BURKHOLDER INTEGRALS MORREY'S PROBLEM AND QUASICONFORMAL MAPPINGS

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joint work with

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"We have a problem…"

yes-or-no

$$f(z) - z \in C_0^\infty(D), \quad p \geqslant 2, \quad df = f_z dz + f_{\bar z} \overline{dz}$$

$$\frac{1}{\pi} \int_D \underbrace{\big(|f_z| - (p-1)|f_{\bar z}| \big) \cdot \big(|f_z| + |f_{\bar z}| \big)^{p-1}}_{B_p(f_z, f_{\bar z})} \leqslant 1 \quad ?$$

✓

Iwaniec conjecture (1982) on Beurling transform

$$\|S\|_p = p - 1$$

interpolation

✗

Morrey's problem (1952) on convexity notions

rank-one $\not\Rightarrow$ *quasiconvex*

Burkholder's martingale theory

Martingale inequality

Burkholder

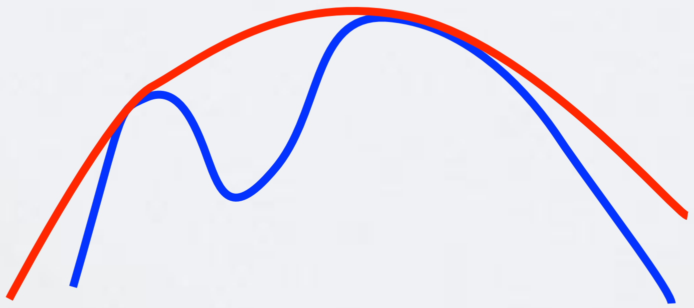
$X_n \prec Y_n$ subordinated martingales

$$\Rightarrow \quad \|X_n\|_p \leqslant (p-1)\,\|Y_n\|_p.$$

$$B_p(z,w) = \big(|z| - (p-1)\,|w|\big)\cdot\big(|z| + |w|\big)^{p-1}$$

$$|z|^p - (p-1)^p\,|w|^p \leqslant c_p\,B_p(z,w)$$

$$\mathbb{E}\,B_p(X_n,Y_n) \leqslant \mathbb{E}\,B_p(X_{n-1},Y_{n-1}) \leqslant \ldots \leqslant 0$$



Rank-one convexity vs quasiconvexity

Morrey

$$\mathbf{E} : \mathbb{R}^{n\times n} \to \mathbb{R}$$

rank $X = 1$

$t \mapsto \mathbf{E}(A + tX)$ *convex*

$$\Leftarrow \quad \int_\Omega \mathbf{E}(Df) \geqslant \int_\Omega \mathbf{E}(A) = \mathbf{E}(A)|\Omega|$$

$$f \in A + C_0^\infty(\Omega, \mathbb{R}^n)$$

local vs *global*

$n \geqslant 3$ Šverák $\not\Rightarrow$

$n = 2$? Faraco-Székelyhidi: "localization"

Burkholder: $B_p(Df) = B_p(f_z, f_{\bar z})$ rank-one concave

$$B_p\,(A) = \left(\frac{p}{2} \det A \; + \; (1 - \frac{p}{2})\,|A|^2 \right) \cdot |A|^{p-2}$$

Beurling transform

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2}\, dm(\zeta)$$

$S(f_{\bar z}) = f_z, \quad f \in W^{1,2}(\mathbb{C})$ L^2-isometry

Conjecture (Iwaniec): $\|S\|_{L^p(\mathbb{C})} = p - 1, \quad p \geqslant 2$

"Win-win" problem:

B_p quasiconcave at $A = Id$

✓ Iwaniec

✗ Morrey

martingale techniques: Bañuelos et al., Volberg-Nazarov,...

