NONSOLVABILITY OF THE ASYMPTOTIC DIRICHLET PROBLEM FOR THE p-LAPLACIAN ON CARTAN-HADAMARD MANIFOLDS

ILKKA HOLOPAINEN

Abstract. We construct, by modifying Borbély's example, a 3-dimensional Cartan-Hadamard manifold M, with sectional curvatures ≤ -1, such that the asymptotic Dirichlet problem for the p-Laplacian is not solvable for any $p > 1$.

1. Introduction

In [13] Greene and Wu conjectured that an n-dimensional Cartan-Hadamard manifold M always carries non-constant bounded harmonic functions if the sectional curvatures of M have an upper bound

$$\mathrm{Sect}_x(P) \le \frac{-C}{\rho(x)^2}$$

outside a compact set for some constant $C > 0$, where $\rho = d(\cdot, o)$ is the distance function to a fixed point $o \in M$ and P is any 2-dimensional subspace of T_xM. Recall that a Cartan-Hadamard manifold is a complete, connected and simply connected Riemannian n-manifold, $n \ge 2$, of non-positive sectional curvature. By the Cartan-Hadamard theorem, the exponential map $\exp_o \colon T_oM \to M$ is a diffeomorphism for every point $o \in M$. In particular, M is diffeomorphic to $\mathbb{R}^n$. It is well-known that M can be compactified by adding a natural geometric boundary, called the *sphere at infinity* (or the *boundary at infinity*) and denoted by $M(\infty)$, so that the resulting space $\bar{M} = M \cup M(\infty)$ equipped with the *cone topology* will be homeomorphic to a closed Euclidean ball; see [12].

The conjecture of Greene and Wu is still open for dimensions $n \ge 3$ whereas it has been verified affirmatively in the two-dimensional case by Hsu and Kendall [19]. The conjecture can be approached by studying the so-called *Dirichlet problem at infinity* (or the *asymptotic Dirichlet problem*) on a Cartan-Hadamard manifold M. Here one asks whether every continuous function on $M(\infty)$ has a (unique) harmonic extension to M. In general, the answer is no since the simplest Cartan-Hadamard manifold $\mathbb{R}^n$ admits no positive harmonic functions other than constants.

The Dirichlet problem at infinity was solved affirmatively by Choi [11] under assumptions that sectional curvatures satisfy $\mathrm{Sect} \le -a^2 < 0$ and any two points in $M(\infty)$ can be separated by convex neighborhoods. Such appropriate convex sets were constructed by Anderson [5] for manifolds of pinched sectional curvature $-b^2 \le \mathrm{Sect} \le -a^2 < 0$. Independently, Sullivan [24] solved the Dirichlet problem at

---

2000 *Mathematics Subject Classification.* 58J32, 53C21, 31C45.

Key words and phrases. p-harmonic function, Dirichlet problem, Cartan-Hadamard manifold.

Work partially supported by the Academy of Finland, project 123633, and the ESF Network HCAA.

infinity under the same pinched curvature assumption by using probabilistic arguments. In [6], Anderson and Schoen presented a simple and direct solution to the Dirichlet problem again in the case of pinched negative curvature. Major contributions to the Dirichlet problem were given by Ancona in a series of papers [1], [2], [3], and [4]. In particular, he was able to replace the curvature lower bound by a bounded geometry assumption that each ball up to a fixed radius is L-bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$ for some fixed $L \geq 1$; see [1]. It is important to notice that the conjecture of Greene and Wu can not be solved merely by studying the Dirichlet problem at infinity. Indeed, in [4] Ancona constructed a 3-dimensional Cartan-Hadamard manifold with sectional curvatures bounded from above by -1 where the asymptotic Dirichlet problem is not solvable. He did not study whether his example carries non-constant bounded harmonic functions. Another example of a (3-dimensional) Cartan-Hadamard manifold, with sectional curvatures ≤ -1, on which the asymptotic Dirichlet problem is not solvable was constructed by Borbély [9]. Furthermore, he showed that his example admits non-constant bounded harmonic functions. Recently, Arnaudon, Thalmaier, and Ulsamer [7] characterized completely bounded harmonic functions on Borbély's example by a careful study of the asymptotic behavior of Brownian motion. Although there are several papers where assumptions on curvature have been weakened by allowing curvature decay (or growth) at certain rate; see e.g. [8], [18], [20], [21], and [22], the role of the curvature lower bound in the asymptotic Dirichlet problem is still far from being understood. To the best of our knowledge, the most general curvature bounds under which the asymptotic Dirichlet problem is solvable are given in the following theorems by Hsu and in Theorems 1.3 and 1.4 below.

