ASYMPTOTIC VARIANCE OF THE BEURLING TRANSFORM

István Prause

UNIVERSITY OF HELSINKI

joint work with

Kari Astala, Oleg Ivrii and Antti Perälä

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BLOCH FUNCTIONS

$\mathcal{B}$ g analytic in $\mathbb{D}$ and

$$\|g\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|g'(z)| < \infty$$

seminorm

Examples for boundary behaviour

$$g(z) = \log(1 - z)$$

$$g(z) = \sum_{n=1}^{\infty} z^{2^n}$$

CONFORMAL MAPS AND BLOCH

$f\colon \mathbb{D} \to \Omega$ conformal $\longrightarrow$ $g = \log f' \in \mathcal{B}$

$\|g\|_{\mathcal{B}} \leqslant 6$

$g = \log f'$

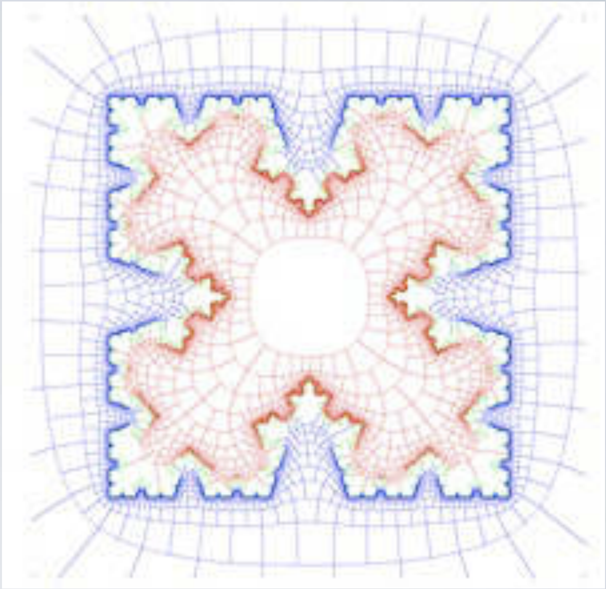
for some conformal map f $\longleftarrow$ $\|g\|_{\mathcal{B}} \leqslant 1$

ISSUE: inherent gap in the constants

MAKAROV'S LIL

$$\limsup_{r\to 1} \frac{|g(r\zeta)|}{\sqrt{\log \frac{1}{1-r} \log\log\log \frac{1}{1-r}}} \leqslant \|g\|_{\mathcal{B}}$$

a.e. $\zeta \in \partial\mathbb{D}$



Courtesy of D. Marshall

Ω simply connected

Makarov's theorem: $\dim \omega = 1$

$$\dim \omega \stackrel{\text{def}}{=} \inf\{\dim E : \omega(E) = 1\}$$

$$h(t) = t \exp C \sqrt{\log \frac{1}{t} \log\log\log \frac{1}{t}}$$

$\omega \ll \Lambda_h$

C is big

$\omega \perp \Lambda_h$ for some Ω

C is small

Przytycki-Urbański-Zdunik

critical $C = \sqrt{\sigma^2(\log f')}$ in "fractal case"

ASYMPTOTIC VARIANCE

$$g \in \mathcal{B}$$

$$\sigma^2(g) = \frac{1}{2\pi} \limsup_{r \to 1} \frac{1}{|\log(1-r)|} \int_{|z|=r} |g|^2 d\theta \qquad \left(\leqslant \|g\|_{\mathcal{B}}^2 \right)$$

Example: $$g(z) = \sum_{n=1}^{\infty} z^{d^n} \qquad \sigma^2 = \frac{1}{\log d}$$

basis to construct singular conformal maps (around t=0)

Makarov, Pommerenke, Rohde, Kayumov, ...

