Research Statement

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My research focuses on problems in functional analysis that come from mathematical physics. In particular, I have worked on proving sharp entropy–energy inequalities on symplectic Riemannian manifolds with applications to the coherent state transforms and coherent state methods in physical problems.

For a probability density ρ on a symplectic Riemannian manifold $\mathcal{M}$, that is, a non-negative measurable function on $\mathcal{M}$ with $\int_{\mathcal{M}} \rho d\mathcal{M} = 1$, the **entropy** is defined as:

$$S(\rho) = -\int_{\mathcal{M}} \rho \ln \rho \ \mathrm{d}\mathcal{M}, \tag{1}$$

where $\mathrm{d}\mathcal{M}$ is the volume element on $\mathcal{M}$. Thus defined, the entropy of a density ρ can be thought of as a measure of its "concentration". If some part of the mass of ρ is very nearly concentrated in a multiple of a Dirac mass, then $S(\rho)$ can be very negative. I am mainly interested in the case in which $\mathcal{M}$ is the phase space of some classical system. In that case, one refers to ρ as a *classical density*, and $S(\rho)$ as its *classical entropy.*

The uncertainty principle limits the extent of possible concentration in phase space: for instance, it prevents both the momentum variables p and the configuration variables q from taking on well-defined values at the same time. Indeed, a quantum mechanical density ρ^Q is a non-negative operator on the Hilbert space $\mathcal{H}$ (defined to be the state space of the quantum system), having unit trace. Then the quantum entropy (or von Neumann entropy) of ρ^Q is defined by:

$$S^Q(\rho^Q) = -\mathrm{Tr}\ \rho^Q \ln \rho^Q \ . \tag{2}$$

Since all of the eigenvalues of ρ^Q lie in the interval $[0, 1]$, it is clear that

$$S^Q(\rho^Q) \geq 0 \ . \tag{3}$$

As Wehrl [Weh] emphasized, when one considers a quantum system and its corresponding classical analogue, not all of the classical probability densities on the phase space $\mathcal{M}$ can correspond to physical densities for the quantum system and one might expect a lower bound on the classical entropy of those probability densities that do correspond to actual quantum states.

There is a natural way to make the correspondence between quantum states and classical probability densities on phase space, which goes back to Schrödinger. It is based on the **coherent state transform**, which is an isometry $\mathcal{L}$ from the quantum state space $\mathcal{H}$ *into* $L^2(\mathcal{M})$, the Hilbert space of square integrable functions on the classical phase space $\mathcal{M}$. Since it is an isometry, if ψ is any unit vector in $\mathcal{H}$,

$$\rho_\psi = |\mathcal{L}\psi|^2$$

is a probability density on $\mathcal{M}$. Wehrl proposed defining the classical entropy of a quantum state ψ in this way (note the the corresponding density matrix has rank one, and hence the von Neumann entropy would be zero, for a "pure state"). The **Wehrl entropy** is defined in terms of the coherent states for the quantum system and is bounded below by the quantum entropy. It has several physically desirable features such as monotonicity, strong subadditivity, and of course, positivity. Wehrl identified the class of probability densities arising through the coherent state transform as the class of *quantum mechanically significant* probability densities on $\mathcal{M}$, and conjectured that corresponding to (3), there should be a lower bound on $S(|\mathcal{L}\psi|^2)$ as ψ ranges over the unit sphere in $\mathcal{H}$.

Specifically, when $\mathcal{H}$ is $L^2(\mathbb{R}, \mathrm{d}x)$, so that the classical phase space is $\mathbb{R}^2$ with its usual symplectic and Riemannian structure, Wehrl conjectured and Lieb [Lie] proved that the lower bound on $S(|\mathcal{L}\psi|^2)$ is attained when ψ is a minimal uncertainty state $\psi_{\min}$, also known as a *Glauber coherent state*. That is:

$$\inf_{\|\psi\|_{\mathcal{H}}=1} S(|\mathcal{L}\psi|^2) = S(|\mathcal{L}\psi|_0^2) \ . \tag{4}$$

