SCHWARZ LEMMA AND QUASICONFORMAL MAPPINGS

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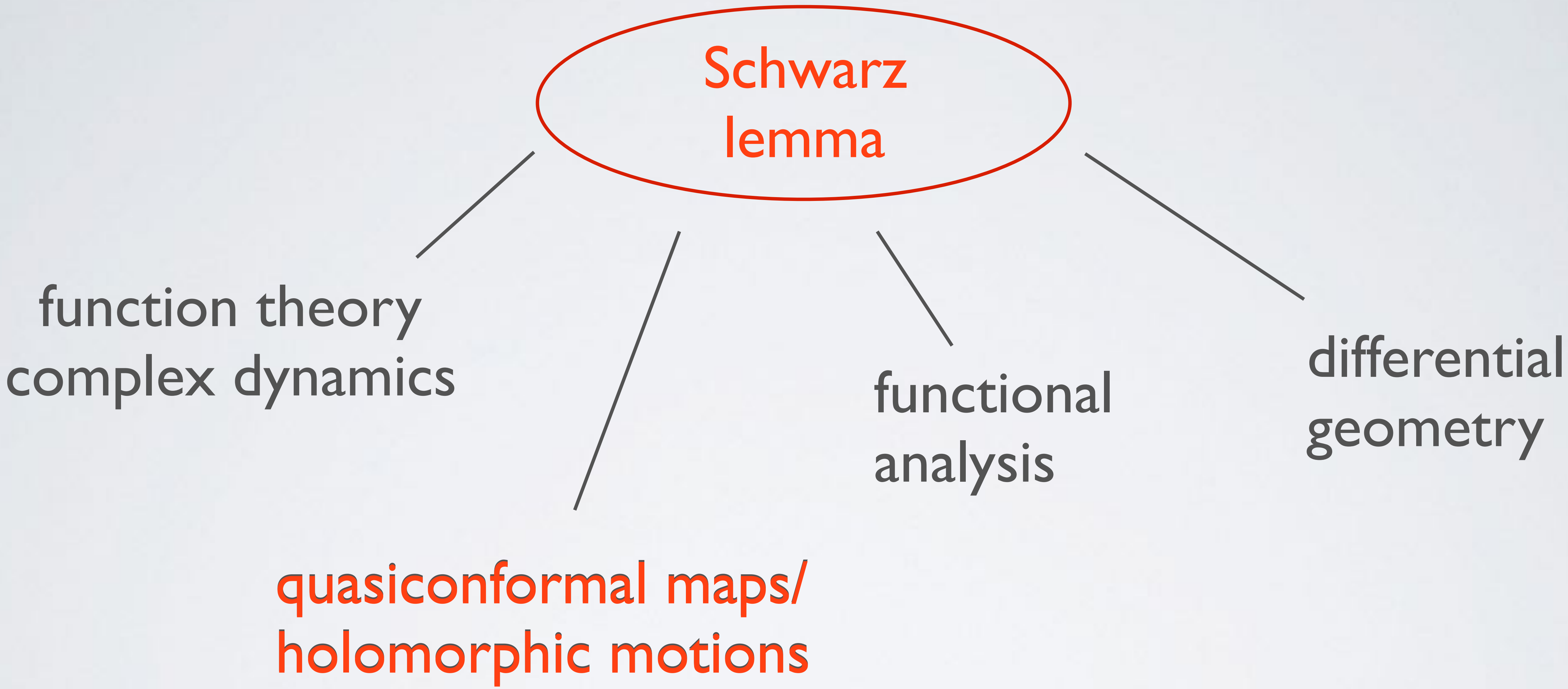
SCHWARZ LEMMA

"A good lemma is worth a thousand theorems."



Lehto's majorant principle for *analytic functionals* on class *S* (univalent) ⇝ *non-analytic* other classes, as well: quasiconformal, bilipschitz

HOLOMORPHIC MOTIONS

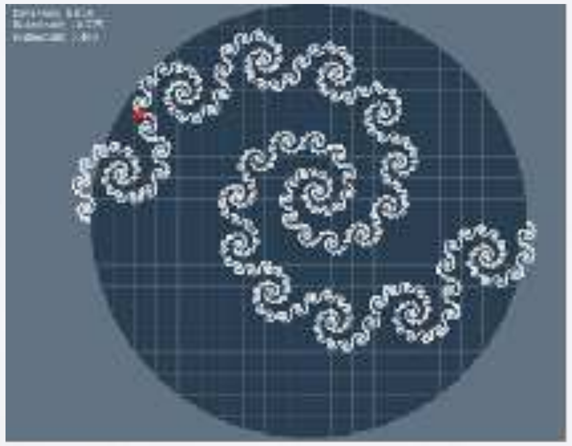
$$\Phi : \mathbb{D} \times E \to \mathbb{C}, \quad E \subset \mathbb{C}$$

- $z \mapsto \Phi(\lambda, z) = \Phi_\lambda(z)$ is injective for all $\lambda \in \mathbb{D}$,
- $\lambda \mapsto \Phi(\lambda, z)$ is holomorphic for all $z \in E$,
- $\Phi(0, z) \equiv z$.