$$g = \log f'$$

INTEGRAL MEANS SPECTRUM

$f\colon \mathbb{D} \to \Omega$ (bounded) conformal map

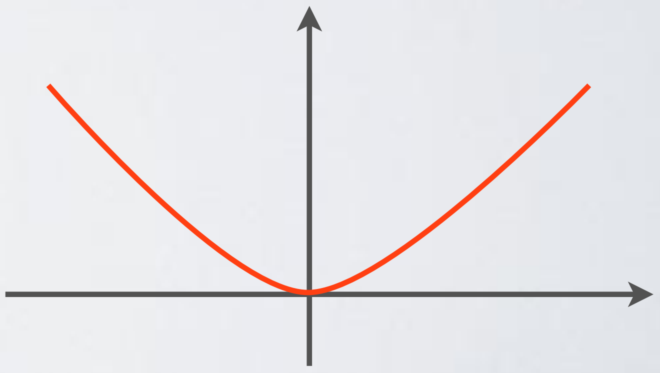
$$\beta_f(t) = \inf \left\{ \beta : \int |f'(re^{i\theta})^t| d\theta = O((1-r)^{-\beta}) \right\}, \quad t \in \mathbb{R}$$

$$B(t) = \sup_{\Omega} \beta_f(t)$$

Universal Spectrum Conjecture

Brennan-Carleson-Jones-Krätzer-...

$$B(t) = \frac{|t|^2}{4}, \quad |t| \leq 2$$



BEURLING TRANSFORM

$$Sf(z) = -\frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)^2} dm(\zeta)$$

$S \circ \partial_{\bar{z}} = \partial_z$ L^2-isometry

$$S\colon L^p \to L^p, 1 < p < \infty$$

$$S\colon L^\infty \to BMO$$

analytic case

$$S\colon L^\infty(\mathbb{D}) \to \mathcal{B}(\mathbb{D}^*)$$

PROBLEM

growth of $S\mu$ in $\mathbb{D}^* = \{|z| > 1\}$

$$\Sigma^2 := \sup_{|\mu| \leq \chi_{\mathbb{D}}} \sigma^2(S\mu) = ?$$

Theorem: $0.879 < \Sigma^2 \leqslant 1$

~~Conjecture:~~ $\Sigma^2 = 1$

Hedenmalm: $\Sigma^2 < 1$

SINGULAR CONFORMAL MAPS

holomorphic motion	$\log(f^{\lambda})' = \lambda \log f'$	$f^{\lambda}_{\bar{z}} = \lambda\, \mu\, f^{\lambda}_{z}$
Bloch function	$\log f'$	$S\mu$
injectivity	f conformal	$\|\mu\|_{\infty} \leqslant 1$
$\sigma^2 \approx c$	$B(t) \sim c\, \frac{t^2}{4}$	H.dim $f^{\lambda}(\mathbb{S}^1) \sim 1 + c\, \frac{\|\lambda\|^2}{4}$
examples	lacunary series	

strong growth → quasicircles → singular welding → conformal map

DIMENSION OF QUASICIRCLES

$$D(k) := \sup\{\text{H.dim}\,\Gamma : \Gamma \text{ is a } K\text{-quasicircle}\} \qquad k = \frac{K-1}{K+1}$$

Smirnov: $D(k) \leqslant 1 + k^2$

Sharpness?

Theorem: $D(k) \geqslant 1 + 0.879\,k^2, \quad k < k_0$

$$\Sigma^2 \leqslant \liminf_{k \to 0} \frac{D(k)-1}{k^2} \qquad \Sigma^2 \leqslant \liminf_{t \to 0} \frac{B(t)}{t^2/4}$$

BLASCHKE PRODUCTS

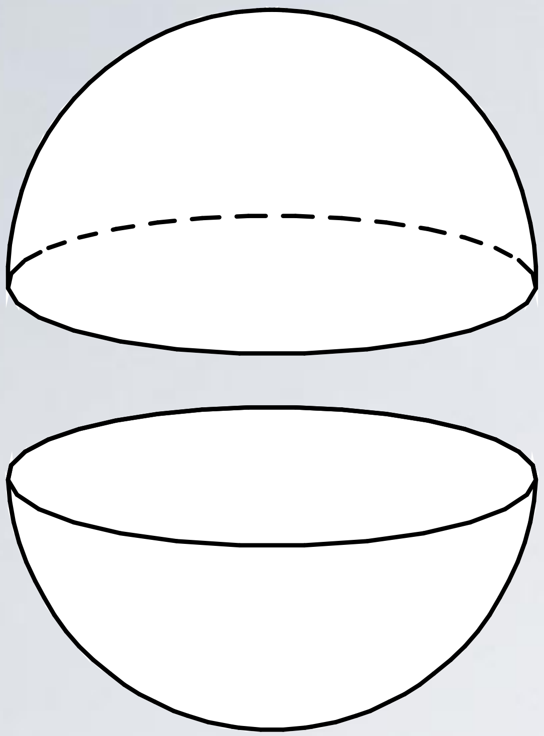
$$B^d = \left. \begin{array}{l} \text{Blaschke products of degree } d \\ \text{with an attracting fixed point} \end{array} \right/ \text{Aut}\,\mathbb{D}$$

