RANDOM TILINGS, ARCTIC CURVES AND A BELTRAMI EQUATION

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joint work with

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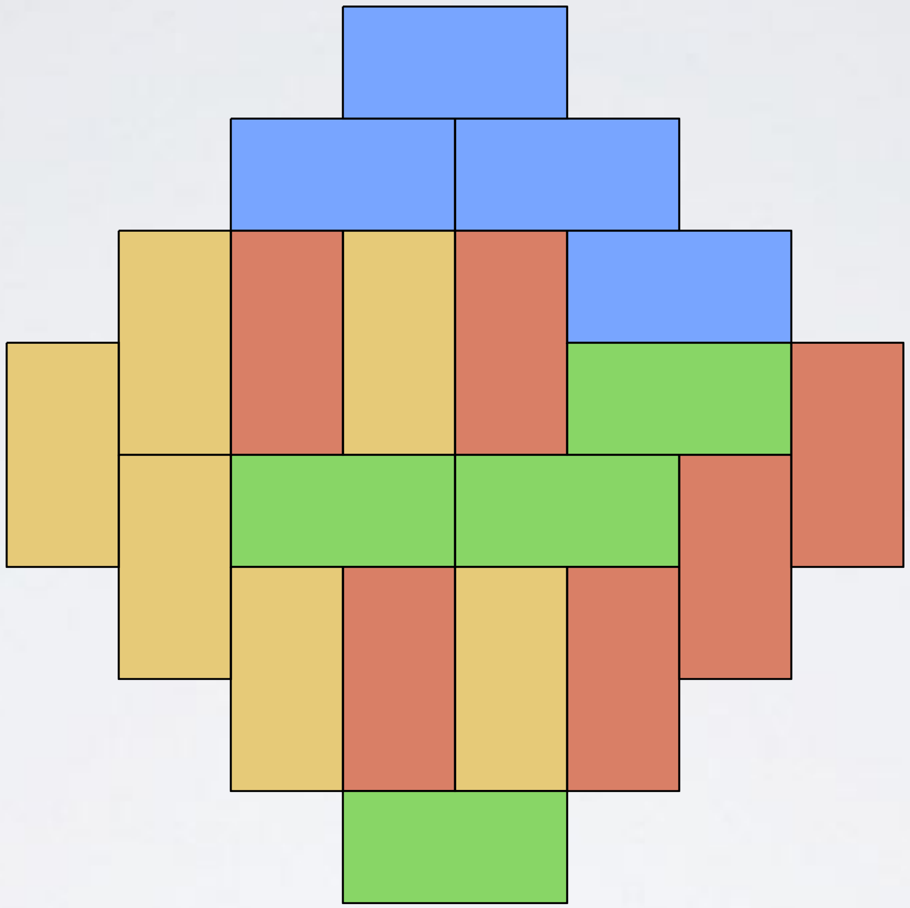
THE AZTEC DIAMOND

