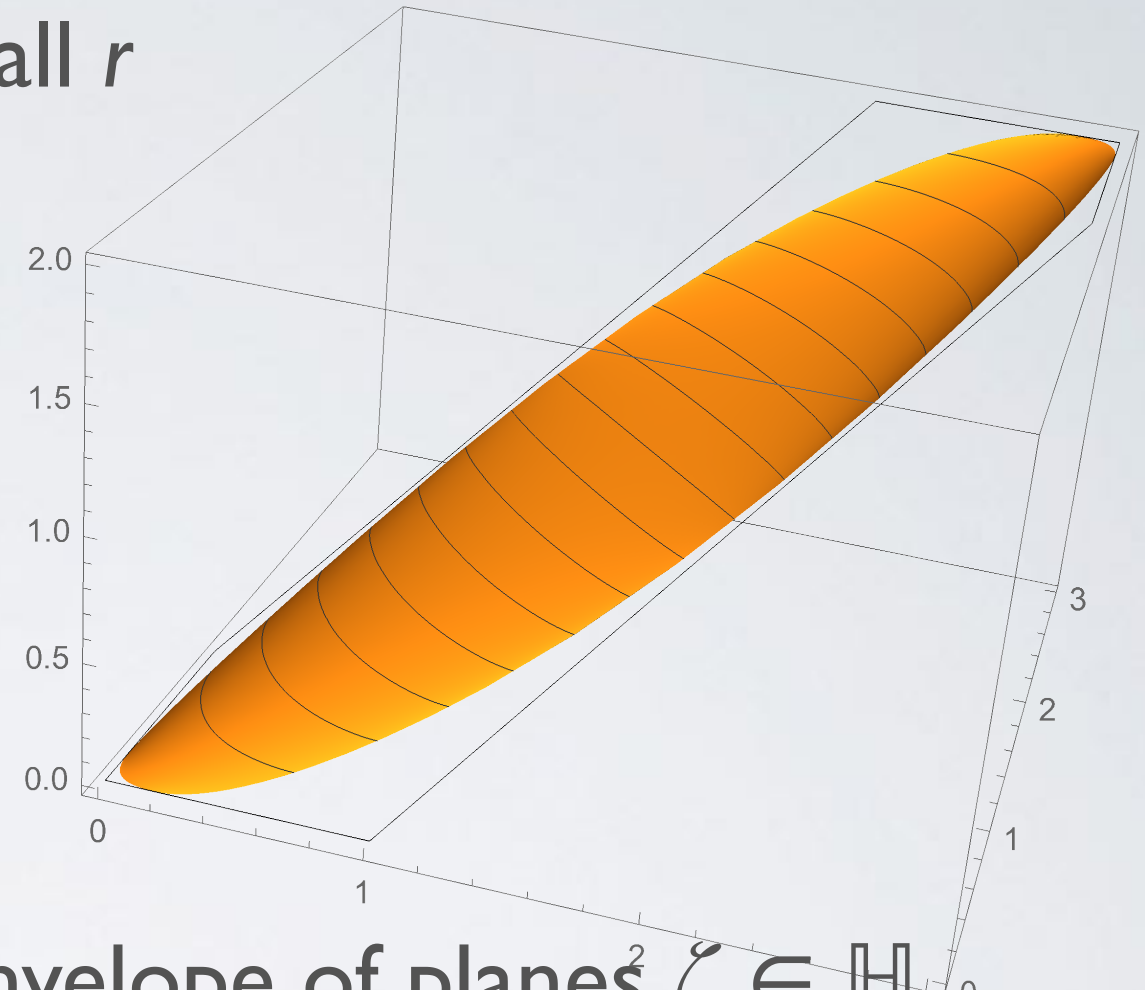


6 facets+ 2 neutral regions



8 intervals on $\partial\mathbb{H}$

full boundary information