Example: $B^2 \cong \mathbb{D}$ $\lambda \in \mathbb{D}$ $f_\lambda(z) = \dfrac{z^2 + \lambda z}{1 + \bar{\lambda} z}$

Julia set = S^1

quasisymmetrically conjugate
to each other

MATING



f_λ

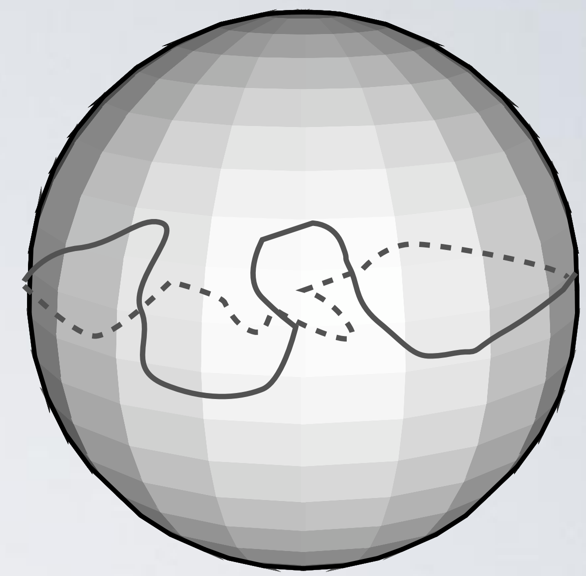
$\bar{f}_\eta$

F rational map

$$F|\Omega_+ \cong f_\lambda$$

$$F|\Omega_- \cong \bar{f}_\eta$$

$$F = [f_\lambda, f_\eta]$$



γ

Jordan curve

Example: d=2

$$F = \frac{z^2 + \lambda z}{1 + \bar{\eta} z}$$

cf. Bers' simultaneous uniformization

WEIL-PETERSSON METRIC

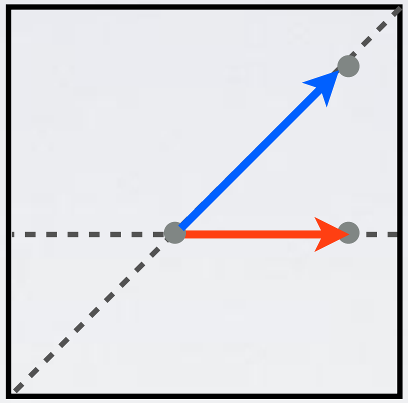
McMullen

$$f_t \in B^d$$

$$F_t = [f_0, f_t]$$

$$J(F_0) = S^1$$

$$m_{0,0} = \text{Lebesgue m.}$$



$$\varphi_t : \mathbb{D}^* \to \mathbb{C}$$

conformal conjugacy

$$-\frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} \dim(m_{t,t}) = 2\frac{d^2}{dt^2}\Big|_{t=0} \dim(J(F_t)) = \sigma^2(v')$$

$$v = \frac{d\varphi_t}{dt}\Big|_{t=0}$$

HOLOMORPHIC MOTIONS

$\{\varphi_t(z)\}, \quad t \in \mathbb{D}$ conformal in $\mathbb{D}^*$

$$v = \dot{\varphi}_0 = \mathcal{C}(\mu) \qquad \mu = \dot{\mu}_0$$

$$v' = \partial\mathcal{C}(\mu) = S\mu \qquad |\mu| \leqslant \chi_{\mathbb{D}}$$

$$\sigma^2(S\mu) \overset{?}{=} 2\frac{d^2}{dt^2}\Big|_{t=0} \dim(\varphi_t(S^1)) \leqslant 1$$