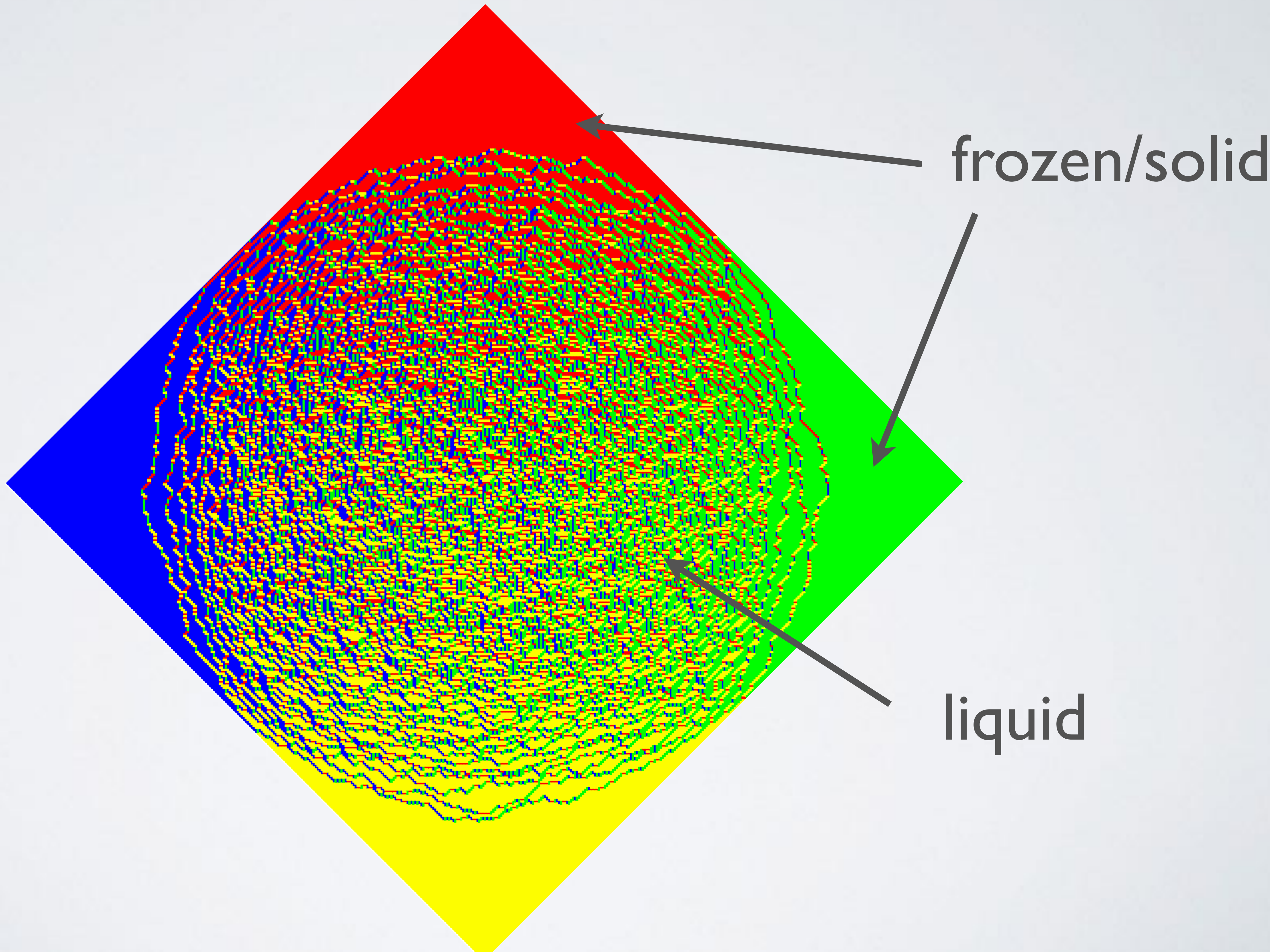


consider a uniformly random tiling by dominoes

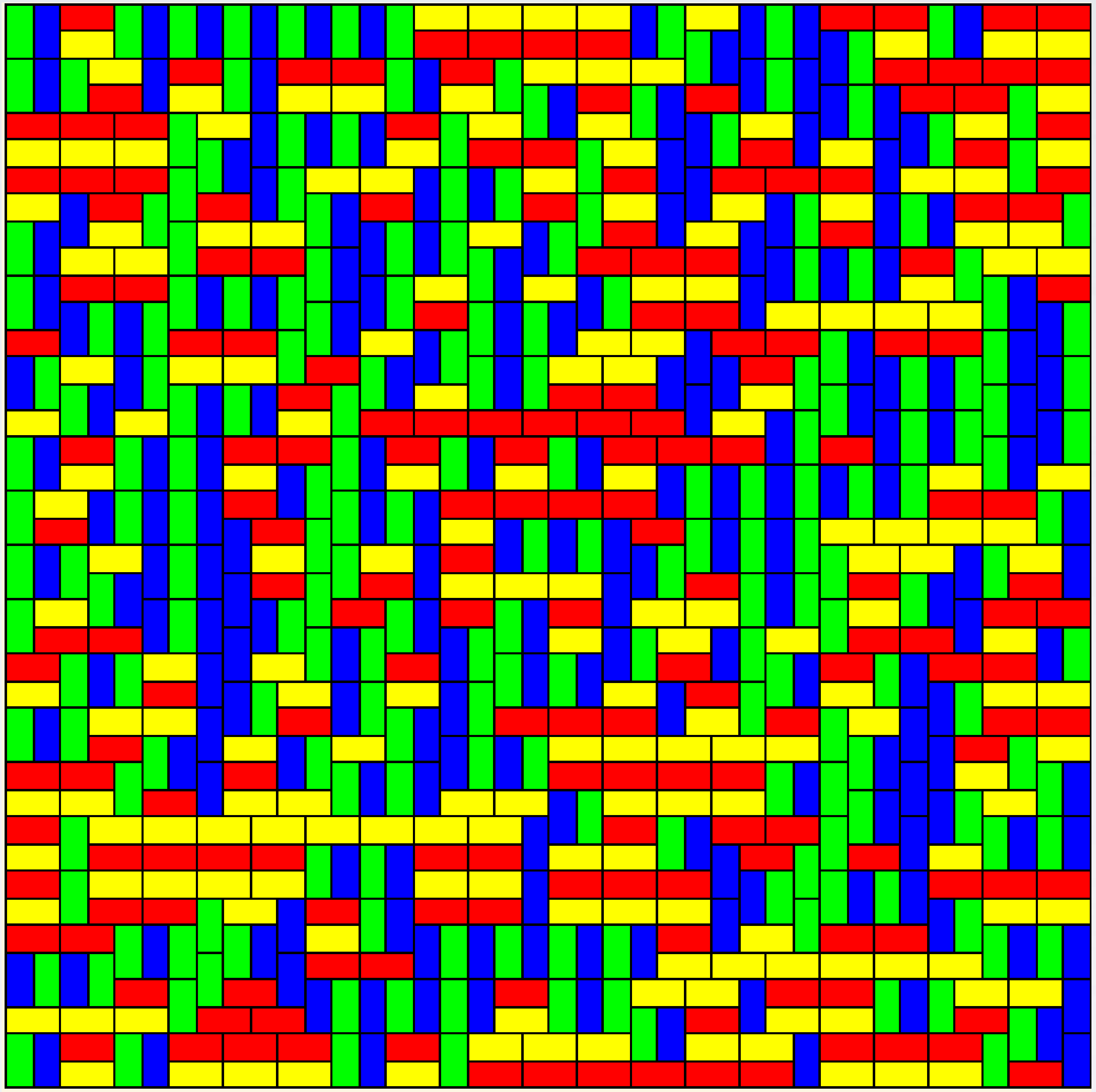
How does it look like?