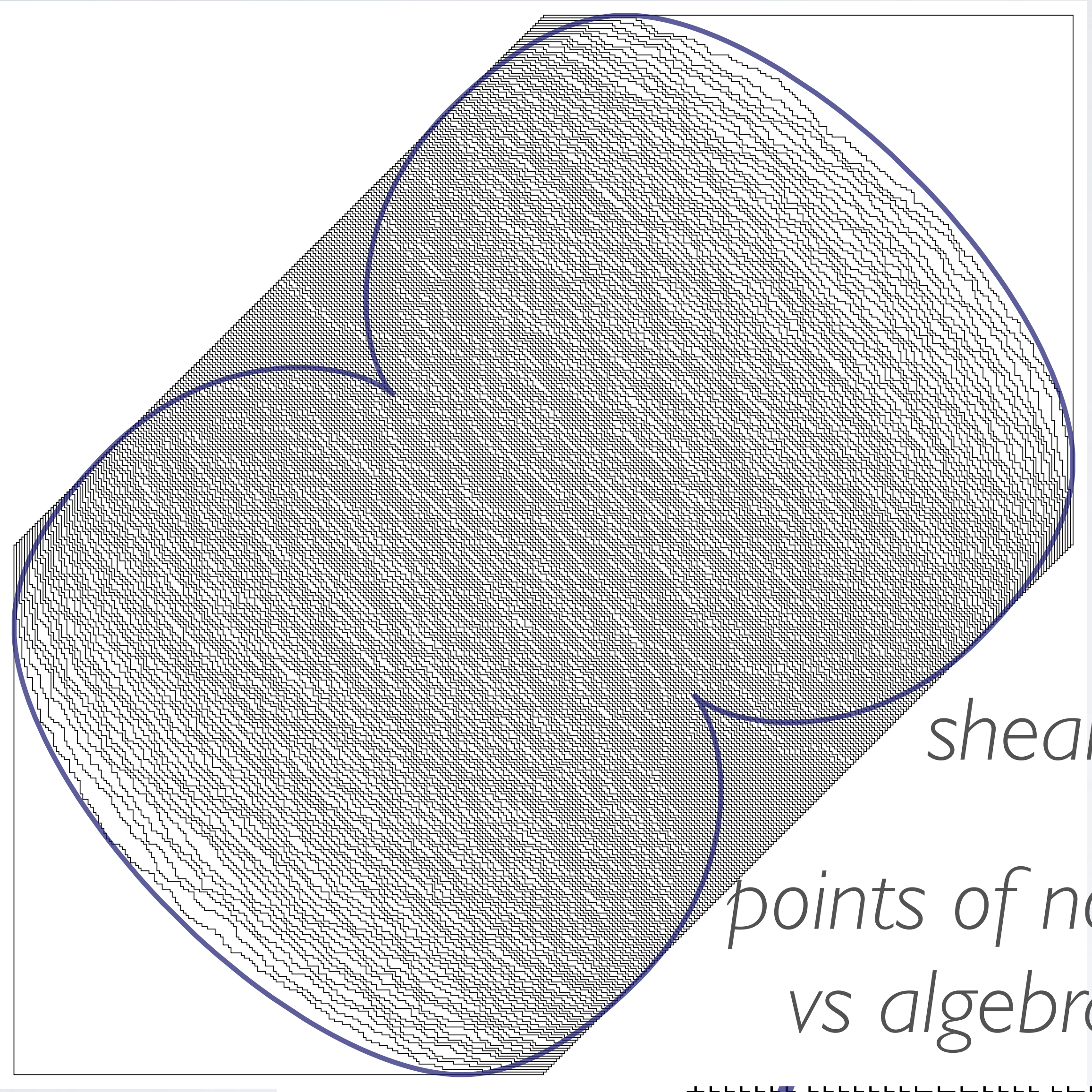
envelope of planes $\zeta \in \mathbb{H}_3$
$(u(\zeta)$ degree 2 cover of $\mathbb{H})$