WR: $\|S\|_p \leqslant 1.575(p-1)$

Main result

"half-yes"

Theorem: $f(z) \in z + C_0^\infty(\Omega)$, $B_p(Df(z)) \geqslant 0,$ $z \in \Omega$

$$\int_\Omega B_p\,(Df) \leqslant \int_\Omega B_p\,(Id) = |\Omega|,$$

$$\frac{1}{|\Omega|}\int_\Omega \big(\,|f_z| - (p-1)\,|f_{\bar z}|\,\big)\cdot\big(\,|f_z| + |f_{\bar z}|\,\big)^{p-1} \leqslant 1$$

trade-off

distortion and exponent

$$|Df|^2 \leqslant K\,J_f \qquad\qquad p \leqslant \frac{2K}{K-1}$$

cf. Astala's *higher integrability*

$|\phi_\lambda|^2\, w \leqslant$ $p=2$ **INTERPOLATE** $p=\infty$ $\phi_0 \equiv 1$

$J(z,f)$ $K=1$

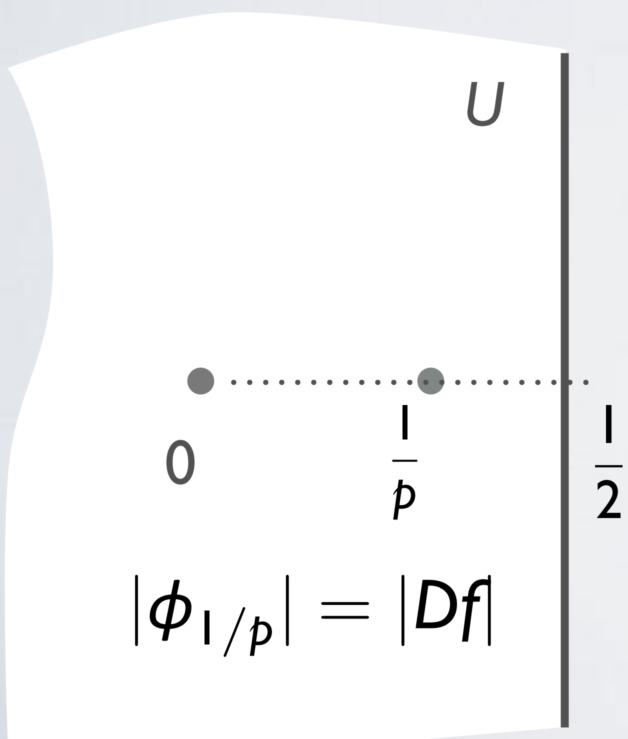
$$f_{\bar z} = \mu\, f_z$$

$$\lambda \in U = \{\mathrm{Re}\,\lambda < 1/2\}$$

$$f^{\lambda}_{\bar z} = \mu_\lambda\, f^{\lambda}_z$$

Ahlfors-Bers

$$\frac{\frac{|\mu|}{\mu}\cdot\mu_\lambda}{1+\frac{|\mu|}{\mu}\cdot\mu_\lambda} = \lambda\, p\,\frac{|\mu|}{1+|\mu|}$$