THE ARCTIC CIRCLE

Jockusch-Propp-Shor

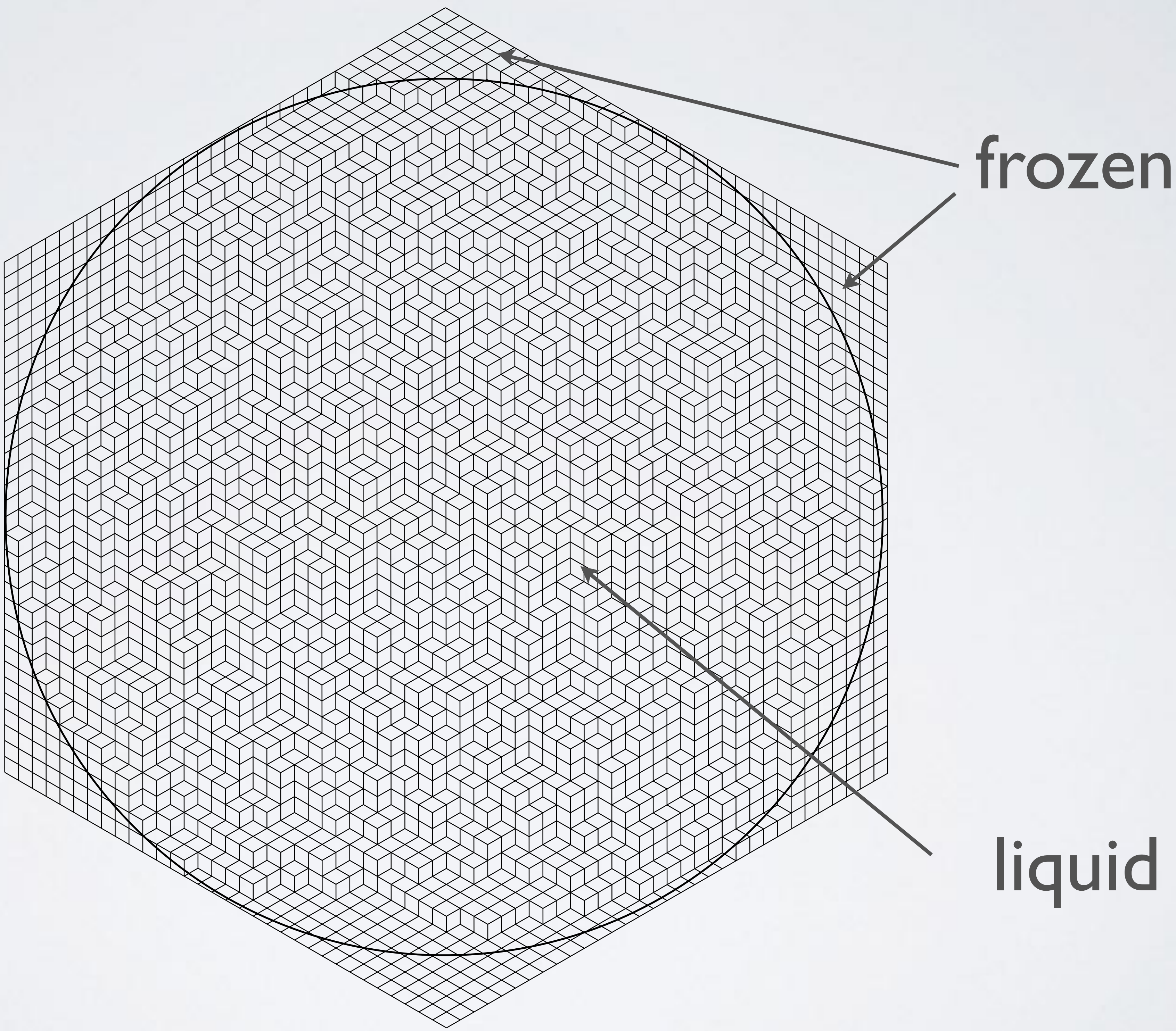


RANDOM TILING OF A SQUARE

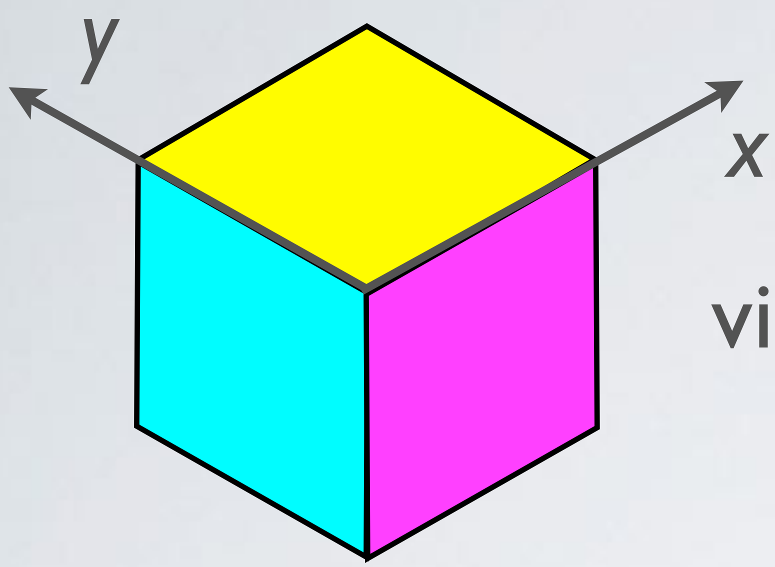


THE ARCTIC CIRCLE AGAIN

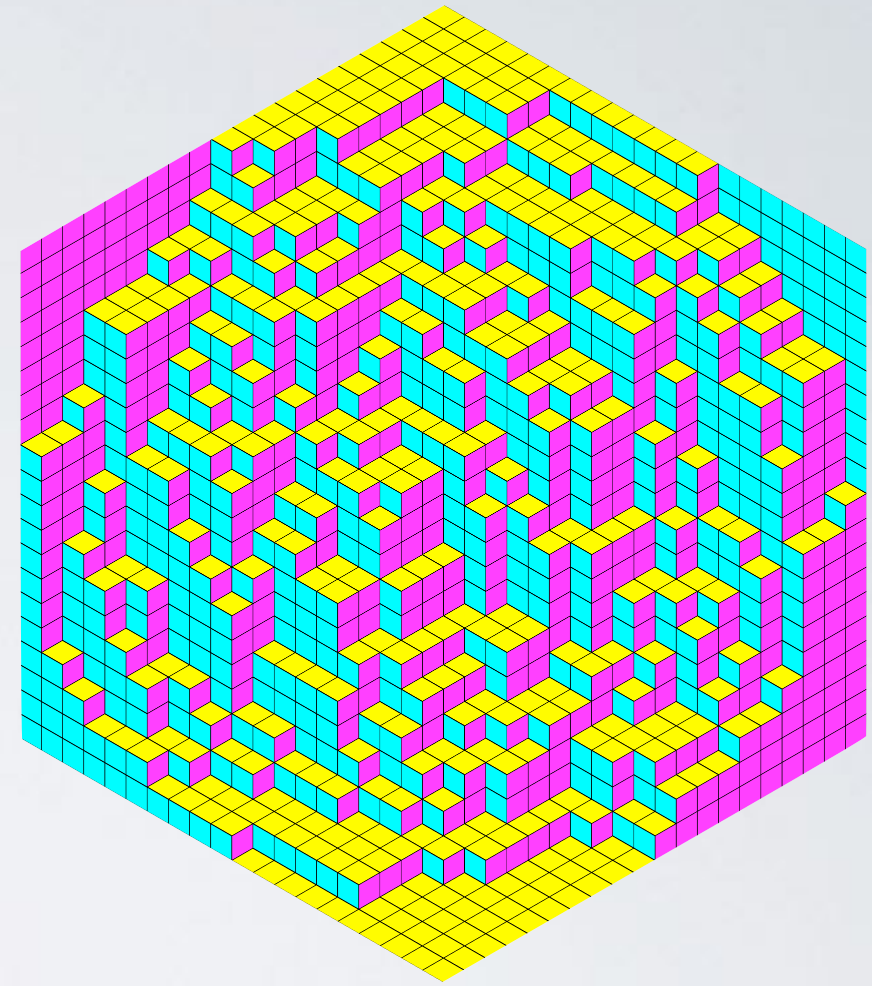
Cohn-Larsen-Propp

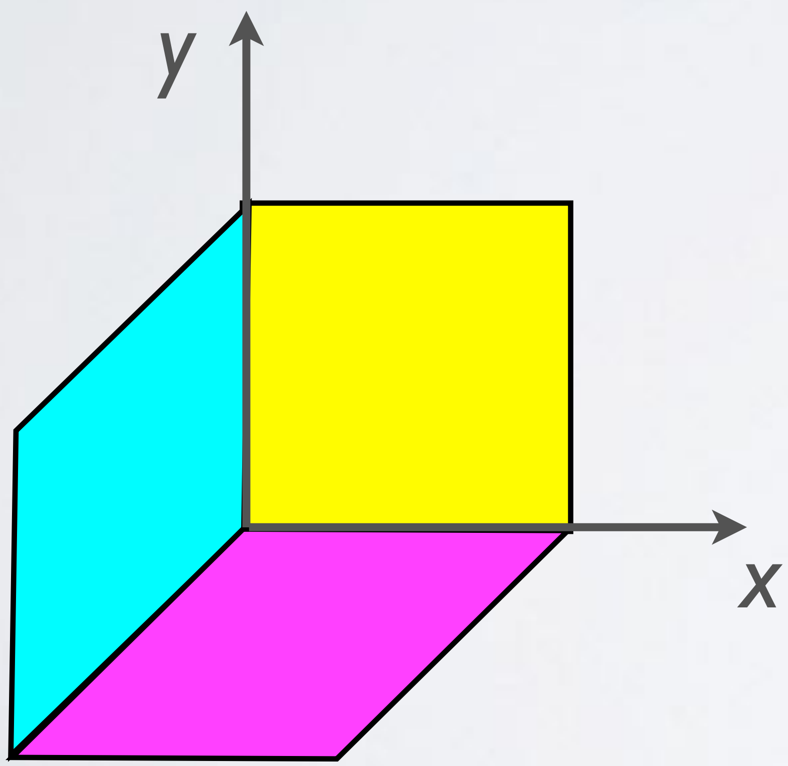


LOZENGE TILINGS



stepped surfaces
viewed from *(1,1,1)*
direction

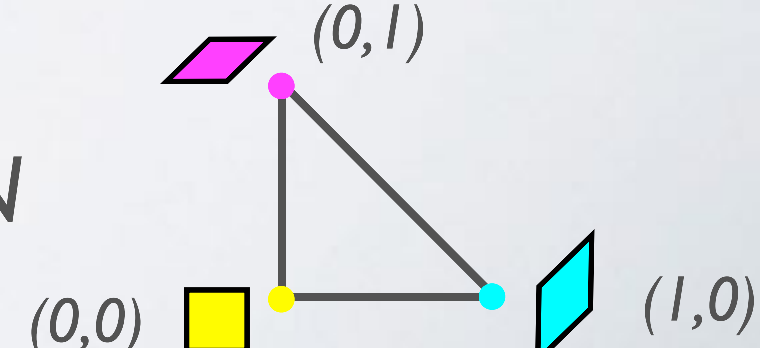




height function

"vertical height" parametrized by *(1,1,1)*
plane in Cartesian coordinates

$\nabla h \in$ vertices of a triangle N

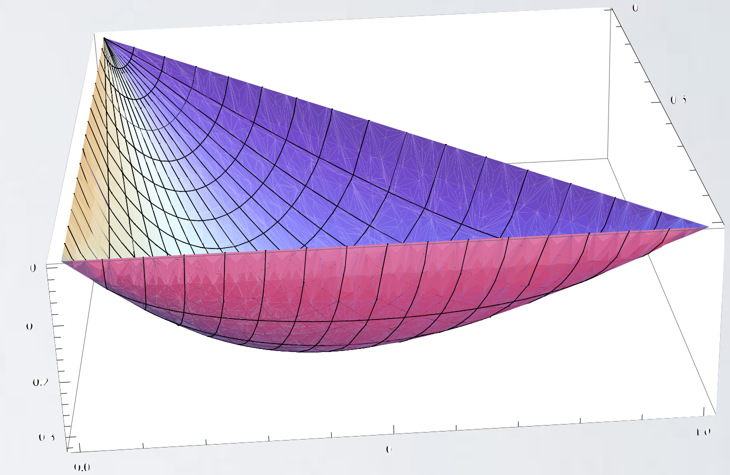


LIMIT SHAPE THEOREM

Cohn-Kenyon-Propp

For a fixed boundary height, as mesh size →0, with probability tending to 1 a random surface lies close to a deterministic surface, called *limit shape*