$$x_3 = s(\zeta)x + t(\zeta)y + c(\zeta)$$

are all *ratios* of linear combinations of *harmonic measures*

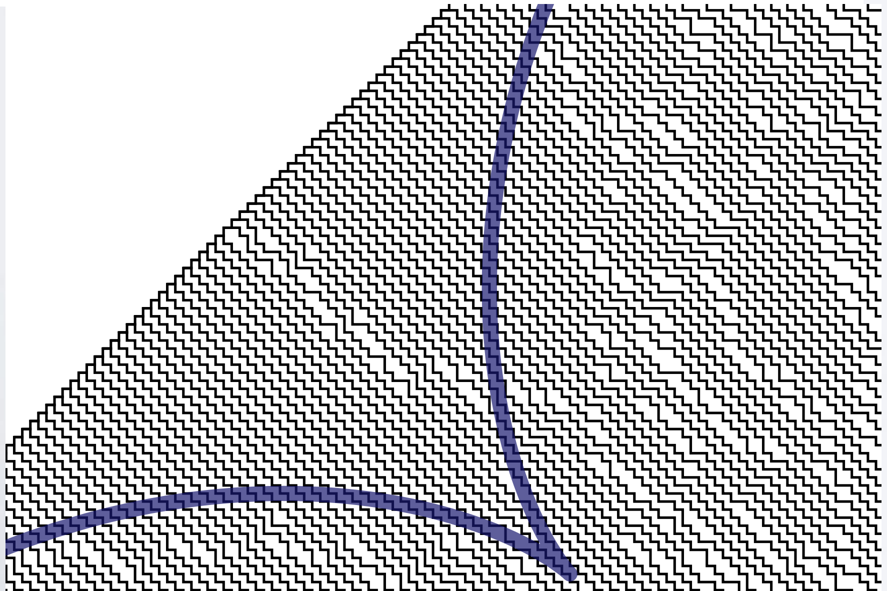
BPP EXAMPLE

large *r*

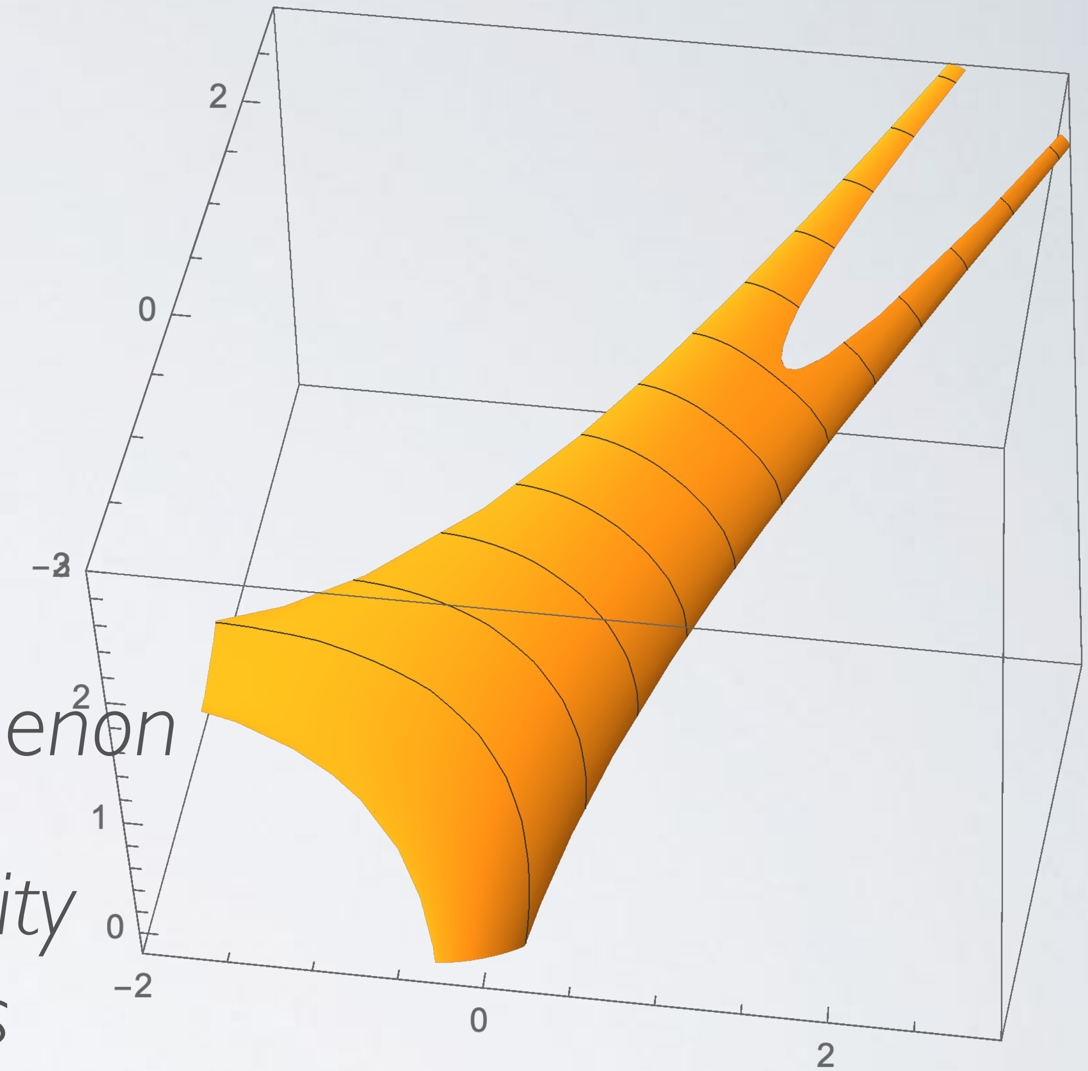


shear phenomenon

points of non-analycity
vs algebraic curves



r=2.5



free energy/Wulff shape

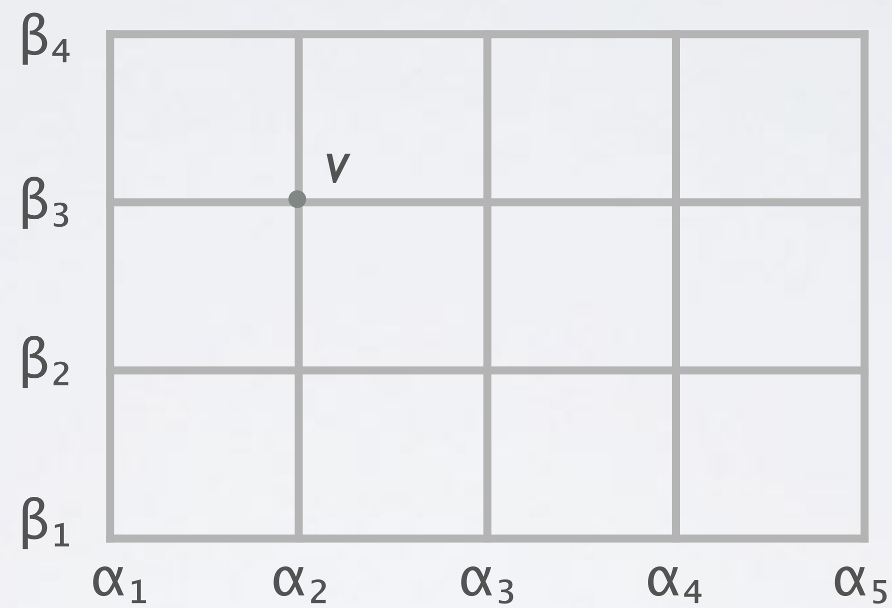
GENUS-ZERO 5-VERTEX MODEL