Mañé-Sad-Sullivan, Slodkowski's λ-lemma:

"holomorphic motions = quasiconformal maps"

$\{\Phi_\lambda(z)\}$ (extends to) an analytic family of qc maps



λ-LEMMA AND QUASISYMMETRY

$$g(\lambda) = \frac{\phi_\lambda(z_1) - \phi_\lambda(z_2)}{\phi_\lambda(z_1) - \phi_\lambda(z_3)}$$

$g\colon \mathbb{D} \to \hat{\mathbb{C}} \setminus \{0, 1, \infty\}$ holomorphic

$\{z_1, z_2, z_3\}$ distinct

Schwarz lemma: contracts in hyperbolic metric

$$\rho(g(\lambda), g(0)) \leqslant \rho_{\mathbb{D}}(0, \lambda) = \log \frac{1 + |\lambda|}{1 - |\lambda|}$$

$$\Rightarrow \quad |g(\lambda)| \leqslant \eta_{|\lambda|}(|g(0)|)$$

For a fixed $\lambda \in \mathbb{D}$ ϕ_λ is quasisymmetric

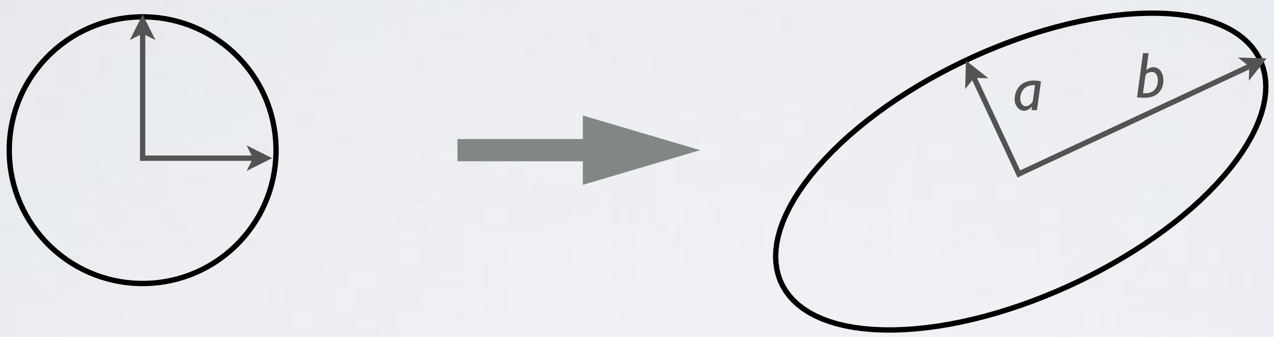
Quasisymmetry: "shapes are preserved" $\eta\colon [0, \infty) \to [0, \infty)$

$$\left|\frac{f(z_1) - f(z_2)}{f(z_1) - f(z_3)}\right| \leqslant \eta\left(\left|\frac{z_1 - z_2}{z_1 - z_3}\right|\right)$$ for any triple

QUASICONFORMAL MAPS

$f\colon \mathbb{C} \to \mathbb{C}$

$f \in W^{1,2}_{loc}(\mathbb{C})$ homeomorphism



$\frac{b}{a} \leqslant K$

Beltrami equation $\quad f_{\bar z} = \mu f_z \quad \|\mu\|_\infty \leqslant k = \frac{K-1}{K+1} < 1$

Corollary: ϕ_λ is $|\lambda|$-quasiconformal

$\lambda \mapsto \mu_\lambda \qquad \mathbb{D} \to L^\infty(\mathbb{C})$

$\mu_0 \equiv 0 \qquad \|\mu_\lambda\|_\infty < 1$

Banach-space valued Schwarz lemma $\qquad \|\mu_\lambda\|_\infty \leqslant |\lambda|$

WHAT IS... a Quasiconformal Mapping?

quasiconformal maps = "complex interpolation class"