The limit shape is described by a *variational principle*



'minimal surface' spanning a wire-frame

$h \colon \Omega \to \mathbb{R}$ Lipschitz

$$\min_h \int_\Omega \sigma(\nabla h), \qquad \nabla h \in N, \quad h_{|\partial\Omega} = h_0$$

analytic, strictly convex surface tension in the interior

singular and degenerates on the boundary

SURFACE TENSION

Kenyon-Okounkov-Sheffield

Kasteleyn
Temperly-Fisher

dimer model "exactly solvable"
determinental formulae
"free fermions"

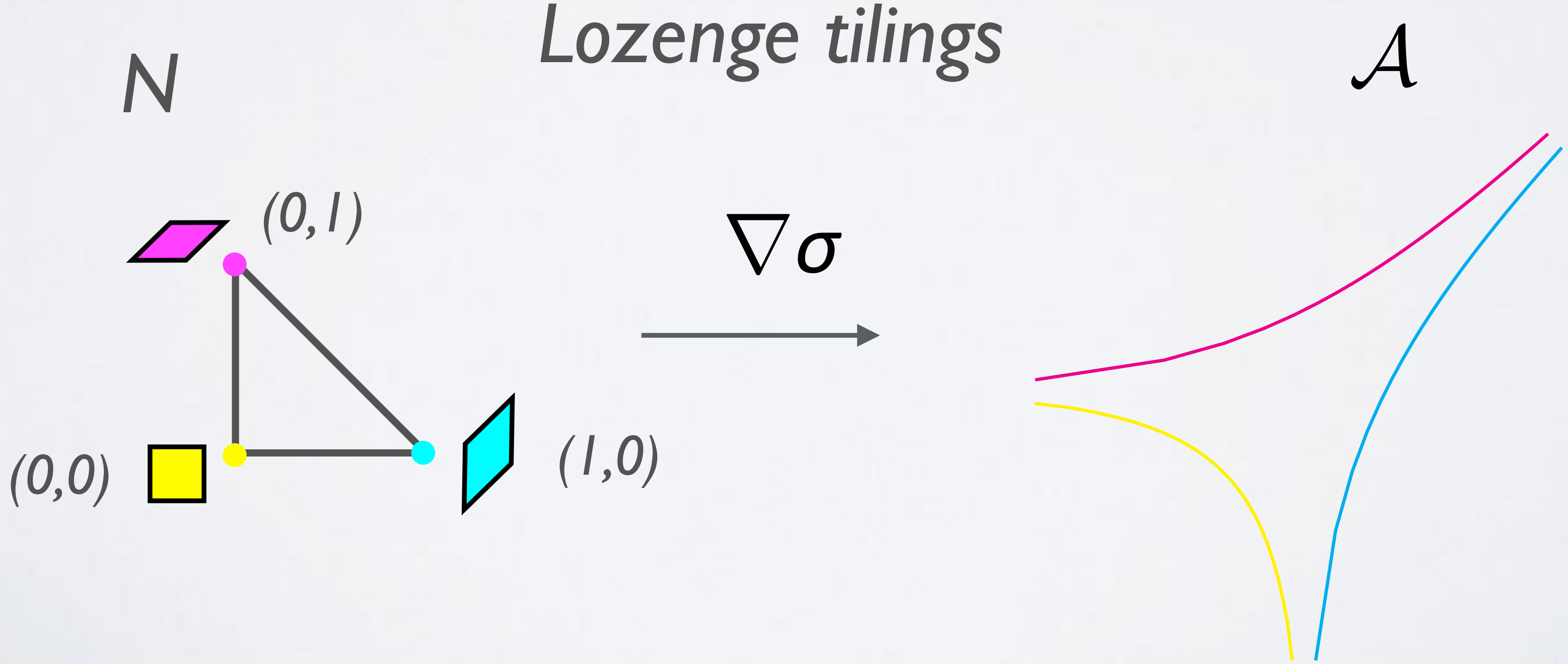
N is a convex polygon (gradient constraint)

Monge-Ampere equation

$\det D^2\sigma \equiv \pi^2$ (except at possible gas points)

σ is piecewise linear on ∂N

Lozenge tilings

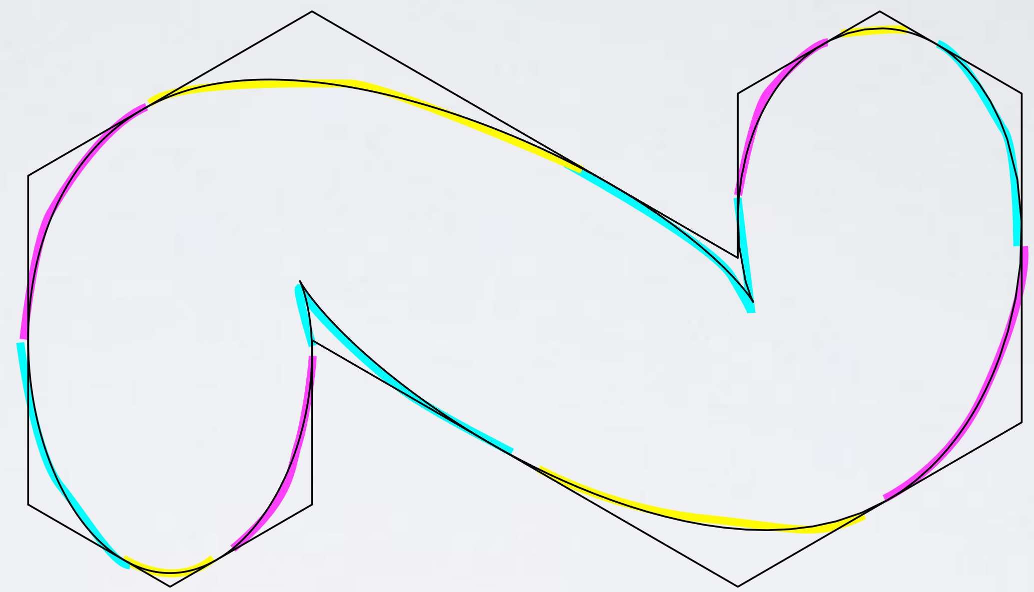


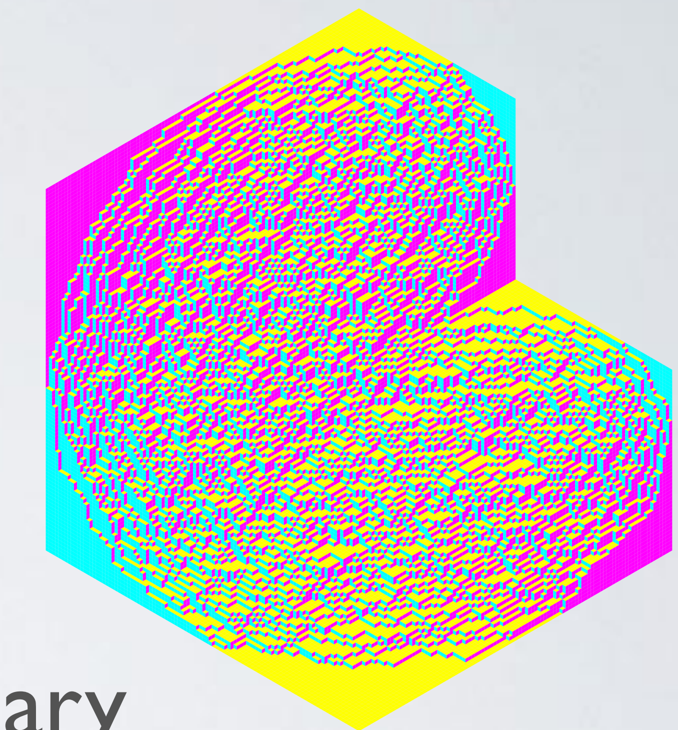
ALGEBRAIC CURVES

Kenyon-Okounkov

frozen boundary = free boundary

cloud curves
inscribed in polygons
(a solution to EL exists)





lozenge model
polygonal boundary
contour in coordinate
directions → algebraic frozen boundary + frozen facets