Kenyon-Prause

staggered model

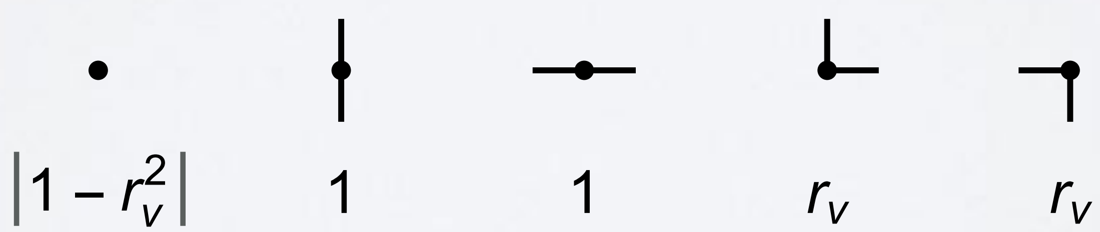
$m_1 \times m_2$ fundamental domain

$\vec{\alpha} = (\alpha_1, \ldots, \alpha_{m_1}) \qquad \vec{\beta} = (\beta_1, \ldots, \beta_{m_2})$



$v = (i, j)$

$r_v = \alpha_i \beta_j$



"small r" *(repulsive)* all $\alpha_i \beta_j < 1$ or all $\alpha_i \beta_j > 1$ "large r" *(attractive)*