$1+k^2$

not true in general

Le-Zinsmeister

$$\sigma^2 = 0$$

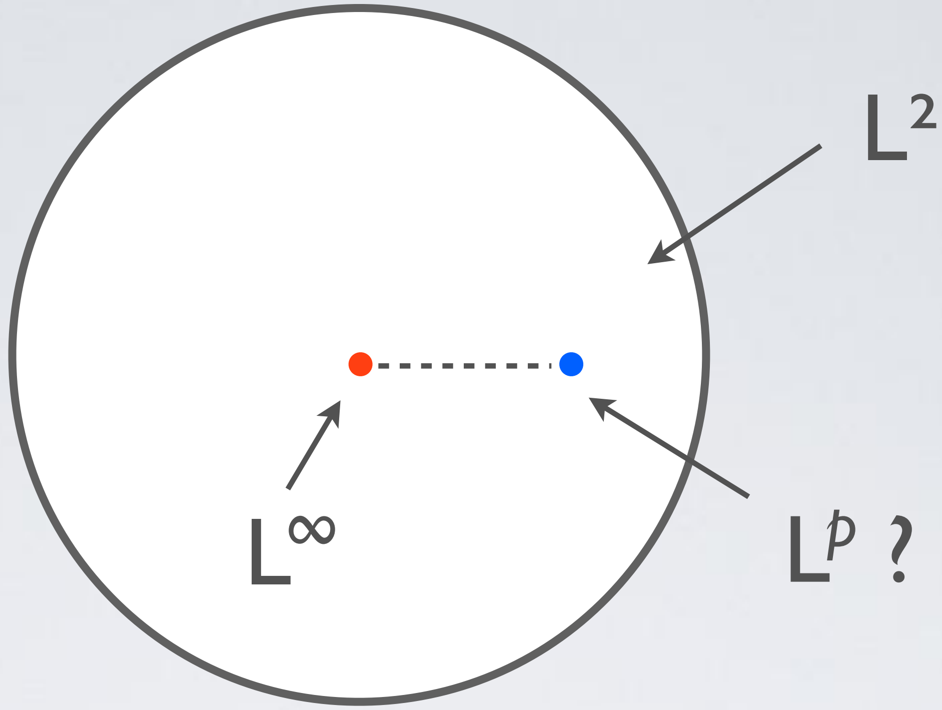
$$\text{M.dim} \geqslant 1 + c\,t^2$$

Bañuelos-Moore

$$\sigma^2 > 0$$

$$\text{H.dim} \equiv 1$$

HOLOMORPHIC INTERPOLATION



Cauchy-Riemann Beltrami

$$f^\lambda_{\bar z} = \lambda \, \mu \, f^\lambda_z \qquad \lambda \in \mathbb{D}$$

$$\lambda \mapsto (f^\lambda)'(z) \neq 0 \quad \text{holomorphic} \qquad z \in \mathbb{D}^*$$

$$\int_{|z|=R} |(f^\lambda)'(z)|^2 |dz|$$

$$(f^\lambda)' = 1 + \lambda \, S\mu + \lambda^2 S\mu S\mu + \ldots$$

INTERPOLATION LEMMA

Astala-Iwaniec-Prause-Saksman

$$0 < p_0, p_1 \leqslant \infty, \quad \theta \in (0,1)$$

$\phi_\lambda(z)$ analytic family $\quad \lambda \in \mathbb{D}$

non-vanishing $\quad \phi_\lambda(z) \neq 0$

antisymmetric $\quad |\phi_\lambda| = |\phi_{-\bar{\lambda}}|$

$$\|\phi_0\|_{p_0} \leqslant M_0$$

$$\|\phi_\lambda\|_{p_1} \leqslant M_1$$

$$\Rightarrow \quad \|\phi_\theta\|_{p_\theta} \leqslant M_0^{\frac{1-\theta^2}{1+\theta^2}} \cdot M_1^{\frac{2\theta^2}{1+\theta^2}}$$

$$\frac{1}{p_\theta} = \frac{1-\theta^2}{1+\theta^2} \cdot \frac{1}{p_0} + \frac{2\theta^2}{1+\theta^2} \cdot \frac{1}{p_1}$$

$$\psi_{\lambda^2} = \sqrt{\phi_\lambda \cdot \phi_{-\lambda}}$$

$$\|\psi_{\lambda^2}\|_{p_1} \leqslant \|\phi_\lambda\|_{p_1}^{1/2} \|\phi_{-\lambda}\|_{p_1}^{1/2} \leqslant M_1$$