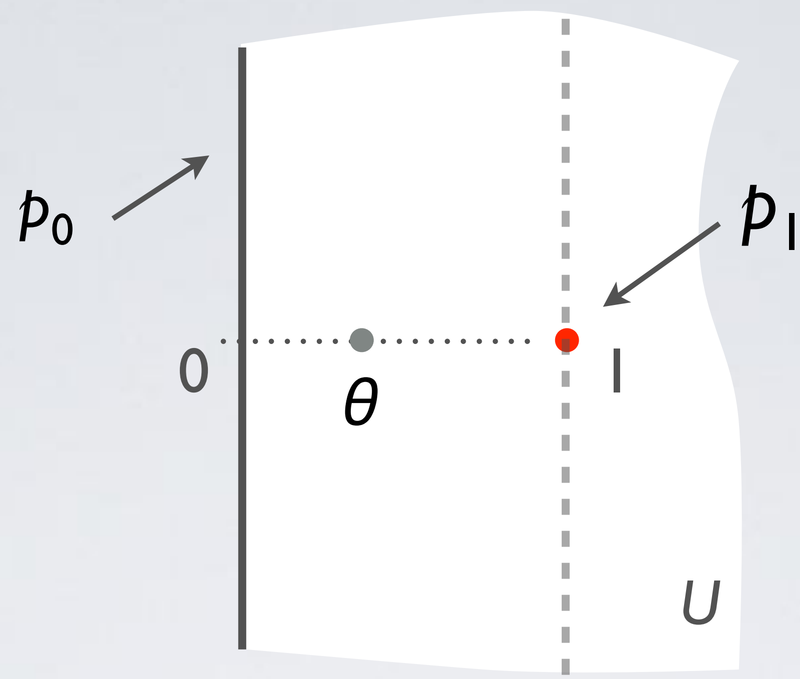


$$\frac{1}{\pi}\int_D \left(1 - p\,\frac{|\mu|}{1+|\mu|}\right) |Df|^p \leqslant 1$$

weight interpolate

$$\phi_\lambda(z) = f^{\lambda}_z(z) + \frac{|\mu(z)|}{\mu(z)} f^{\lambda}_{\bar z}(z)$$

Interpolation lemma



$$0 < p_0, p_1 \leqslant \infty, \qquad \theta \in (0,1)$$

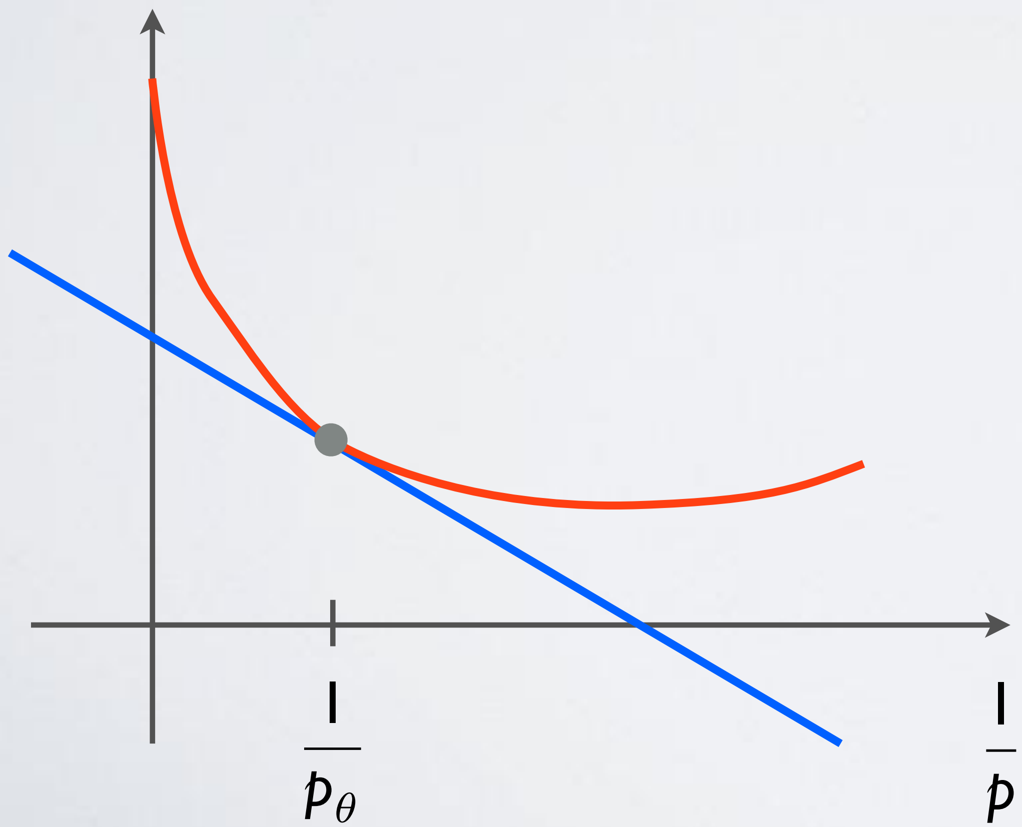
$\phi_\lambda(z)$ analytic family, $\lambda \in U = \{\mathrm{Re}\,\lambda > 0\}$