SURFACE TENSION

(small r)

convex in $\mathcal{N}$

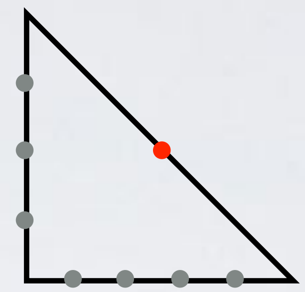
strictly convex in $\mathring{\mathcal{N}}'$

piecewise linear on $\partial\mathcal{N}$

$\sigma|_{\mathcal{N}\setminus\mathcal{N}'} \equiv 0$

non-uniqueness

$$\sqrt{\det D^2\sigma} = \frac{1}{\pi}(\arg u)^2$$



$u \in \mathbb{H}$

conformal coordinate

trivial potential

(large r)

convex in $\mathcal{N}$

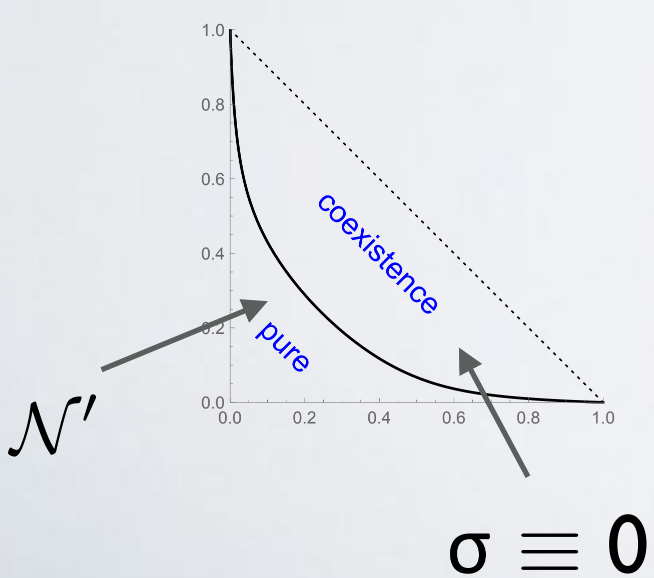
strictly convex in $\mathring{\mathcal{N}}$

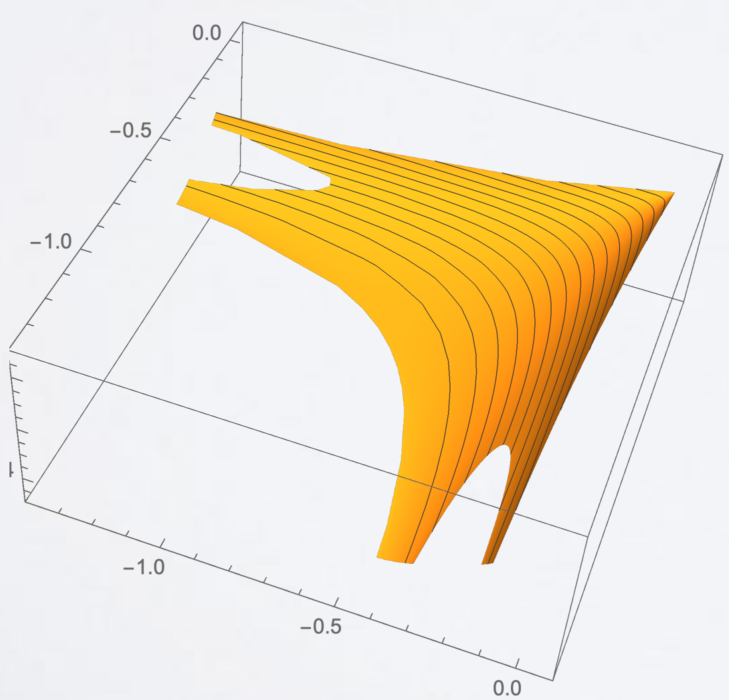
piecewise linear on $\partial\mathcal{N}$