Why algebraic curves ? What is their geometry? Other dimer models? Universality?

construct the minimizer vs prove things about the minimizer

NON-LINEAR BELTRAMI EQUATION

$\operatorname{div} A(\nabla u) = 0 \qquad \nabla v = *A(\nabla u) \qquad v_x = -A_2(\nabla u), \quad v_y = A_1(\nabla u)$

$F = u + iv \qquad F_{\bar z} = \mathcal{H}(F_z)$ Hodge dual

$$F_{z\bar z} = \frac{\mathcal{H}_z(F_z)}{1 - |\mathcal{H}_{\bar z}(F_z)|^2} F_{zz} + \frac{\mathcal{H}_{\bar z}(F_z)\overline{\mathcal{H}_z(F_z)}}{1 - |\mathcal{H}_{\bar z}(F_z)|^2}\overline{F_{zz}}$$

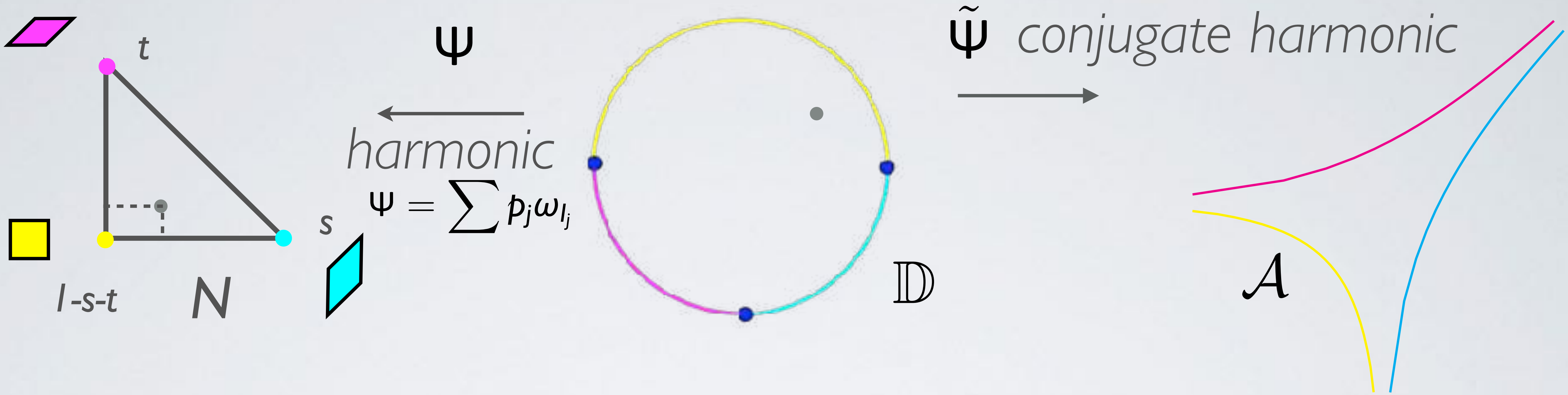
$f := F_z \qquad f_{\bar z} = \mu(f) f_z + \nu(f)\overline{f_z} \qquad |\mu| + |\nu| < 1$

$A = \nabla\sigma, \quad \det D^2\sigma \equiv 1 \quad \longleftrightarrow \quad \mu$ is complex analytic, $\nu \equiv 0$

$$f_{\bar z} = \mu_\sigma(f) f_z$$

$\hat f := \mu_\sigma(f) \qquad \bar\partial \hat f = \hat f \partial \hat f$ universal equation

HARMONIC COORDINATES



$\mathcal{L}$ *liquid region:* $\quad \mathrm{div}\,(\nabla\sigma \circ \nabla h) = 0 \qquad \nabla h \in N^\circ \setminus \mathcal{G}$

$$f = \Psi^{-1} \circ \nabla h$$

$f \colon \mathcal{L} \to \mathbb{D}$

$\bar{\partial} f = f \partial f$

⟷ complex Burgers equation

Kenyon-Okounkov

DeSilva-Savin: C^1-regularity away from the obstacles

$\nabla h \colon \mathcal{L} \to N^\circ \setminus \mathcal{G}$ proper for (oriented) extremal boundary values

→ f proper

THE BELTRAMI EQUATION

Astala-Duse-Prause-Zhong

$f\colon \mathcal{L} \to \mathbb{D}$

$\mathcal{L} \subset \mathbb{C}$

$\bar{\partial} f = f \partial f$

bounded simply connected

(or finitely connected)

$f \in W^{1,2}_{loc}(\mathcal{L})$

→ $f \in C^{\infty}(\mathcal{L})$

(self-improves)

properness assumption: $f(z) \to \partial\mathbb{D}$ as $z \to \partial\mathcal{L}$

→

$f \in C(\overline{\mathcal{L}})$ generically on bdry Hölder exponent 1/2

$\partial\mathcal{L}$ is algebraic '*cloud curve*'

tangent relations

limit shape is a *crystal surface* rounded rough part (liquid) + flat facets (frozen and gas)

Pokrovsky-Talapov transition $|h(x) - h(x_0) - \langle p_0, x - x_0 \rangle| \simeq |x - x_0|^{3/2}$

bonus: frozen boundaries are *universal*

HODOGRAPH TRANSFORM

$$f = B \circ g^{-1}$$

$g \colon \mathbb{D} \to \mathcal{L}$ homeomorphism $\bar\partial g = -B(z)\overline{\partial g}$

$B \colon \mathbb{D} \to \mathbb{D}$ analytic + proper ➡ finite Blaschke product

$h = g + B\bar{g}$ is holomorphic in $\mathbb{D}$

$$(1 - |B|^2)g = h - B\bar{h}$$

$$h(z) - B(z)\overline{h(z)} \to 0, \quad z \to \zeta \in \partial\mathbb{D}$$

extend by reflection $h(z) = B(z)\overline{h(1/\bar{z})}$ $\longleftrightarrow$ $g(z) = g(1/\bar{z})$

$h(z) - h(1/\bar{z}) \to 0, \quad |z| \to 1$ $\bar\partial h = 0$ weakly across $\partial\mathbb{D}$

➡ h is a rational map of $\hat{\mathbb{C}}$ (properness $\rightsquigarrow$ continuity)

RATIONAL PARAMETRIZATION

$$g(z) = \frac{h(z) - B(z)\overline{h(z)}}{1 - |B(z)|^2} = \frac{\frac{h(z)}{B(z)} - \overline{h(z)}}{\frac{1}{B(z)} - \overline{B(z)}} = \frac{\frac{h(z)}{B(z)} - \frac{h(1/\bar{z})}{B(1/\bar{z})}}{\frac{1}{B(z)} - \frac{1}{B(1/\bar{z})}}$$

$\rightarrow r(\zeta), \quad z \rightarrow \zeta \in \partial\mathbb{D}$ $\qquad r = \frac{(h/B)'}{(1/B)'}$ $\qquad \overline{r(1/\bar{z})} = \frac{h'}{B'}$