Theorem 1.1. [18, Theorem 1.1] *Let M be a Cartan-Hadamard manifold. Suppose that there exist a positive constant a and a positive and non-increasing function h with $\int_0^\infty th(t)\,dt < \infty$ such that*

$$-h\big(\rho(x)\big)^2 e^{2a\rho(x)} \leq \operatorname{Ric}_x \quad \textit{and} \quad \operatorname{Sect} \leq -a^2.$$

Then the Dirichlet problem at infinity for M is solvable.

Theorem 1.2. [18, Theorem 1.2] *Let M be a Cartan-Hadamard manifold. Suppose that there exist positive constants r_0, $\alpha > 2$, and $\beta < \alpha - 2$ such that*

$$-\rho(x)^{2\beta} \leq \operatorname{Ric}_x \quad \textit{and} \quad \operatorname{Sect}_x \leq -\frac{\alpha(\alpha-1)}{\rho(x)^2}$$

for all $x \in M$, with $\rho(x) \geq r_0$. Then the Dirichlet problem at infinity for M is solvable.

The proofs of these results rely on probabilistic arguments which involve proving the angular convergence of transient Brownian motion.

The asymptotic Dirichlet problem has been studied also in a more general context of p-harmonic and $\mathcal{A}$-harmonic functions. In the case of the p-Laplacian, $1 < p < \infty$, Pansu [23] showed the existence of nonconstant bounded p-harmonic functions with finite p-energy on Cartan-Hadamard manifolds of pinched curvature $-b^2 \leq \operatorname{Sect} \leq -a^2$ for $p > (n-1)b/a$. The Dirichlet problem at infinity for the p-Laplacian was solved in [15] on Cartan-Hadamard manifolds of pinched negative sectional curvature by modifying the direct approach of Anderson and Schoen [6]. Recall that a *p-harmonic function* u in an open subset U of a Riemannian manifold M is a continuous (weak) solution of the p-Laplace equation

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)=0.$$

More precisely, u belongs to the local Sobolev space $W^{1,p}_{\text{loc}}(U)$ and

$$\int_U \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \, dm = 0 \tag{1.1}$$

for every $\varphi \in C_0^\infty(U)$. Note that (1.1) is the Euler-Lagrange equation for the (p-energy) variational integral

$$\int_U |\nabla u|^p \, dm. \tag{1.2}$$

Furthermore, a function $u \in W^{1,p}_{\text{loc}}(U)$ is called a *p-subsolution* in U if

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)\ge 0$$

weakly in U, that is

$$\int_U \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \, dm \le 0$$

for every non-negative $\varphi \in C_0^\infty(U)$. Similarly, a function $v \in W^{1,p}_{\text{loc}}(U)$ is called a *p-supersolution* in U if $-v$ is a p-subsolution in U. In [17] the author and Vähäkangas studied the asymptotic Dirichlet problem for the p-Laplacian and the p-regularity of a point x_0 at infinity on a Cartan-Hadamard manifold M under a curvature assumption

$$-b\big(\rho(x)\big)^2 \le \operatorname{Sect}_x \le -a\big(\rho(x)\big)^2$$

in $U\cap M$, where U is a neighborhood of $x_0 \in M(\infty)$. Here $a, b\colon [0,\infty)\to[0,\infty)$, $b \ge a$, are smooth functions subject to certain growth conditions which are irrelevant for the current paper. For the purpose of this paper we just single out the following two special cases of functions a and b.

Theorem 1.3. [17, Corollary 3.22] *Let $\phi > 1$ and $\varepsilon > 0$. Let $x_0 \in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology. Suppose that*

$$-\rho(x)^{2\phi-4-\varepsilon} \le \operatorname{Sect}_x \le -\frac{\phi(\phi-1)}{\rho(x)^2} \tag{1.3}$$

for every $x \in U\cap M$. Then x_0 is a p-regular point at infinity for every $p \in \big(1, 1+(n-1)\phi\big)$.