non-vanishing $\phi_\lambda(z) \neq 0$

$$\|\phi_\lambda\|_{p_0} \leqslant M_0\, e^{c\,\mathrm{Re}\lambda} \qquad \|\phi_1\|_{p_1} \leqslant M_1 \quad \Rightarrow \quad \|\phi_\theta\|_{p_\theta} \leqslant M_0^{1-\theta} \cdot M_1^\theta \qquad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

cf. Riesz-Thorin

duality	⇝	*log-convexity*
change p		freeze p
subharmonic		harmonic
Hadamard		Harnack



$$\log \|\phi_\theta\|_p \geqslant A \cdot \frac{1}{p} + B$$

Proof of the lemma

$$\log \|\phi_\lambda\|_p \geqslant A \cdot \frac{1}{p} + B(\lambda)$$

non-vanishing ⇝ harmonic

$$A \cdot \frac{1}{p_0} + B(\theta) \leqslant \theta \left(A \cdot \frac{1}{p_0} + B(1) \right)$$

Harnack

$$\log \|\phi_\theta\|_{p_\theta} = A \cdot \frac{1}{p_\theta} + B(\theta) = A \cdot \frac{1}{p_0} + B(\theta) + \theta \cdot A \left(\frac{1}{p_1} - \frac{1}{p_0} \right)$$

$$\leqslant \theta \left(A \cdot \frac{1}{p_1} + B(1) \right) \leqslant \theta \log \|\phi_1\|_{p_1} \leqslant 0 \qquad \square$$

Corollaries

sharp integrability estimates

LlogL integrability: $f(z) \in z + C_0^\infty(\Omega)$, *homeomorphism*,

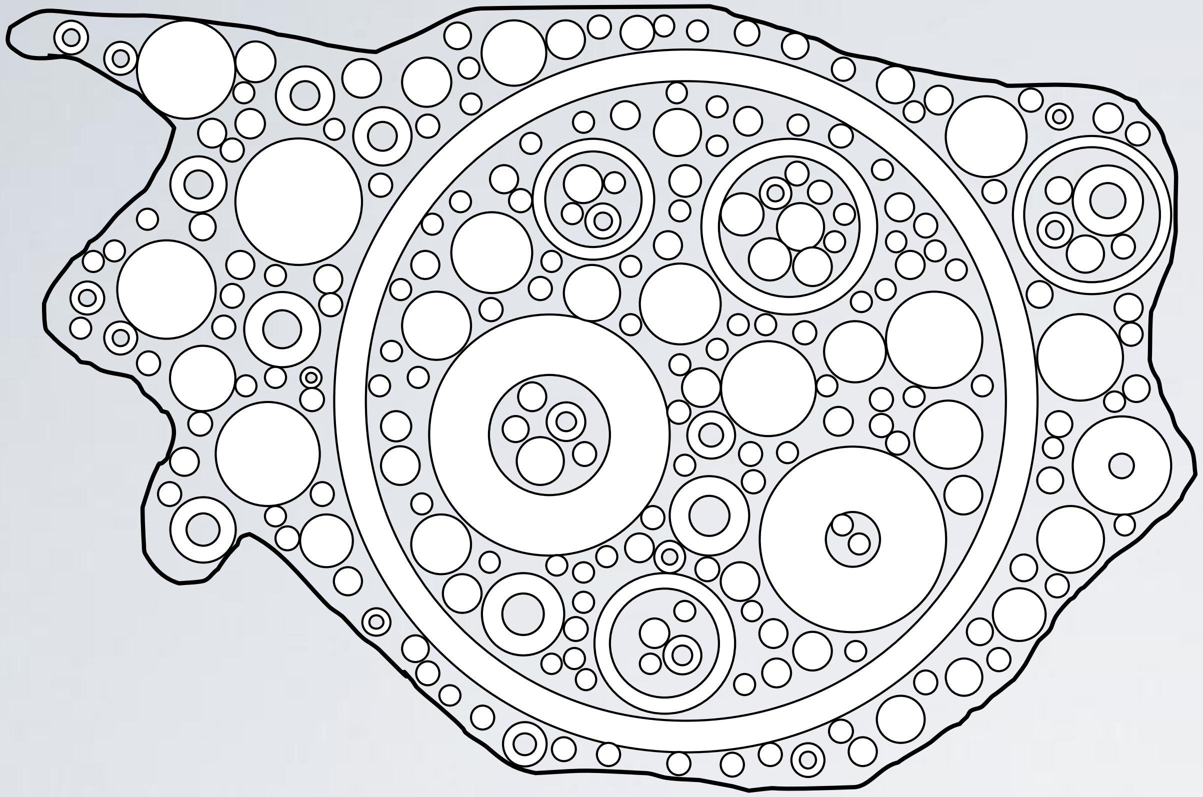
$$\int_\Omega \left(1 + \log|Df(z)|^2\right) J(z,f) \leqslant \int_\Omega |Df(z)|^2. \quad \textit{Proof:} \quad \left.\frac{dB_p}{dp}\right|_{p=2} \quad \square$$