$z\,r'(z)$ self-reflective with respect to *B(z)*

$$\frac{B(z)}{z^2}\overline{r'(1/\bar{z})} = r'(z)$$

tangent vector $i\zeta r'(\zeta) = iA(\zeta)\sqrt{B(\zeta)}$

real, sign changes at cusps

d/2 − 1/2 # cusps = 1

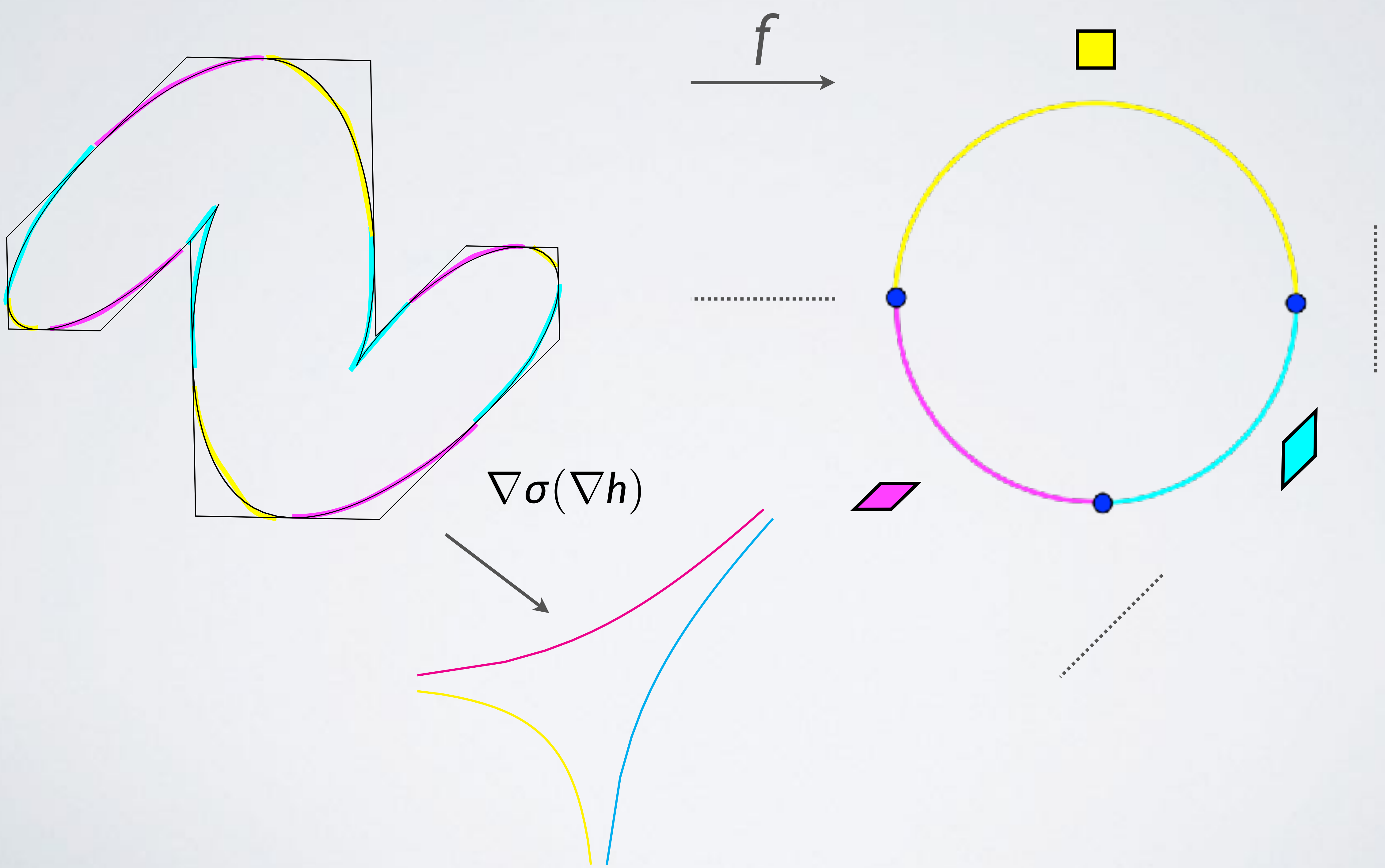
→ boundary is algebraic & locally convex except at *d-2* cusps *cloud curve* geometry