Theorem 1.4. [17, Corollary 3.23] *Let $k > 0$ and $\varepsilon > 0$. Let $x_0 \in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology. Suppose that*

$$-\rho(x)^{-2-\varepsilon}e^{2k\rho(x)} \le \operatorname{Sect}_x \le -k^2 \tag{1.4}$$

for every $x \in U\cap M$. Then x_0 is a p-regular point at infinity for every $p \in (1,\infty)$.

Roughly speaking, the p-regularity of $x_0 \in M(\infty)$ means that, at the point x_0, the Dirichlet problem for the p-Laplacian is solvable with continuous boundary data; see [17] and [26] for the details. In particular, the Dirichlet problem at infinity for the p-Laplacian is solvable if every point $x_0 \in M(\infty)$ is p-regular. It is important to notice that the probabilistic methods based on the asymptotic behavior of Brownian motion are no longer available in the non-linear setting of the p-Laplacian ($p \ne 2$). The method in [17] is a refinement of the arguments used in [6] and [15]. The idea is to take a continuous function h on the sphere at infinity and extend it radially to $\bar{M} \setminus \{o\}$. The extended function is then smoothened by a convolution type procedure. By a small perturbation the smoothened function yields a p-supersolution that can be used as a barrier function if the original function h is chosen properly.

In [26] Vähäkangas generalized the method and results due to Cheng [10] and showed that $x_0 \in M(\infty)$ is p-regular if it has a neighborhood V in the cone topology such that the radial sectional curvatures in $V \cap M$ satisfy a pointwise pinching condition

$$|\mathrm{Sect}_x(P)| \le C|\mathrm{Sect}_x(P')|$$

for some constant C and have an upper bound

$$\mathrm{Sect}_x(P) \le -\frac{\phi(\phi-1)}{\rho^2(x)}$$

for some constant $\phi > 1$ with $p < 1 + \phi(n-1)$. Above P and P' are any 2-dimensional subspaces of T_xM containing the (radial) vector $\nabla\rho(x)$. It is worth observing that no curvature lower bounds are needed here. In fact, Vähäkangas considered even a more general case of $\mathcal{A}$-harmonic functions (of type p), i.e. continuous weak solutions to the equation

$$-\operatorname{div}\mathcal{A}(\nabla u) = 0,$$

where $V \mapsto \mathcal{A}(V)$ is a measurable vector field whenever V is and $\langle \mathcal{A}(V), V\rangle \approx |V|^p$. We refer to the book [14] by Heinonen, Kilpeläinen, and Martio for the non-linear potential theory associated with p-harmonic and $\mathcal{A}$-harmonic functions. Recently, Vähäkangas generalized theorems 1.3 and 1.4 to cover the case of $\mathcal{A}$-harmonic functions as well; see [25, Corollary 3.7, Corollary 3.8, Remark 3.9]. It is worth noting that the method used in the proofs of 1.3 and 1.4 is available for p-harmonic functions only since it involves computations of the p-Laplacian of (smooth) functions. To obtain the corresponding results for $\mathcal{A}$-harmonic functions, Vähäkangas uses PDE-type methods like Caccioppoli inequalities.

Finally, in [16] the author together with Lang and Vähäkangas studied the Dirichlet problem at infinity for p-harmonic functions in a very general setting of Gromov hyperbolic metric measure spaces. The metric spaces studied in [16] do not have, in general, a manifold structure not to mention a smooth structure. Therefore, p-harmonic functions can not, in general, be defined as solutions of an equation like (1.1) but rather as minimizers of a variational integral such as (1.2).

The purpose of this paper is twofold. The primary motivation is to study whether there are Cartan-Hadamard manifolds with sectional curvatures ≤ -1 on which the Dirichlet problem at infinity is not solvable for the p-Laplacian for *any* $p \in (1,\infty)$. Another goal is to raise questions on the role of curvature lower bounds in the asymptotic Dirichlet problem. In particular, since there are three different kind of proofs for the solvability of the asymptotic Dirichlet problem under essentially the same curvature bounds (1.3) and (1.4), it would be interesting to know whether these bounds are sharp. In this paper we are able to settle the first problem.