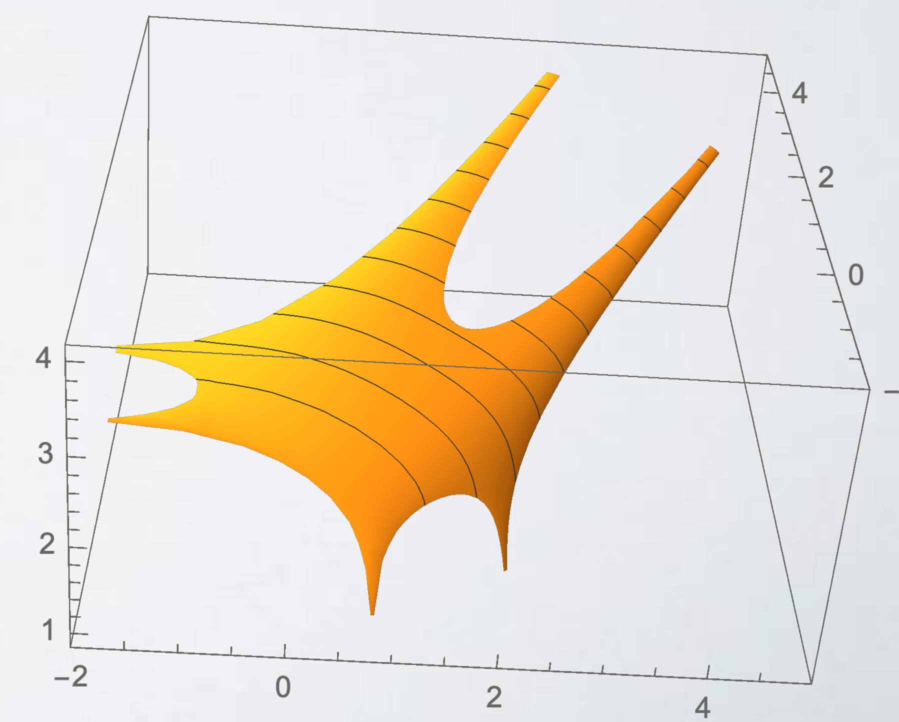
slope discontinuity at **(1/2,1/2)**

unique minimizer

$$\sqrt{\det D^2\sigma} = \frac{1}{\pi}(2\pi - \arg u)^2$$







DARBOUX INTEGRABILITY

Thm: In any component of the liquid region the tangent planes to the limit shape can be parametrized by a complex ζ

$$\nabla h(x,y) = (s(u), t(u)) \qquad h(x,y) - \nabla h(x,y)\cdot(x,y) = G(\zeta)/\theta(u)$$

with $u(\zeta)$ holomorphic, $G(\zeta)$ harmonic, $\theta(u) = \begin{cases} \arg u & \text{(r<1)} \\ 2\pi - \arg u & \text{(r>1)} \end{cases}$

$$\sqrt{\kappa} = \sqrt[4]{\det D^2\sigma}$$

Corollary: In any component of the liquid region

$$(s\theta)_\zeta x + (t\theta)_\zeta y + G_\zeta - \theta_\zeta h(x,y) = 0$$

(shear phenomenon)