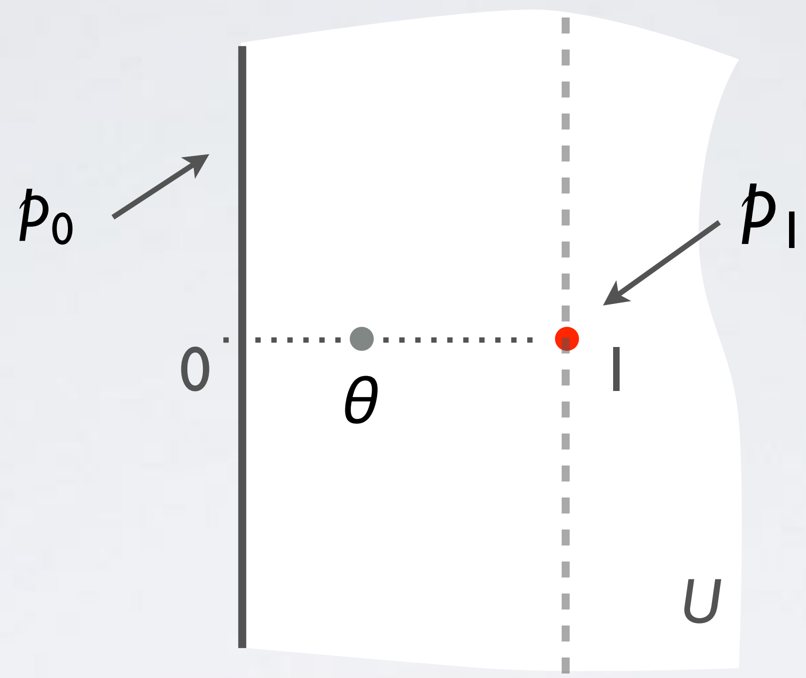
$$f^{\lambda}_{\bar z} = \lambda \mu f^{\lambda}_{z} \qquad \lambda \in \mathbb{D}$$

cf. Mañé-Sad-Sullivan, Slodkowski's λ-lemma

homeomorphisms		L^2
quasiconformal maps	complex interpolation	L^p
diffeomorphisms		L^∞

INTERPOLATION LEMMA

Astala-Iwaniec-Prause-Saksman



$$0 < p_0, p_1 \leqslant \infty, \qquad \theta \in (0, 1)$$

$\phi_\lambda(z)$ analytic family, $\quad \lambda \in U = \{\mathrm{Re}\,\lambda > 0\}$

non-vanishing $\quad \phi_\lambda(z) \neq 0$

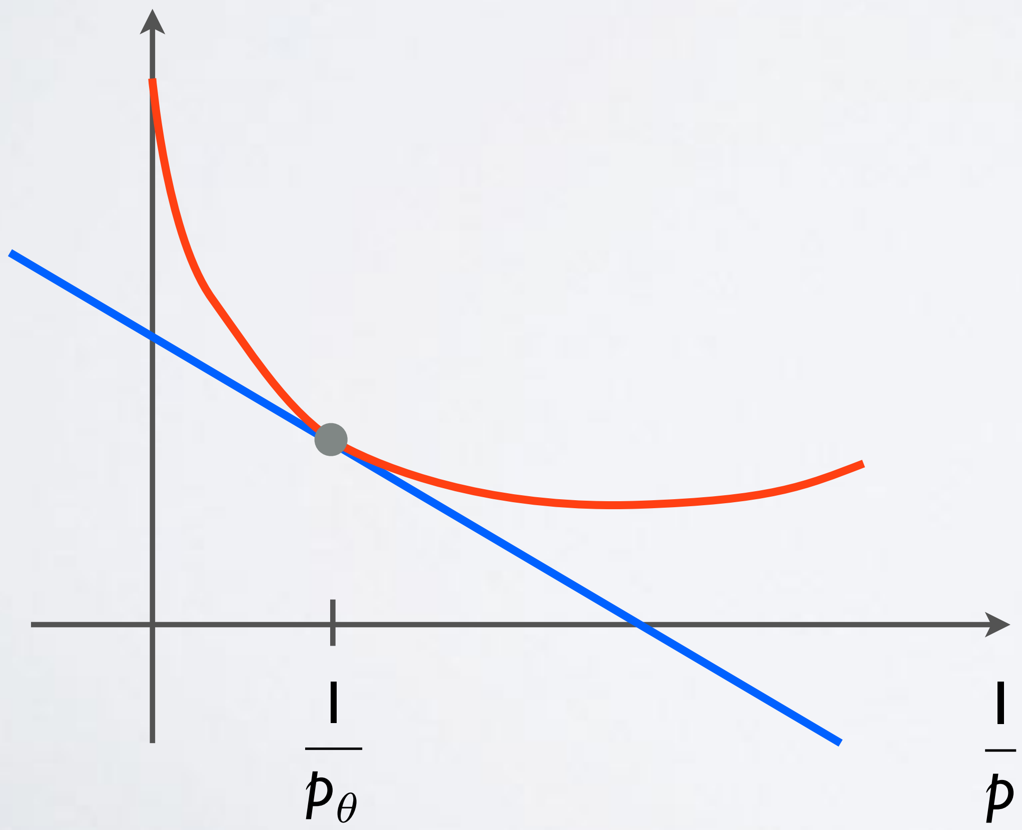
$$\|\phi_\lambda\|_{p_0} \leqslant M_0 \qquad \qquad \|\phi_\theta\|_{p_\theta} \leqslant M_0^{1-\theta} \cdot M_1^\theta$$