Theorem 1.5. *There exists a 3-dimensional Cartan-Hadamard manifold M with sectional curvatures ≤ -1 such that, for any $p > 1$, the asymptotic Dirichlet problem for the p-Laplacian is not solvable, but there are non-constant bounded p-harmonic functions on M.*

The question about the possible sharpness of the curvature bounds in (1.3) and (1.4) remains open. In fact, there is a large gap between the curvature lower bound in the example above and the curvature lower bound (1.4) in Theorem 1.4. We show at the end of the paper that

$$\operatorname{Sect}_{x_i}(P_i) \le -\exp\left(\tfrac{1}{2}\exp(2\rho(x_i))\right)$$

for a sequence of points $x_i \in M$ tending to infinity and for a sequence of 2-planes $P_i \in T_{x_i}M$.

Our paper owes much to the paper [9] by Borbély. Indeed, the construction of the manifold M and the idea for the proof of the existence of non-trivial bounded p-harmonic functions on M are essentially due to him. However, there are some differences. First of all, computations and estimates of p-Laplacians in Sections 4 and 8 are more involved than those for the Laplacian in [9]. Secondly, we want to fix concretely the "initial condition" for the function g that determines the Riemannian metric; see (3.2) and (6.12). Finally, since we look for an example of a Cartan-Hadamard manifold M, with $\operatorname{Sect} \le -1$, where the asymptotic Dirichlet problem is not solvable for any $1 < p < \infty$, the construction of functions q_a that yield the p-subsolutions φ_a (cf. Theorem 2.3) is slightly more complicated than in [9].

2. Main results

Our main result, Theorem 1.5, follows from the following since the condition (a) below clearly implies that a non-constant bounded p-harmonic function can not have a continuous extension to $x_0 \in M(\infty)$.

Theorem 2.1. *There exists a 3-dimensional Cartan-Hadamard manifold M with sectional curvatures ≤ -1 and a point $x_0 \in M(\infty)$ such that*

(a) *for all bounded p-harmonic functions u in M and for all (cone) neighborhoods U of x_0,*

$$\inf_M u = \inf_{U\cap M} u, \quad \sup_M u = \sup_{U\cap M} u, \quad \text{and}$$

(b) *there are non-constant bounded p-harmonic functions on M.*

Theorem 2.2. *There exists a 3-dimensional Cartan-Hadamard manifold M with sectional curvatures ≤ -1, a point $x_0 \in M(\infty)$, and a family of functions u_a, $a \in \mathbb{R}$, in $\bar{M}$ that are p-harmonic on M, $0 \le u_a \le 1$, and satisfy*

(a) $u_a | M(\infty) = \chi_{\{x_0\}}$,

(b)

$$\lim_{y\to x} u_a(y) = 0$$

for all $a \in \mathbb{R}$ and $x \in M(\infty) \setminus \{x_0\}$, and

(c)

$$\lim_{a\to-\infty} u_a(x) = 1$$

for all $x \in M$.

Theorem 2.1 and Theorem 2.2 will follow from the following theorem.

Theorem 2.3. *There exists a 3-dimensional Cartan-Hadamard manifold M with sectional curvatures ≤ -1, a point $x_0 \in M(\infty)$, and families of functions φ_a and ψ_a, $a \in \mathbb{R}$, in $\bar{M}$ such that φ_a is a continuous p-subsolution in M, ψ_a is a continuous p-supersolution in M, $0 \le \varphi_a \le \psi_a \le 1$, and that*

(a) $\varphi_a|M(\infty) = \psi_a|M(\infty) = \chi_{\{x_0\}}$,

(b)
$$\lim_{y\to x} \psi_a(y) = 0$$
for all $a \in \mathbb{R}$ and $x \in M(\infty) \setminus \{x_0\}$, and

(c)
$$\lim_{a\to -\infty} \varphi_a(x) = 1$$
for all $x \in M$.

Since $0 \le \varphi_a \le \psi_a \le 1$ we also have

(b')
$$\lim_{y\to x} \varphi_a(y) = 0$$
for all $a \in \mathbb{R}$ and $x \in M(\infty) \setminus \{x_0\}$, and

(c')
$$\lim_{a\to -\infty} \psi_a(x) = 1$$
for all $x \in M$.