$$|\phi_\theta| = |\psi_{\theta^2}|$$

ANTISYMMETRIC MAPS

$$\mathrm{dist}(\mathrm{spt}\,\mu, \mathbb{S}^1) > \varepsilon \qquad \|\mu\|_\infty \leqslant 1 - \delta$$

$$\tilde{\mu}(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)\frac{\bar{z}^2}{z^2}} \qquad f^\lambda_{\bar{z}} = \lambda\,(\mu - \tilde{\mu})\, f^\lambda_z \qquad \{0,1,\infty\}\text{ fixed}$$

$$\phi_\lambda(z) := z\,\frac{\partial f_\lambda(z)}{f_\lambda(z)} \qquad \phi_\lambda(z) = \overline{\phi_{-\bar{\lambda}}(1/\bar{z})}$$

antisymmetric interpolation for $\phi_\lambda(z) \quad z \in \mathbb{S}^1$

$$p_0 = p_1 = 2$$

Key lemma: $$\frac{1}{2\pi}\int_{|z|=1}\left|\frac{(f^k)'(z)}{f^k(z)}\right|^2 |dz| \leqslant C(\delta)^{k^2}\,\varepsilon^{-\frac{2k^2}{1+k^2}}$$

$$\mu = \frac{\mu - \tilde{\mu}}{2} + \frac{\mu + \tilde{\mu}}{2} \qquad \sigma^2(S\mu) \leqslant \frac{1}{4}(2+2) = 1$$

WP METRIC at z^d

Ruelle

$$F_t = z^d + tz \qquad t \in \mathbb{D}$$

$$\varphi_t(z^d) = \varphi_t^d(z) + t\varphi_t(z) \qquad \varphi_0(z) = z$$

$$z \in \mathbb{D}^*$$

$$v = \dot{\varphi}_0$$

$$v(z) = v_0(z) + \frac{1}{dz^{d-1}} v(z^d), \quad v_0(z) = -\frac{1}{d} z^{2-d}$$

$$v(z) = \sum_{k=0}^{\infty} v_k(z), \quad v_{k+1}(z) = \frac{1}{dz^{d-1}} v_k(z^d)$$

$$v(z) = -\frac{z}{d} \sum_{n=0}^{\infty} \frac{z^{-(d-1)d^n}}{d^n}, \qquad |z| > 1.$$

WP METRIC at z^d

$$v'(z) = \frac{(d-1)}{d} \cdot \sum_{n \geq 0} z^{-(d-1)d^n} + b_0, \quad b_0 \in \mathcal{B}_0{}^*$$

$$\sigma^2(v') = \frac{(d-1)^2}{d^2 \log d}$$

$$\text{H.dim}\, J(F_t) = 1 + \sigma^2(v')|t|^2 + \mathcal{O}(|t|^3)$$

$J(F_t)$ is a quasicircle (φ_t has |t|-qc extension)

Degree	λ-lemma	Explicit repr.
$d = 2$	$0.3606\ldots$	$0.3606\ldots$
$d = 3$	$0.4045\ldots$	$0.5394\ldots$
$d = 4$	$0.4057\ldots$	$0.6441\ldots$
$d = 20$	$0.3012\ldots$	$0.8791\ldots$

Lower bounds for Σ^2

EXPLICIT REPRESENTATION

$$\mathcal{C}\mu(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{\mu(w)}{z-w}\,dm(w)$$

Find $\quad \mu,\ \operatorname{spt}\mu \subset \mathbb{D}, \quad v = \mathcal{C}\mu \quad \& \quad \min \|\mu\|_\infty$

pull-back $\qquad \mu^*(z) = \left((z^d)^*\mu\right)(z) = \mu(z^d)\dfrac{\bar{z}^{d-1}}{z^{d-1}}$

Lemma: $\qquad \mathcal{C}\Big((z^d)^*\mu\Big)(z) = \dfrac{1}{dz^{d-1}}\Big\{\mathcal{C}\mu(z^d) - \mathcal{C}\mu(0)\Big\}, \quad z \in \mathbb{C}$

Proof: $\qquad$ take $\ \bar{\partial} \quad$ and $\quad z \to \infty \quad \square$