$$\Rightarrow$$

$$\|\phi_1\|_{p_1} \leqslant M_1 \qquad \qquad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

cf. RIESZ-THORIN

duality	⇝	*log-convexity*
change p		freeze p
subharmonic		harmonic
Hadamard		Harnack



$$\log \|\phi_\theta\|_p \geqslant A \cdot \frac{1}{p} + B$$

PROOF OF THE LEMMA

$$\log \|\phi_\lambda\|_p \geqslant A \cdot \frac{1}{p} + B(\lambda)$$

non-vanishing ⇝ harmonic

$$A \cdot \frac{1}{p_0} + B(\theta) \leqslant \theta \left(A \cdot \frac{1}{p_0} + B(1) \right)$$

Harnack

$$\log \|\phi_\theta\|_{p_\theta} = A \cdot \frac{1}{p_\theta} + B(\theta) = A \cdot \frac{1}{p_0} + B(\theta) + \theta \cdot A \left(\frac{1}{p_1} - \frac{1}{p_0} \right)$$

$$\leqslant \theta \left(A \cdot \frac{1}{p_1} + B(1) \right) \leqslant \theta \log \|\phi_1\|_{p_1} \leqslant 0 \qquad \square$$

VARIATION OF DIM OF JULIA SETS

Astala, Ransford

$(q_\lambda)_\lambda,\ \lambda \in D$ analytic family of rational maps, $d \geqslant 2$

D simply connected, q_λ hyperbolic

Theorem: $1/\dim J_\lambda = \inf_{u \in \mathcal{H}} u(\lambda)$ $\quad \mathcal{H}$ a collection of harmonic functions

Corollary:
$$\frac{1/\dim(J_{\lambda_1}) - \frac{1}{2}}{1/\dim(J_{\lambda_2}) - \frac{1}{2}} \leqslant \exp \rho_D(\lambda_1, \lambda_2)$$

Pf: $\dim \leqslant 2,\quad u - \dfrac{1}{2} \geqslant 0 \quad \xrightarrow{\text{Harnack}} \quad \dfrac{u(\lambda_1) - \frac{1}{2}}{u(\lambda_2) - \frac{1}{2}} \leqslant \exp \rho_D(\lambda_1, \lambda_2)$

THERMODYNAMICS

variational principle (Ruelle, Bowen)

$$\dim(J_\lambda) = \sup_{\mu \in \mathcal{M}_\lambda} \frac{h_\mu(q_\lambda)}{\int_{J_\lambda} \log |q'_\lambda| d\mu}$$

$$\varphi_\lambda : J_{\lambda_0} \to J_\lambda$$

Mañé-Sad-Sullivan

$$q_\lambda = \varphi_\lambda \circ q_{\lambda_0} \circ \varphi_\lambda^{-1}$$

holomorphic motion

$$\frac{1}{\dim J_\lambda} = \inf_{\mu \in \mathcal{M}_{\lambda_0}} \frac{\int_{J_{\lambda_0}} \log |q'_\lambda \circ \varphi_\lambda| d\mu}{h_\mu(q_{\lambda_0})} \longleftarrow$$

harmonic

EXAMPLE: $q_\lambda = z^d + \lambda z, \quad \lambda \in \mathbb{D}$

Cor: $\dim J(q_\lambda) \leqslant 1 + |\lambda|$ Smirnov ⇝ $1 + \left(\frac{|\lambda|}{2}\right)^2 + \ldots$

Ruelle: $\dim J(q_\lambda) = 1 + \frac{(d-1)^2}{d^2 \log d} \left(\frac{|\lambda|}{2}\right)^2 + \cdots$

$\varphi_\lambda : \mathbb{D}^* \to \mathbb{C}$ conformal conjugacy has

Astala-Ivrii-Perälä-Prause: $\frac{d^{1/(d-1)}}{2} |\lambda| + \mathcal{O}(|\lambda|^2)$ qc extension

efficient representation of $\dot{\varphi}_0 = \mathcal{C}\mu$ — spt $\mu \subset \mathbb{D}$ & min $\|\mu\|_\infty$