Müller: *LlogL* integrability under J(z,f)≥0

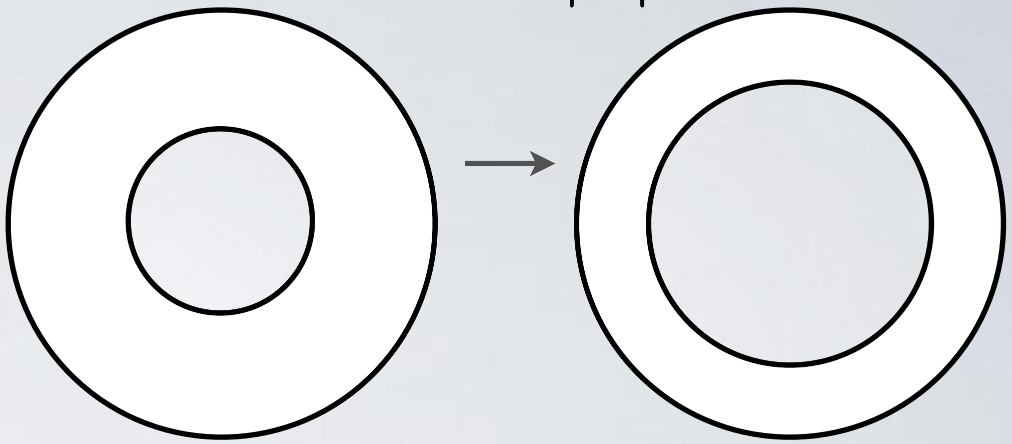
Exp integrability: $|\mu(z)| \leqslant \chi_D(z), \quad z \in \mathbb{C}$

$$\frac{1}{\pi}\int_D \left(1 - |\mu|\right) e^{|\mu| + \mathrm{Re}\, S\mu} \leqslant 1. \quad \textit{Proof:} \quad \left.\frac{dB_p}{dp}\right|_{p=\infty} \quad \square$$