→ $f = B \circ g^{-1}$ continuous up to the boundary

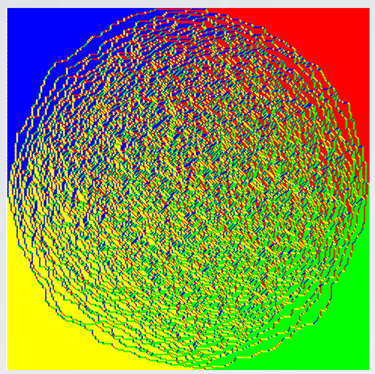
3d-3 real parameters

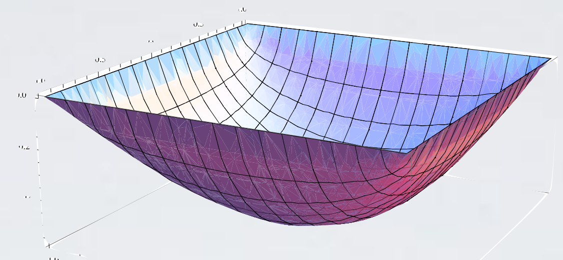
BOUNDARY VALUE RECONSTRUCTION

boundary values governed by tangents

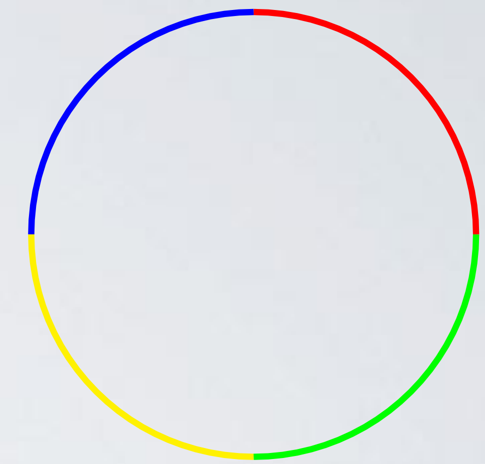


BACK TO THE ARCTIC CIRCLE





surface tension



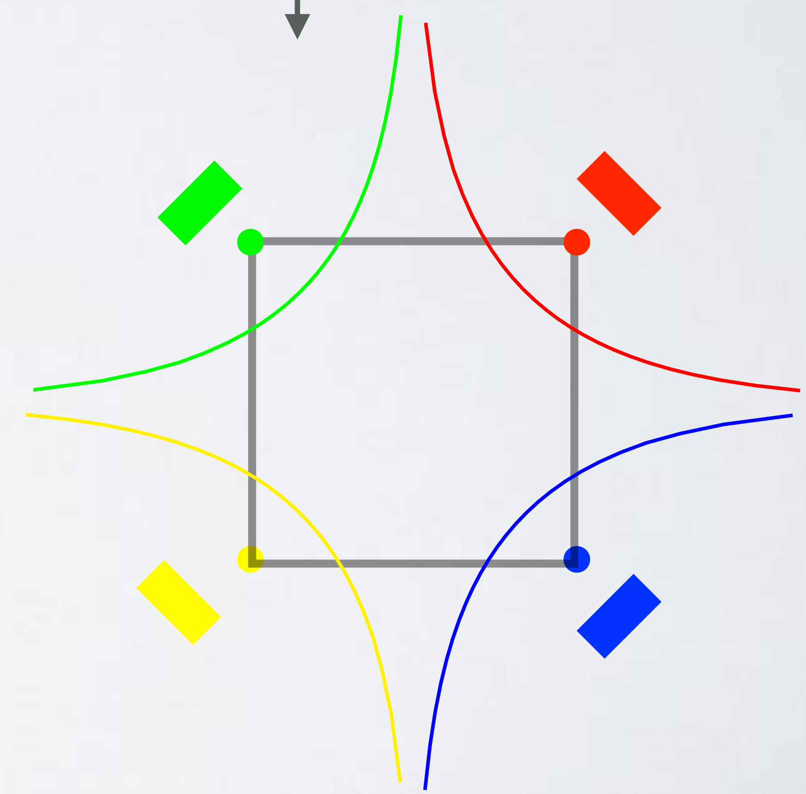
$$f\colon \mathcal{L} \to \mathbb{D}$$

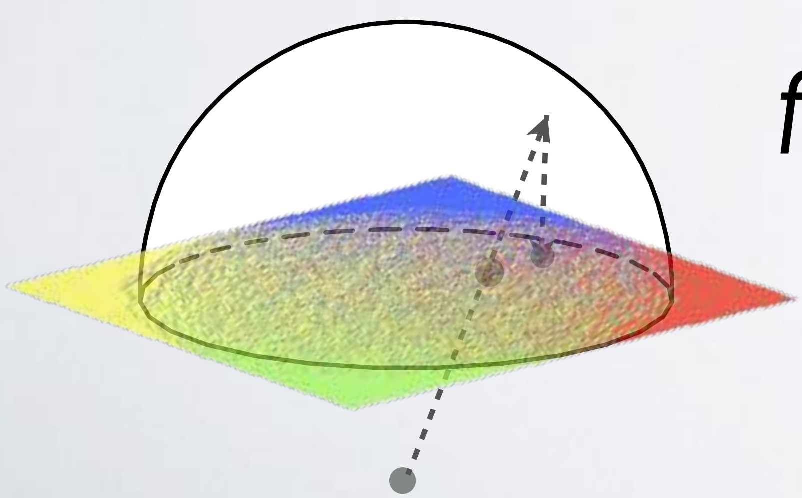
$$\bar{\partial} f = f^2 \partial f$$

intrinsic EL equation

$$\nabla h = \Psi \circ f$$

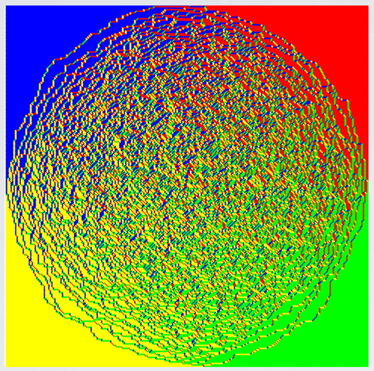
Ψ harmonic

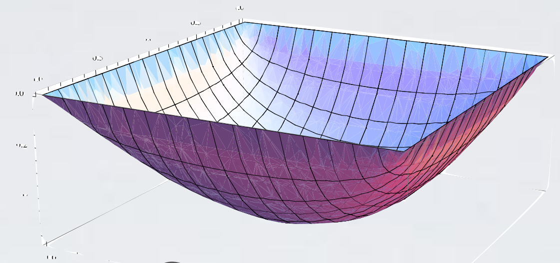




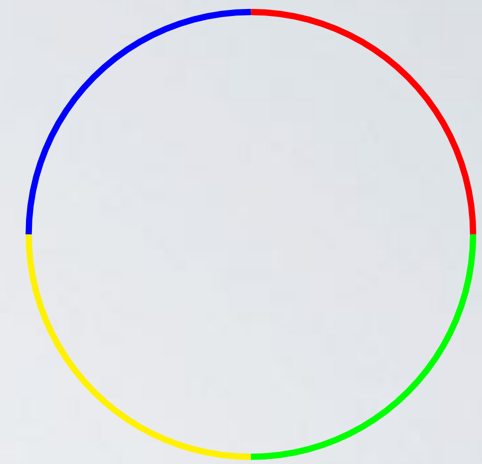
$$f^{-1}(z) = \frac{2z}{1 + |z|^2}$$

BACK TO THE ARCTIC CIRCLE





surface tension



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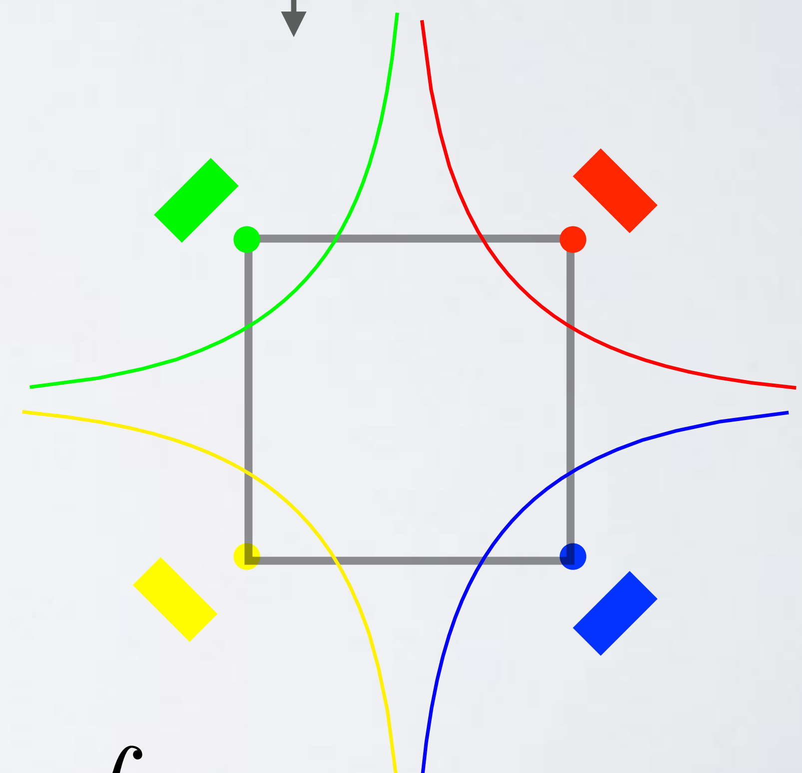


Euler-Lagrange *inclusion* makes sense *everywhere*

if there is a div-free vectorfield Φ s.t.

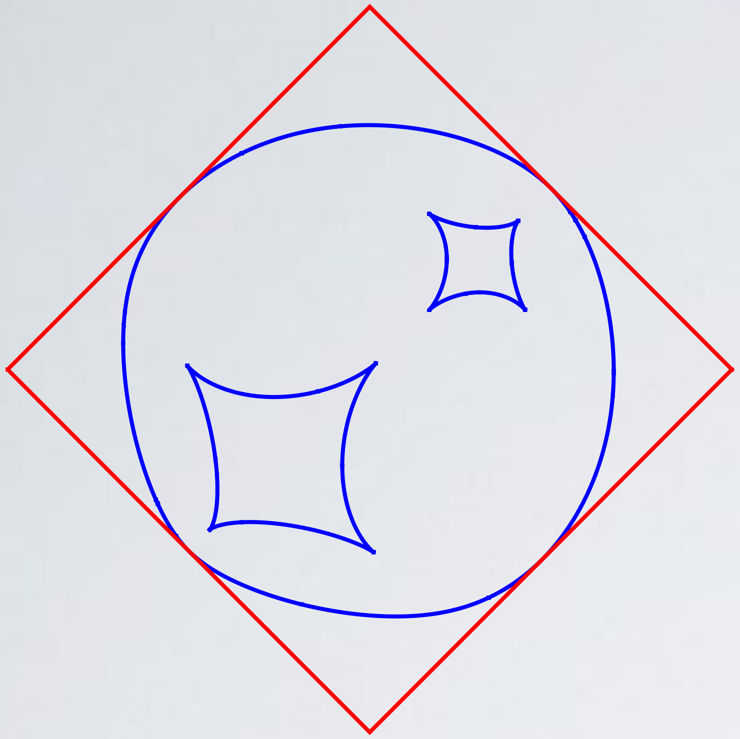
$\Phi \in \partial\sigma(\nabla h)$ a.e. in Ω

then *h* is a minimiser

$$\int_\Omega \sigma(\nabla u) - \sigma(\nabla h) \geqslant \int_\Omega \Phi \cdot (\nabla(u-h)) = -\int_\Omega (u-h)\,\mathrm{div}\Phi = 0$$