Cor: $J(q_\lambda)$ is a k-quasicircle with dimension > *1+0.879* k^2 for k small *(d=20)*

ON DIM OF QUASICIRCLES

Smirnov: $\dim \leqslant 1 + k^2$

Astala-Ivrii-Perälä-Prause: For any holomorphic family of conformal maps $\{\varphi_\lambda\}_{\lambda \in \mathbb{D}}$ $\varphi_0 = \mathrm{id}$

“asymptotic variance” $\sigma^2(\dot{\varphi}_0') \leqslant 1$

$$\sigma^2(g) = \frac{1}{2\pi} \limsup_{r \to 1} \frac{1}{|\log(1-r)|} \int_{|z|=r} |g|^2 d\theta$$

Hedenmalm: $\sigma^2 \leqslant 1 - \varepsilon_0$

Ivrii: $\dim < 1 + k^2$

HOLOMORPHIC INTERPOLATION

variation of dimension	Riesz-Thorin
holomorphic motions	holomorphic exponent
dimension	norm
variational principle	duality
apriori bounds	endpoint estimates
Harnack's inequality	Hadamard's three lines theorem

INTERPOLATION THEME

interpolate	p-norm of gradient	complex powers	complex powers
end-point estimates	Jacobian null-Lagrangian	dim ≤ 2	B(2)=1
maximum principle	Harnack	Schwarz lemma	Nevanlinna-Pick in bidisk
application	quasiconvexity	stretching and rotation spectrum	twisting of the Riemann map

MARTINGALE INEQUALITY

Burkholder

$X_n \prec Y_n$ subordinated martingales

$$|X_n - X_{n-1}| \leq |Y_n - Y_{n-1}| \text{ a.s.}$$

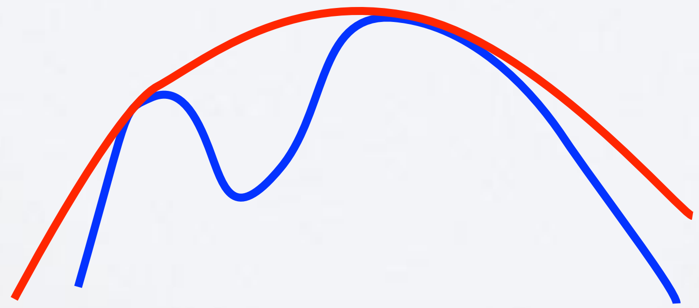
$p \geqslant 2$

$$\Rightarrow \quad \|X_n\|_p \leqslant (p-1)\,\|Y_n\|_p.$$

$$B_p(z,w) = \left(|z| - (p-1)\,|w|\right) \cdot \left(|z| + |w|\right)^{p-1}$$

$$|z|^p - (p-1)^p\,|w|^p \;\leqslant\; c_p\, B_p(z,w)$$

$$\mathbb{E}\, B_p(X_n, Y_n) \leqslant \mathbb{E}\, B_p(X_{n-1}, Y_{n-1}) \leqslant \ldots \leqslant 0$$



Rank-one convexity vs Quasiconvexity

local Morrey *global*

$$\mathbf{E}: \mathbb{R}^{n\times n} \to \mathbb{R}$$

$\operatorname{rank} X = 1$

$t \mapsto \mathbf{E}(A + tX)$ *convex*

$$\Leftarrow \quad \int_\Omega \mathbf{E}(Df) \geqslant \int_\Omega \mathbf{E}(A) = \mathbf{E}(A)|\Omega|$$

$$f \in A + C_0^\infty(\Omega, \mathbb{R}^n)$$

(ellipticity of Euler-Lagrange)

(lower semicontinuity)

$n \geqslant 3$ Šverák $\not\Rightarrow$

$n = 2$? Faraco-Székelyhidi: “localization”

Burkholder: $B_p(Df) = B_p(f_z, f_{\bar{z}})$ rank-one concave

$$B_p(A) = \left(\frac{p}{2} \det A + (1 - \frac{p}{2}) |A|^2 \right) \cdot |A|^{p-2}$$