Natural analogues of the Wehrl conjecture can be formulated for other state spaces and other coherent state transforms. In fact, the Wehrl conjecture was first generalized to the $SU(2)$ coherent states by Lieb. The irreducible unitary representations of $SU(2)$ are indexed by a half-integer j, which is the *quantum number* in this context. Lieb conjectured that the Wehrl entropy is minimized by coherent states generated from the least weight vectors in the various unitary representations of $SU(2)$. Thus there is a conjectured lower bound for each value of the quantum number j. Although a proof of the full conjecture (i.e. for all values of j) is still awaited, Schupp [Sch] proved it for $j = 1$ and $j = 3/2$. Later Bodmann [Bod] deduced a lower bound for the Wehrl entropy of $SU(2)$ coherent states, for which the high spin asymptotics coincided with the conjectured estimate up to, but not including, terms of the first and higher orders in the inverse of spin quantum number j.

Bodmann did this by proving a sharp L^p bound on the range of the coherent state transform. This led to a proof of an analogue of Lieb's conjecture for certain *Renyi entropies*: for any $p > 1$ and any classical density ρ, define its Renyi entropy as

$$S_p(\rho) = \frac{1}{p-1} \ln\left(\|\rho\|_p\right) \ , \tag{5}$$

where $\|\rho\|_p$ is the L^p norm of ρ. Then it is easy to see that

$$\lim_{p\to 1} S_p(\rho) = S(\rho) \ .$$

Bodmann derived his bound on Renyi entropies from a Sobolev type inequality and a Fisher information identity, which is another type of concentration bound on the range of the coherent state transform. The **Fisher information** $I(\rho)$ of a probability density ρ on $\mathcal{M}$ is defined by:

$$I(\rho) = \int_{\mathcal{M}} |\nabla \ln \rho|^2 \rho \ \mathrm{d}\mathcal{M} = 4 \int_{\mathcal{M}} |\nabla \sqrt{\rho}|^2 \ \mathrm{d}\mathcal{M} \ .$$

For the Glauber coherent state transform, Carlen [Car] proved that all classical densities on $\mathbb{R}^2$ arising through the coherent state transform had the *same* finite value of the Fisher information. He then used that together with the logarithmic Sobolev inequality (cf. [Gro]) to give a new proof of Wehrl's conjecture, and to show that the lower bound in (4) is attained only for Glauber coherent states. Bodmann proved an analogue of Carlen's result for Fisher information, and used it, together with a sharp Sobolev inequality (instead of the sharp logarithmic Sobolev inequality) to obtain his Renyi information bounds.

The problem of investigating an analogue of the Lieb-Wehrl conjecture for the group $SU(1,1)$ was suggested to me by my advisor Professor Eric Carlen. The representations of $SU(1,1)$ belonging to a discrete series, are labeled by a half-integer k, the relevant *quantum number* in this context. While the classical phase space for SU(2) is the sphere S^2, for $SU(1,1)$ the classical phase space is the hyperbolic plane H^2, or equivalently, the unit disk . It is natural to conjecture that, here too, the coherent states generated by the least-weight vector of the representation provide a lower bound on the entropy, as in Lieb's conjecture for $SU(2)$. In a paper titled "Optimal Concentration for $SU(1,1)$ Coherent State Transforms and An Analogue of the Lieb-Wehrl Conjecture for $SU(1,1)$" (submitted to *Communications in Mathematical Physics*), I proved that this is indeed asymptotically true, in the semi-classical limit. To arrive at this result, I proved a number of theorems concerning analysis in H^2, that are of independent interest. Specifically, a new sharp Sobolev inequality as well as a sharpened entropy–energy inequality in H^2 have been derived. The **Sobolev inequality** is

$$\|f\|_q^q + \frac{4}{kq(kq-2)} \int |\nabla |f|^{q/2}|^2 d\nu \geq \left(\frac{2k-1}{kq-1}\right) \left(\frac{kp-1}{2k-1}\right)^{q/p} \left(\frac{kq-1}{kq-2}\right) \|f\|_p^q,$$

where $p = q + 1/k$, $q \geq 2$, $kq > 2$ and the measure $d\nu$ is a constant times the standard measure on H^2, obtained from the Poincare metric; all of the cases of equality have been determined. To prove the sharpness of the Sobolev inequality mentioned above, I needed to prove and use a uniqueness result for radial solutions of a semi-linear Poisson equation on the hyperbolic plane. The nature of this equation on H^2 is substantially different from that of similar equations which have been investigated in the past. The methods developed in my paper may well be useful for other uniqueness problems.