Proof of Theorem 2.2 assuming Theorem 2.3. Let M, $x_0 \in M(\infty)$, and the families $\{\varphi_a\}$ and $\{\psi_a\}$ be as in Theorem 2.3. Let $\Omega_i \Subset M$, $i \in \mathbb{N}$, be an exhaustion of M by p-regular open sets. For each $a \in \mathbb{R}$ and $i \in \mathbb{N}$, let $u_{a,i} \in C(M)$ be the unique function that is p-harmonic in Ω_i and coincides with φ_a in $\bar{M} \setminus \Omega_i$. By the usual comparison principle [14, Lemma 3.18], we have $\varphi_a \le u_{a,i} \le \psi_a$ in $\bar{M}$. Thus the sequence $(u_{a,i})$ is uniformly bounded and by a standard reasoning involving Hölder-continuity estimates and the Ascoli-Arzela theorem we obtain a subsequence of $(u_{a,i})$ that converges locally uniformly to a function u_a which is p-harmonic in M satisfies $\varphi_a \le u_a \le \psi_a$ in $\bar{M}$, and hence conditions (a)-(c) in Theorem 2.2. □

Proof of Theorem 2.1 assuming Theorem 2.3. Let M, $x_0 \in M(\infty)$, and the families $\{\varphi_a\}$ and $\{\psi_a\}$ be as in Theorem 2.3. Condition (b) follows from Theorem 2.2. To prove (a), suppose that h is a bounded p-harmonic function on M, U is a cone neighborhood of x_0, and let
$$b = \inf_M h, \quad \text{and} \quad B = \inf_{U\cap M} h.$$
Then $b \le B$ and we claim that $b = B$. For each $a \in \mathbb{R}$ an auxiliary continuous p-subsolution
$$f_a = b + (B-b)\varphi_a$$
satisfies, for all $x \in M(\infty) \setminus \{x_0\}$,
$$\begin{aligned}\liminf_{\substack{y\to x\\ y\in M}} \big(h(y) - f_a(y)\big) &= \liminf_{\substack{y\to x\\ y\in M}} \big(h(y) - b + (b-B)\varphi_a(y)\big)\\ &\ge \liminf_{\substack{y\to x\\ y\in M}} \big(h(y) - b\big) + \liminf_{\substack{y\to x\\ y\in M}} (b-B)\varphi_a(y) \ge 0.\end{aligned}$$

Furthermore,

$$\begin{aligned}\liminf_{\substack{y\to x_0\\ y\in M}}\big(h(y)-f_a(y)\big) &= \liminf_{\substack{y\to x_0\\ y\in M}}\big(h(y)-b-(B-b)\varphi_a(y)\big)\\ &= \liminf_{\substack{y\to x\\ y\in M}}\Big((B-b)\big(1-\varphi_a(y)\big)+h(y)-B\Big)\\ &\ge \liminf_{\substack{y\to x_0\\ y\in M}}\big(h(y)-B\big)\ge 0.\end{aligned}$$

Hence

$$\liminf_{\substack{y\to x\\ y\in M}} h(y) \ge \limsup_{\substack{y\to x\\ y\in M}} f_a(y) \tag{2.1}$$

for all $x\in M(\infty)$. It follows from the comparison principle that $h\ge f_a$ in M for all $a\in\mathbb{R}$. To be precise, suppose on the contrary that $h(y)<f_a(y)-\varepsilon$ for some $y\in M$ and $\varepsilon>0$. Let A be the y-component of the set $\{x\in M\colon h(x)<f_a(x)-\varepsilon\}$. Then A is an open set with a compact closure $\bar{A}\subset M$ by (2.1) and continuity of $h-f_a$. On the other hand, $h=f_a-\varepsilon$ on ∂A, and therefore $h\ge f_a-\varepsilon$ in A by the comparison principle leading to a contradiction. Since $\lim_{a\to-\infty}\varphi_a(x)=1$ for all $x\in M$, we obtain

$$h(x)\ge \lim_{a\to-\infty} f_a(x)=B$$

for all $x\in M$. Hence $b\ge B$, and so $b=B$.

To complete the proof, we just apply the above to the bounded p-harmonic function $-h$ and obtain

$$\sup_M h = -\inf_M(-h) = -\inf_{U\cap M}(-h)=\sup_{U\cap M} h.$$

□

Remark 2.4. As is seen in the proof above, only the family $\{\varphi_a\}$ is needed in order to get the non-solvability of the asymptotic Dirichlet problem.