Quasiconvexity result

Astala-Iwaniec-Prause-Saksman

$$B_p(z,w) = \left(|z| - (p-1)\,|w|\right)\cdot\left(|z| + |w|\right)^{p-1} \qquad p \geqslant 2$$

$$B_p(Df) = B_p(f_z, f_{\bar z})$$

Theorem: $f(z) \in z + C_0^\infty(\Omega), \quad B_p(Df) \geqslant 0, \; z \in \Omega$

$$\int_\Omega B_p\,(Df) \leqslant \int_\Omega B_p\,(Id) = |\Omega|$$

full quasiconvexity $\longrightarrow$

$$\|S\|_{L^p(\mathbb{C})} = p - 1$$

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2}\,dm(\zeta)$$

$$|\phi_\lambda|^2\, w \leqslant$$

$p=2$ **INTERPOLATE** $p=\infty$

$J(z,f)$ $K=1$

$$\phi_0 \equiv 1$$

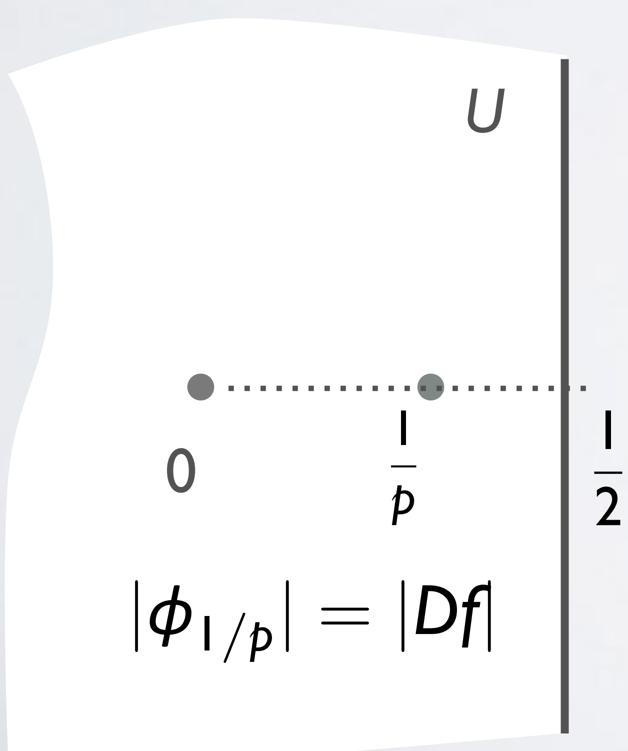
$$f_{\bar z} = \mu\, f_z$$

$$\lambda \in U = \{\mathrm{Re}\,\lambda < 1/2\}$$

$$f^\lambda_{\bar z} = \mu_\lambda\, f^\lambda_z$$

Ahlfors-Bers

$$\frac{\frac{|\mu|}{\mu}\cdot\mu_\lambda}{1+\frac{|\mu|}{\mu}\cdot\mu_\lambda} = \lambda\, p\,\frac{|\mu|}{1+|\mu|}$$



$$\frac{1}{\pi}\int_D \left(1 - p\,\frac{|\mu|}{1+|\mu|}\right) |Df|^p \leqslant 1$$

weight interpolate

$$\phi_\lambda(z) = f^\lambda_z(z) + \frac{|\mu(z)|}{\mu(z)}\, f^\lambda_{\bar z}(z)$$

Stretching and Rotation

stretching exponent

$$\alpha(z_0) = \lim_{|z-z_0|=r_n\to 0} \frac{\log|f(z)-f(z_0)|}{\log|z-z_0|}$$

$$\alpha > 0$$

radial stretching

$$\frac{z}{|z|}\,|z|^{\alpha}$$

K-quasiconformal

$$K = \max\{\alpha, 1/\alpha\}$$

rate of rotation

$$\gamma(z_0) = \lim_{r_n\to 0} \frac{\arg(f(z_0+r_n)-f(z_0))}{\log|f(z_0+r_n)-f(z_0)|}$$

$$\gamma \in \mathbb{R}$$

logarithmic spiral

$$z\,|z|^{i\gamma}$$

L-bilipschitz

$$L - \frac{1}{L} = \max\{\gamma, -\gamma\}$$

STRETCHING vs ROTATION

harmonic dependence *"conjugate harmonic"*

stretching	rotation
quasiconformal	**bilipschitz**
Grötzsch problem	John's problem
Hölder exponent	rate of spiralling
log $J(z,f) \in$ BMO	arg $f_z \in$ BMO
higher integrability	exponential integrability
multifractal spectrum	