I then proved the following Fisher information identity:

$$\int |\nabla |\mathcal{L}\psi|^{q/2}|^2 d\nu = \frac{1}{4} kq \int |\mathcal{L}\psi|^q d\nu,$$

where q is a positive number such that $kq > 2$. As mentioned above, an identity like this was first proved by Carlen [Car] for coherent state transforms associated with the Glauber coherent states.

The sharp Sobolev inequality and the Fisher information identity were used to derive an L^p norm estimate, which in turn led to a lower bound for the Wehrl entropy of coherent state transforms via a convexity argument, the result being:

$$S(|\mathcal{L}\psi(\zeta)|^2) \geq 2k \ln\left(1 + \frac{1}{2k-1}\right). \tag{6}$$

The conjectured bound, on the other hand, is:

$$S(|\mathcal{L}\psi(\zeta)|^2) \geq \frac{2k}{2k-1}. \tag{7}$$

It is seen that for high values (this gives us the semi-classical limit) of the quantum number k, the lower bound (6) coincides with the analogue (7) of the Lieb-Wehrl conjecture, up to but not including terms of first and higher orders in k^{-1}.

The methods used to bound the entropy also served to produce a new, sharpened entropy–energy inequality for functions on H^2. An entropy–energy inequality is an inequality of the form

$$-S(\rho) \leq \Phi_M(I(\rho)) \ , \tag{8}$$

for some function Φ. Since the Fisher information can be expressed in terms of an energy integral as shown before, the entropy–energy terminology is natural. For a given Riemannian manifold

M, the entropy–energy problem is to determine the least function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ for which (8) is true. There has been a lot of investigation of entropy–energy inequalities for various Riemannian manifolds (see [Bec], [Heb], [Rot] for example). Though there has been significant progress, many questions are still open.

In the case of H^2 , Beckner proved [Bec] that the entropy–energy inequality for H^2 holds with the same Φ as in $\mathbb{R}^2$. That is,

$$\Phi_{H^2}(t) \leq \Phi_{\mathbb{R}^2}(t),$$

for all $t \geq 0$. This result is asymptotically sharp in the sense that

$$\lim_{t \to 0} \frac{\Phi_{H^2}(t)}{\Phi_{\mathbb{R}^2}(t)} = 1 \ .$$

However, the inequality is actually strict, and significantly so, for large t. I proved in my paper an improved bound, $\Phi_{H^2}(t) < \Phi_{\mathbb{R}^2}(t)$ and gave sharpened estimates on $\Phi_{H^2}(t)$.

It is interesting to observe how sharp bounds on the Fisher information of coherent state transforms can lead to sharp Sobolev type inequalities in a larger function space, which can then be used to derive entropy–energy inequalities on various symplectic Riemannian manifolds that are classical phase spaces, e.g., the sphere and the hyperbolic plane. These manifolds are determined by the groups for which we construct the coherent states. It seems natural to ask: for which other groups having unitary irreducible representations in spaces of holomorphic functions, can one obtain bounds on the Fisher information of the coherent state transforms and formulate analogues of the Lieb-Wehrl conjecture? This is one of the issues my research currently focuses on.

Another area that interests me is that of applications of coherent state methods to physical problems. The coherent state methods have proved quite useful in a number of physical problems [Per]. In most such cases, the coherent state methods reduce the quantum problems to their classical analogues. A few examples of such problems are: parametric excitation of a quantum oscillator (this can be solved using the $SU(1,1)$ coherent state method), spin motion in a variable magnetic field ($SU(2)$ coherent state method is very useful here), Boson pair production in a variable homogeneous external field etc. [Per]. One thus expects that the entropy and Fisher information bounds mentioned before, would prove useful in a variety of physical problems and this issue forms a significant part of my current research.

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