Many extremals



$$g(z) = \rho(|z|)\,\frac{z}{|z|}$$



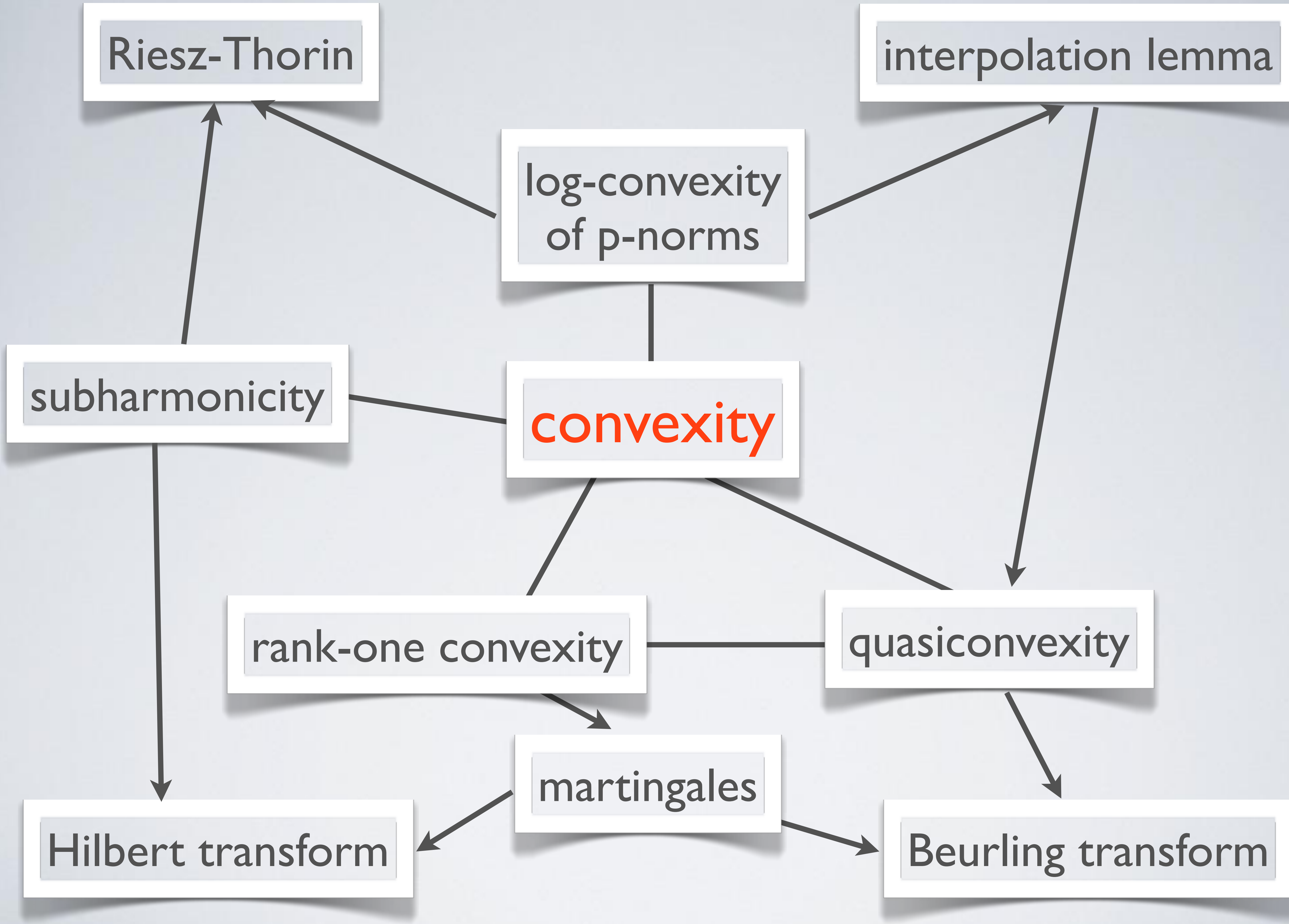
expanding

$$\frac{\rho(t)}{t} \geqslant \dot{\rho}(t), \quad \rho(t) = o(t^{1-\frac{2}{p}})$$

B_p linear on rank-one connections

Baernstein-Montgomery-Smith

$$\int_{B(0,R)} B_p(Dg) = \pi \int_0^R \left(\frac{[\,\rho(t)\,]^p}{t^{p-2}} \right)' \mathrm{d}t = \pi R^2$$


Riesz-Thorin
interpolation lemma
log-convexity of p-norms
subharmonicity
convexity
rank-one convexity
quasiconvexity
martingales
Hilbert transform
Beurling transform