EXPLICIT REPRESENTATION

Building block: $\mu_0(z) := \left(\bar{z}/|z|\right)^{d-3} \chi_{A(r,\rho)}$ $\rho^d = r$

$$\mathcal{C}\mu_0(z) = \frac{2}{d-1}\left(\rho^{d-1} - r^{d-1}\right) z^{2-d}, \quad |z| > 1$$

$$\mathcal{C}\mu_0 = \alpha_d \, v_0$$

$$\mu_{k+1} := \mu_k^*, \quad \mathcal{C}\mu_{k+1}(z) = \alpha_d \frac{1}{dz^{d-1}} v_k(z^d) = \alpha_d \, v_{k+1}(z)$$

disjoint spt

$$\mu := \sum_{k=0}^{\infty} \mu_k, \quad \mathcal{C}\mu = \alpha_d \, v$$

optimize over r $-\alpha_d^{-1} = \frac{d^{1/(d-1)}}{2} \leqslant 1$

optimize over d (d=20) → $\Sigma^2 \geqslant \sigma^2(S\mu) = 0.879\ldots$

QUASICONFORMAL EXTENSION

$$F_t = z^d + tz \qquad t \in \mathbb{D}$$

$\varphi_t : \mathbb{D}^* \to \mathbb{C}$ conformal conjugacy

λ-lemma: H_t $|t|$-qc extension

Thm: $\frac{d^{1/(d-1)}}{2}|t| + \mathcal{O}(|t|^2)$ qc extension

Proof: $\mu_{H_t} = t\tilde{\mu} + \dots$

$\tilde{\mu} - \alpha_d^{-1}\mu$ is infinitesimally trivial → $\mu_{N_t} = t(\tilde{\mu} - \alpha_d^{-1}\mu) + \dots$

$N_t|_{\mathbb{D}^*} = \mathrm{id}$

correct H_t by $H_t \circ N_t^{-1}$ $\qquad \mu_{H_t \circ N_t^{-1}} = \alpha_d^{-1} t\mu + \dots \qquad \square$

Corollary: $J(F_t)$ is a k-quasicircle with dimension > $1 + 0.879\, k^2$ for k small $(d=20)$

FRACTAL APPROXIMATION

Polynomial perturbation: For any $\varepsilon > 0$, there exists

$$z^d + t(a_{d-2}z^{d-2} + a_{d-3}z^{d-3} + \cdots + a_0), \quad t \in (-\varepsilon_0, \varepsilon_0)$$

and $\varepsilon_0 > 0$ s.t. the Julia set J_t is a k(t)-quasicircle with

$$\text{H.dim}\, J_t \geq 1 + (\Sigma^2 - \varepsilon)k(t)^2, \quad t \in (-\varepsilon_0, \varepsilon_0)$$

Corollary $$\Sigma^2 \leqslant \liminf_{k \to 0} \frac{D(k) - 1}{k^2} \leqslant 1$$

periodize & truncate

$$\mu = \mu_0 + \mu_0^* + \mu_0^{**} + \ldots$$

$\mathcal{C}\mu_0$ has finitely many terms

FUCHSIAN CASE

McMullen

$X_t = \mathbb{D}/G_t \subset \mathcal{T}_g$

Bers' simultaneous uniformization

$\Gamma_t = [G_0, G_t]$

quasifuchsian group

$$\sigma^2(v') = 2\frac{d^2}{dt^2}\Big|_{t=0} \dim(\Lambda(\Gamma_t)) = \frac{2}{3}\,\frac{\|\dot{X}_0\|^2_{WP}}{\mathrm{area}(X_0)} \leqslant \frac{2}{3}\|\dot{X}_0\|^2_T$$

no fractal approximation

$$\|\phi\|^2_T = \left(\int_{X_0} \frac{|\phi|}{\rho^2}\rho^2\right)^2 \leqslant \int_{X_0}\rho^2 \cdot \int_{X_0}\frac{|\phi|^2}{\rho^4}\rho^2 = \mathrm{area}(X_0)\|\phi\|^2_{WP}$$

pairing $\quad \langle \phi, \mu\rangle = \int_X \phi dz^2 \cdot \mu \frac{\overline{dz}}{dz}$

cf. Håkan Hedenmalm's talk