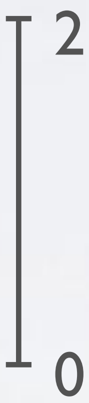
MULTIFRACTAL SPECTRA

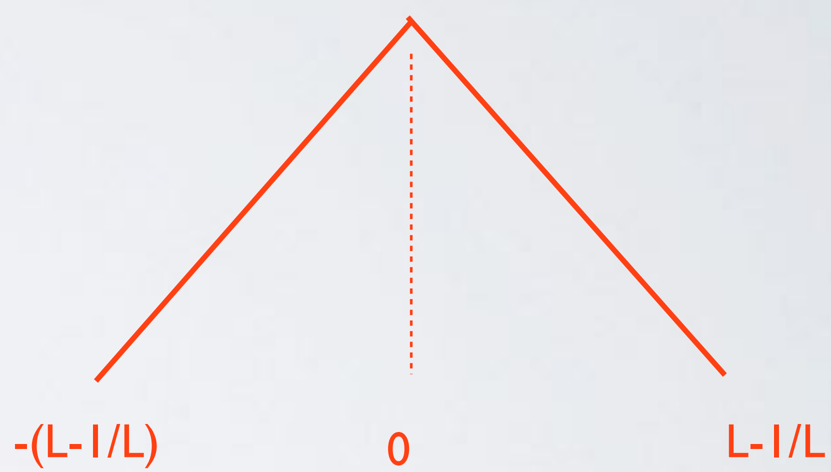
Astala-Iwaniec-Prause-Saksman

K-quasiconformal

$f\colon \mathbb{C} \to \mathbb{C}$

L-bilipschitz

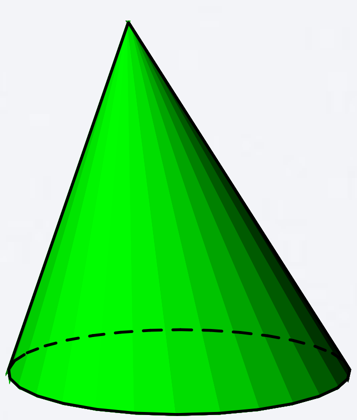




$$\dim_H\{z \in \mathbb{C} : \alpha(z) = \alpha\} \leq 1 + \alpha - \frac{|1-\alpha|}{k}$$

$$\dim_H\{z : \gamma(z) = \gamma\} \leq 2 - \frac{2L}{L^2 - 1}|\gamma|$$

$$\alpha(z_0) = \lim_{|z-z_0|=r_n\to 0} \frac{\log|f(z) - f(z_0)|}{\log|z - z_0|}$$



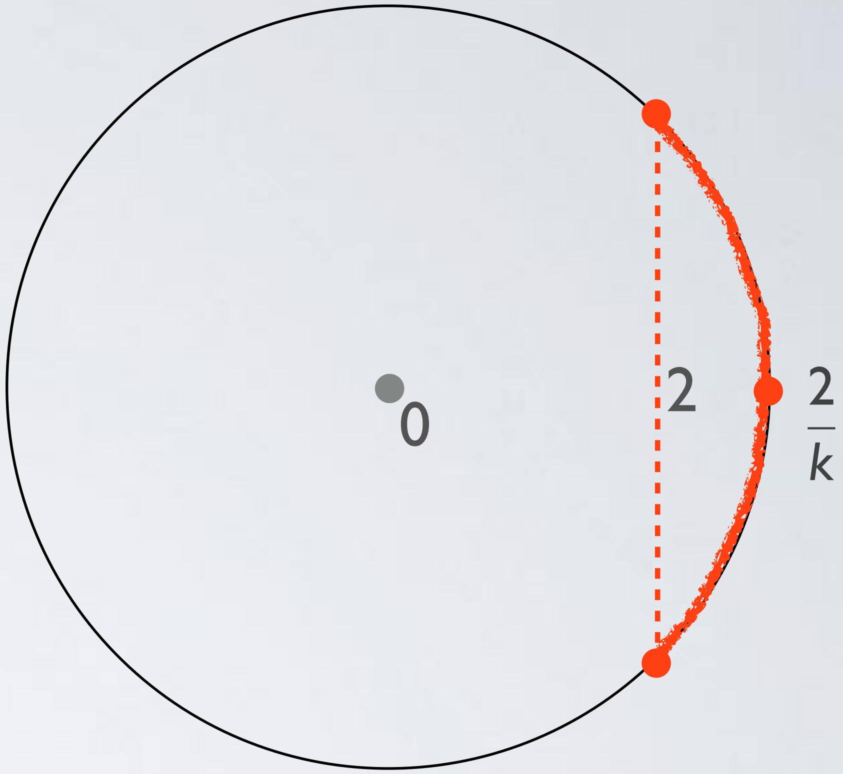
$$\sigma = \alpha(1 + i\gamma)$$

$$\gamma(z_0) = \lim_{r_n\to 0} \frac{\arg(f(z_0 + r_n) - f(z_0))}{\log|f(z_0 + r_n) - f(z_0)|}$$