Arctic curves of the octahedron equation

Philippe Di Francesco[1] **and Rodrigo Soto-Garrido**[2]



same Aztec diamond

same gradient constraint (square)

$$\det D^2\sigma \equiv \pi^2$$

except two interior points (Dirac mass)

$$
\begin{aligned}
P(u, v) = {} & 603\,358\,073\,569\,688\,095\,393\,738\,000u^{14} + 1822\,971\,422\,522\,481\,873\,814\,304\,800vu^{13} + 302\,414\,835\,014\,281\,399\,576\,977\,600u^{13} \\
& + 7658\,013\,562\,515\,635\,323\,215\,886\,000v^2u^{12} + 626\,386\,479\,045\,976\,264\,625\,165\,760vu^{12} + 65\,648\,625\,922\,043\,130\,480\,407\,960u^{12} \\
& + 8502\,660\,801\,885\,990\,861\,442\,260\,800v^3u^{11} - 4016\,291\,377\,989\,674\,598\,197\,523\,840v^2u^{11} - 8955\,889\,812\,423\,159\,779\,663\,425\,824vu^{11} \\
& - 648\,516\,348\,371\,464\,166\,524\,636\,080u^{11} + 24\,870\,815\,123\,962\,290\,558\,144\,794\,000v^4u^{10} - 961\,218\,355\,287\,663\,519\,951\,292\,800v^3u^{10} \\
& - 6515\,407\,606\,857\,043\,381\,218\,037\,200v^2u^{10} - 172\,367\,781\,226\,698\,452\,854\,372\,560vu^{10} + 1099\,108\,080\,544\,208\,467\,044\,202\,281u^{10} \\
& + 8664\,424\,609\,796\,383\,599\,417\,068\,000v^5u^9 - 17\,028\,590\,399\,764\,390\,279\,389\,912\,000v^4u^9 - 11\,010\,604\,932\,056\,552\,215\,403\,730\,080v^3u^9 \\
& + 8631\,816\,024\,097\,405\,173\,283\,346\,160v^2u^9 + 6130\,342\,332\,781\,365\,103\,023\,636\,918vu^9 + 300\,038\,159\,586\,641\,951\,467\,587\,240u^9 \\
& + 48\,101\,067\,368\,389\,417\,583\,947\,124\,400v^6u^8 - 8874\,912\,221\,343\,735\,420\,641\,284\,800v^5u^8 - 30\,642\,500\,612\,723\,566\,034\,952\,420\,120v^4u^8 \\
& + 4089\,814\,979\,226\,593\,490\,453\,920\,400v^3u^8 + 2890\,833\,100\,949\,061\,663\,021\,542\,421v^2u^8 - 1537\,862\,122\,247\,709\,326\,673\,670\,200vu^8 \\
& - 1276\,791\,684\,735\,224\,437\,145\,235\,252u^8 + 165\,638\,160\,476\,224\,209\,249\,648\,000v^7u^7 - 15\,185\,547\,478\,970\,793\,846\,022\,129\,920v^6u^7 \\
& + 10\,619\,324\,480\,232\,243\,252\,222\,805\,440v^5u^7 + 17\,121\,927\,414\,923\,400\,963\,351\,428\,640v^4u^7 - 4642\,084\,019\,946\,561\,205\,079\,466\,936v^3u^7 \\
& - 4201\,308\,893\,745\,605\,384\,096\,673\,600v^2u^7 + 97\,128\,658\,780\,698\,750\,571\,038\,384vu^7 + 16\,658\,271\,644\,450\,437\,458\,125\,640u^7 \\
& + 48\,101\,067\,368\,389\,417\,583\,947\,124\,400v^8u^6 - 15\,185\,547\,478\,970\,793\,846\,022\,129\,920v^7u^6 - 54\,696\,534\,109\,775\,129\,942\,931\,200\,352v^6u^6 \\
& + 8498\,087\,480\,515\,562\,992\,290\,313\,440v^5u^6 + 20\,848\,735\,934\,263\,779\,279\,940\,738\,242v^4u^6 - 2928\,090\,072\,842\,649\,426\,783\,830\,400v^3u^6 \\
& - 1125\,942\,030\,946\,106\,640\,101\,862\,864v^2u^6 + 881\,693\,827\,811\,784\,667\,334\,364\,120vu^6 + 410\,818\,358\,444\,129\,895\,320\,450\,118u^6 \\
& + 8664\,424\,609\,796\,383\,599\,417\,068\,000v^9u^5 - 8874\,912\,221\,343\,735\,420\,641\,284\,800v^8u^5 + 10\,619\,324\,480\,232\,243\,252\,222\,805\,440v^7u^5 \\
& + 8498\,087\,480\,515\,562\,992\,290\,313\,440v^6u^5 - 16\,598\,910\,777\,434\,586\,615\,901\,305\,852v^5u^5 - 3118\,943\,690\,894\,703\,413\,913\,413\,040v^4u^5 \\
& + 4436\,727\,620\,735\,139\,576\,883\,870\,032v^3u^5 + 378\,779\,090\,210\,933\,672\,213\,353\,800v^2u^5 - 894\,275\,420\,028\,329\,313\,474\,734\,772vu^5 \\
& - 28\,143\,830\,188\,642\,461\,399\,955\,080u^5 + 24\,870\,815\,123\,962\,290\,558\,144\,794\,000v^{10}u^4 - 17\,028\,590\,764\,390\,279\,389\,912\,000v^9u^4 \\
& - 30\,642\,500\,723\,566\,034\,952\,420\,120v^8u^4 + 17\,121\,927\,923\,400\,963\,351\,428\,640v^7u^4 + 20\,848\,735\,263\,779\,279\,940\,738\,242v^6u^4 \\
& - 3118\,943\,690\,894\,703\,413\,913\,413\,040v^5u^4 - 6585\,025\,120\,215\,513\,060\,415\,620\,600v^4u^4 + 224\,576\,822\,600\,011\,254\,994\,156\,440v^3u^4 \\
& + 730\,062\,356\,407\,169\,871\,489\,508\,026v^2u^4 - 87\,999\,348\,446\,432\,687\,845\,418\,760vu^4 - 39\,991\,576\,579\,826\,072\,416\,315\,884u^4 \\
& + 8502\,660\,801\,885\,990\,861\,442\,260\,800v^{11}u^3 - 961\,218\,355\,287\,663\,519\,951\,292\,800v^{10}u^3 - 11\,010\,604\,932\,056\,552\,215\,403\,730\,080v^9u^3 \\
& + 4089\,814\,979\,226\,593\,490\,453\,920\,400v^8u^3 - 4642\,084\,019\,946\,561\,205\,079\,466\,936v^7u^3 - 2928\,090\,072\,842\,649\,426\,783\,830\,400v^6u^3 \\
& + 4436\,727\,620\,735\,139\,576\,883\,870\,032v^5u^3 + 224\,576\,822\,600\,011\,254\,994\,156\,440v^4u^3 - 771\,752\,886\,154\,129\,578\,670\,446\,744v^3u^3 \\
& + 54\,105\,975\,565\,681\,638\,845\,373\,840v^2u^3 + 158\,742\,939\,499\,283\,087\,522\,192\,736vu^3 + 3181\,828\,983\,737\,934\,822\,021\,000u^3 \\
& + 7658\,013\,562\,515\,635\,323\,215\,886\,000v^{12}u^2 - 4016\,291\,377\,989\,674\,598\,197\,523\,840v^{11}u^2 - 6515\,407\,606\,857\,043\,381\,218\,037\,200v^{10}u^2 \\
& + 8631\,816\,024\,097\,405\,173\,283\,346\,160v^9u^2 + 2890\,833\,100\,949\,061\,663\,021\,542\,421v^8u^2 - 4201\,308\,893\,745\,605\,384\,096\,673\,600v^7u^2 \\
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& + 54\,105\,975\,565\,681\,638\,845\,373\,840v^3u^2 - 205\,856\,416\,682\,486\,477\,443\,753\,704v^2u^2 - 4541\,013\,871\,098\,771\,634\,821\,000vu^2 \\
& - 417\,838\,190\,775\,940\,873\,949\,175u^2 + 1822\,971\,422\,522\,481\,873\,814\,304\,800v^{13}u + 626\,386\,479\,045\,976\,264\,625\,165\,760v^{12}u
\end{aligned}
$$

LIFE BEYOND THE ARCTIC CIRCLE