all holomorphic functions

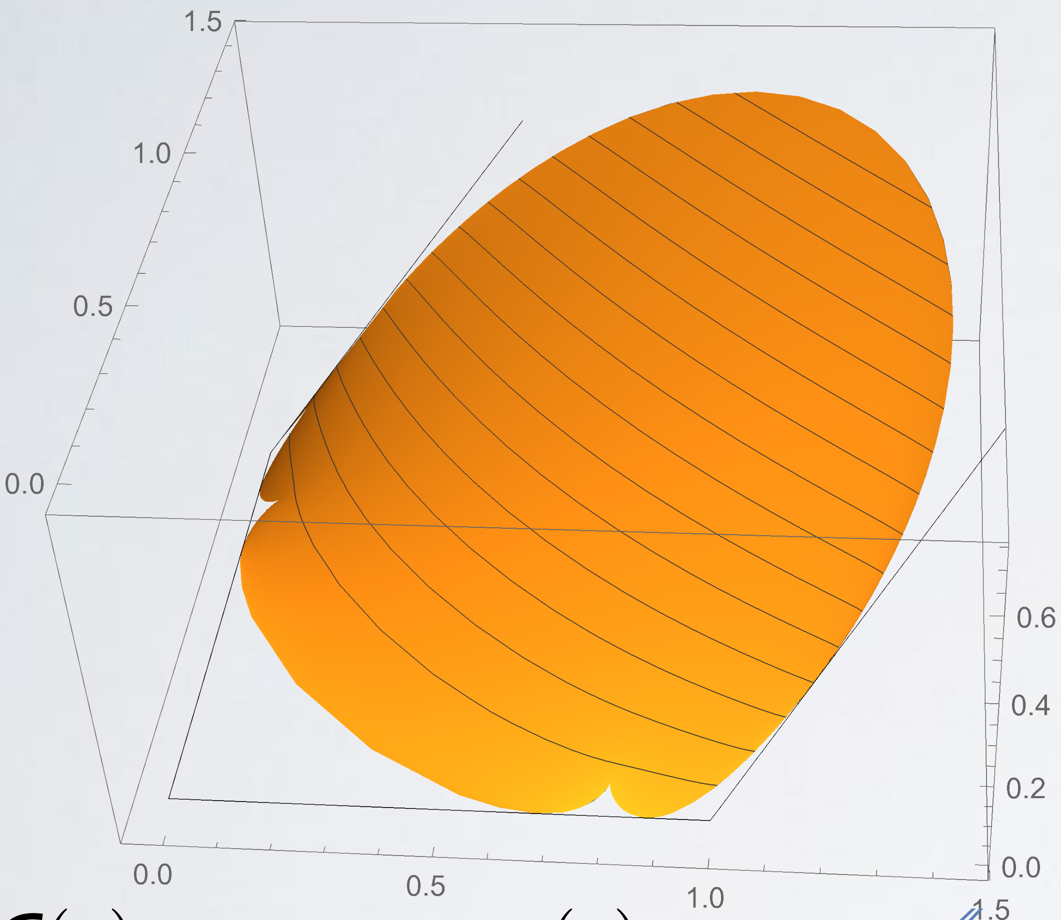
2x2 EXAMPLES

"semi-boxed plane partition"