TWISTING OF THE RIEMANN MAP

Thm (P): $f\colon \mathbb{D} \to \Omega$ conformal with k-qc extension

$$\int_{\mathbb{D}} \left|(f')^t\right| < \infty \qquad |t| < \frac{2}{k},\ \mathrm{Re}\, t \geqslant 2$$



"Circular" Brennan's conjecture: true for every $|t| < \dfrac{2}{k}$

(Becker-Pommerenke)

holomorphic amplification from t=2 case

Schwarz lemma

⇓

3-point Nevanlinna-Pick interpolation problem in the bidisk

HOLOMORPHIC MOTION IN $\mathbb{D}^2$

$$R > 1 \qquad\qquad |\mu| \leqslant \chi_{\{|z|\geqslant R\}}$$

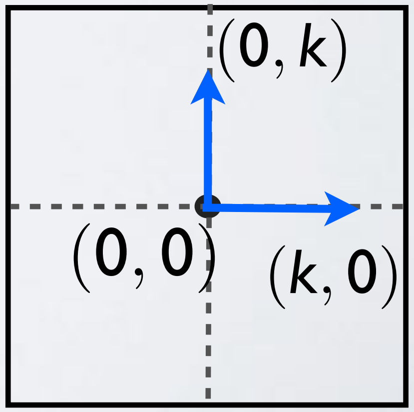
$$\mu_{\lambda,\eta}(z) = \begin{cases} \lambda\mu(z) \text{ for } |z| > 1, \\ \eta\mu(1/\bar{z}) \text{ for } |z| < 1. \end{cases} \qquad (\lambda,\eta) \in \mathbb{D}^2$$

$$\bar{\partial} f_{\lambda,\eta} = \mu_{\lambda,\eta}\, \partial f_{\lambda,\eta} \qquad \{0, 1, \infty\} \text{ -normalization}$$

$$f'(z) \rightsquigarrow \frac{z f'(z)}{f(z)}$$

$$\varphi_{\lambda,\eta}(z) = \frac{z\, f_{\lambda,\eta}{}'(z)}{f_{\lambda,\eta}(z)}, \qquad z \in \mathbb{S}^1$$

$$f_{\lambda,\eta}(z) = \frac{1}{\overline{f_{\bar{\eta},\bar{\lambda}}(1/\bar{z})}} \longrightarrow \varphi_{\lambda,\eta} = \overline{\varphi_{\bar{\eta},\bar{\lambda}}}$$



CLASSICAL LÖWNER CHAINS

Löwner, Kufarev, Pommerenke

Any univalent function $f\colon \mathbb{D} \to \mathbb{C}$, $\quad f(0) = 0,\ f'(0) = 1$

is the initial element $(f = f_0)$ of some

classical *radial Löwner chain* $\quad (f_t)_{t \geqslant 0} \subset \mathrm{Hol}(\mathbb{D}, \mathbb{C})$

- f_t is univalent in $\mathbb{D}$ $\quad f_t(0) = 0,\ f_t'(0) = e^t$
- $f_s(\mathbb{D}) \subset f_t(\mathbb{D}), \quad s \leqslant t$

Löwner equation

$$\dot{f_t} = -G(z,t) f_t{}' \qquad \text{a.e. in } t$$

$$G(z,t) = -zp(z,t)$$

classical Herglotz function $\quad p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$

- $p(z, \cdot)$ is measurable
- $p(\cdot, t)$ holomorphic with $\mathrm{Re}\, p \geqslant 0, \quad p(0,t) = 1$

QUASICONFORMAL EXTENSIONS

Becker: $p(\mathbb{D}, t) \subset U(k) = \left\{\zeta : \left|\frac{\zeta - 1}{\zeta + 1}\right| \leqslant k\right\}$, a.e. t

→ each f_t has (an explicit) k-quasiconformal extension to $\overline{\mathbb{C}}$

reverse direction: Schwarz lemma? No

Includes many other sufficiency criteria

Bracci-Contreras-Díaz-Madrigal-Gumenyuk (non-autonomous framework)

Löwner chains from time-dependent Herglotz vector fields

$$G(z,t) = (\tau(t) - z)(1 - \overline{\tau(t)}z)p(z,t)$$

$\tau : [0, \infty) \to \overline{\mathbb{D}}$ measurable

$t \mapsto p(0,t) \in L^1_{loc}$

Becker's condition still ensures k-qc extendability

Becker radial ($\tau \equiv 0$) Gumenyuk-Hotta chordal ($\tau \equiv 1$) Hotta general

Gumenyuk-Prause short proof via holomorphic motions $p \hookrightarrow p_\lambda$