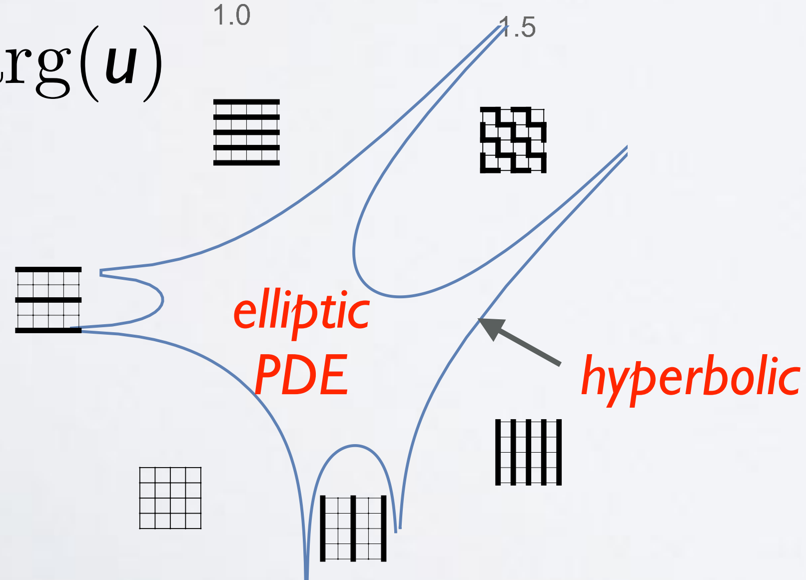
$$u = \zeta$$

large r

$$(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (2, 5/4)$$

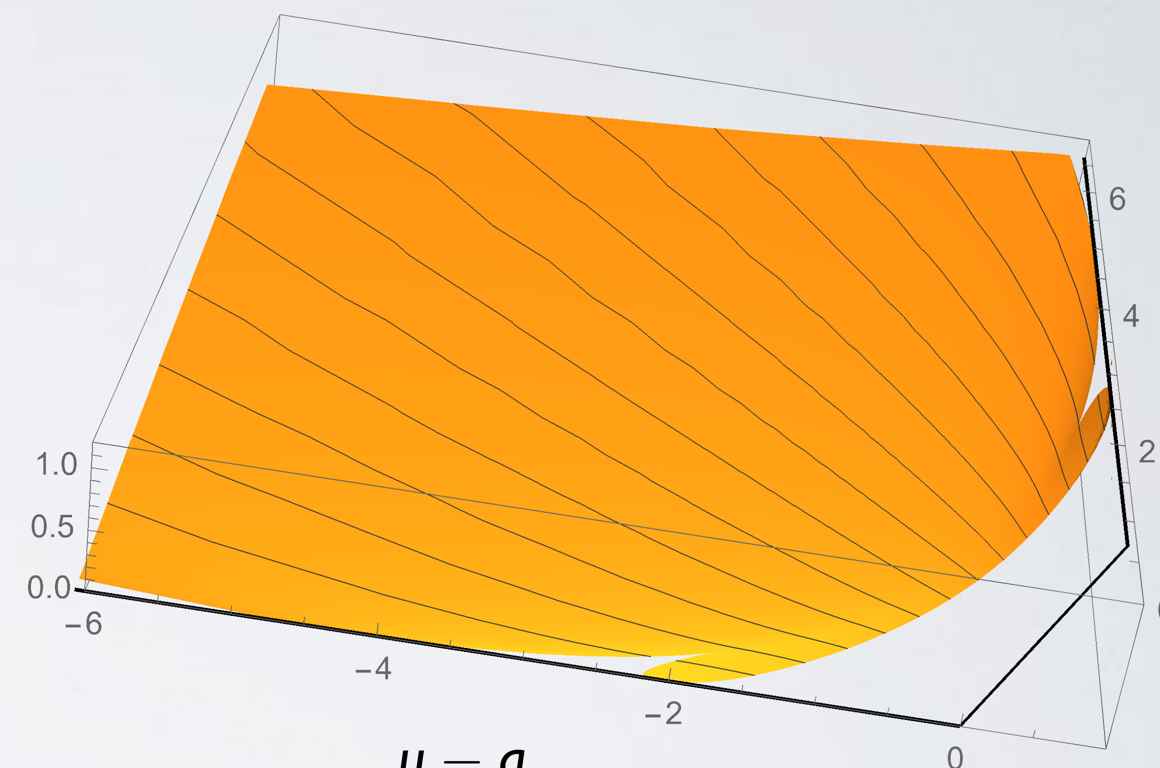


$$G(u) = -\pi - \arg(u)$$

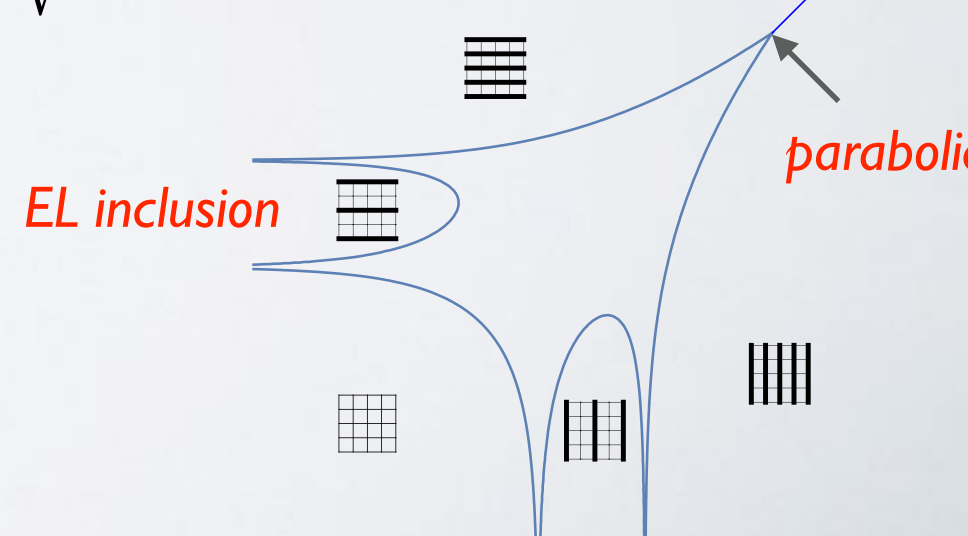


small r

$$(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = (4/5, 1/4)$$



$$G(u) = \arg\left(\frac{u - a}{\sqrt{(u + \alpha_1^2)(u + \alpha_2^2)}}\right)$$



DARBOUX HIERARCHY

free fermionic	constant Hessian det
5-vertex	trivial potential
six-vertex	?
…	?

LIFE BEYOND THE ARCTIC CIRCLE