3. Construction of M: main idea

The construction of the Riemannian manifold M is up to some minor modifications (mostly in notation) essentially due to Borbély [9]; see also [4], and [7]. Thus we first express the Riemannian metric of the hyperbolic 3-space $\mathbb{H}^3$ by using "cylindrical" or, more precisely, Fermi coordinates along a geodesic and then we modify the metric in certain direction. The modification of the metric will be done in such a way that neither the sphere at infinity nor the cone topology changes. To start the construction, let us consider the upper half space model

$$\mathbb{H}^3=\{(x^1,x^2,x^3)\in\mathbb{R}^3\colon x^3>0\}$$

for $\mathbb{H}^3$ equipped with the hyperbolic metric $ds^2_{\mathbb{H}^3}$ of constant sectional curvature -1. The sphere at infinity, $\mathbb{H}^3(\infty)$, can be realized as a one-point compactification of the x^1x^2- plane or, more concretely, as the union of the x^1x^2-plane and the "common endpoint $(x^1,x^2,+\infty)$" of all vertical geodesics. We fix a point at infinity $x_0\in\mathbb{H}^3(\infty)$ and a unit speed geodesic L terminating at x_0. We will denote by L also the image $L(\mathbb{R})$. We can assume without loss of generality that L is the positive x^3-axis $\{(0,0,x^3)\colon x^3>0\}$, $L(0)=(0,0,1)$, and that $x_0=L(+\infty)$ corresponds to the point $(0,0,0)$. Next we introduce Fermi coordinates (s,r,ϑ) along L. For this purpose, let $x\in\mathbb{H}^3\setminus L$ be an arbitrary point. Then there exists a unique point $L(s)$

on L closest to x. This determines uniquely the first Fermi coordinate s of x. The second coordinate r is defined as the distance $r = d(x, L(s)) = \operatorname{dist}(x, L)$. Finally, the third Fermi coordinate ϑ of x is the unique angle $\vartheta \in [0, 2\pi[$ obtained from the polar coordinate representation $x^1 = t\cos\vartheta,\ x^2 = t\sin\vartheta$ of $x = (x^1, x^2, x^3)$. For points $x \in L$, the third Fermi coordinate ϑ is not defined, whereas $r = 0$ and s is determined by $L(s) = x$. Nevertheless, we equip $L(s)$ with Fermi coordinates $(s, 0, *)$. On $\mathbb{H}^3 \setminus L$, the vector fields

$$S = \frac{\partial}{\partial s}, \quad R = \frac{\partial}{\partial r}, \quad \text{and} \quad \Theta = \frac{\partial}{\partial \vartheta}$$

form a frame, with $\{ds, dr, d\vartheta\}$ as a coframe. Furthermore, S, R, and Θ are commuting as coordinate vector fields, i.e. their Lie brackets vanish:

$$[S, R] = [S, \Theta] = [R, \Theta] = 0.$$

To write the (original) hyperbolic metric of $\mathbb{H}^3$ in this frame, we first observe that S, R, and Θ are orthogonal with respect to the Euclidean metric, and therefore also with respect to the hyperbolic metric. Hence we need to find just their norms. It is straightforward to observe that a surface $\{(s, r, \vartheta) \colon r = \text{constant}\}$ is an Euclidean cone around L with the tip at $x_0 = L(+\infty)$ and with the opening angle

$$\alpha = \alpha(r) = 2\arctan(e^r) - \frac{\pi}{2}.$$

Conversely,

$$r(\alpha) = \log\Big(\tan\big(\frac{\alpha}{2} + \frac{\pi}{4}\big)\Big).$$

Using these we obtain the following formulae for the hyperbolic norms

$$\begin{aligned} \|R\|_{\mathbb{H}^3} &= 1, \\ \|S\|_{\mathbb{H}^3} &= \cosh r, \\ \|\Theta\|_{\mathbb{H}^3} &= \sinh r, \end{aligned}$$

and therefore the hyperbolic metric of $\mathbb{H}^3$ in Fermi coordinates is given by

$$ds^2_{\mathbb{H}^3} = dr^2 + \cosh^2 r\, ds^2 + \sinh^2 r\, d\vartheta^2.$$

The Riemannian manifold M is then obtained from $\mathbb{H}^3$ by modifying the metric in Θ-directions as

$$ds^2_M = dr^2 + \cosh^2 r\, ds^2 + g^2(s, r)\, d\vartheta^2, \tag{3.1}$$

where $g \colon \mathbb{R} \times [0, +\infty[\to \mathbb{R}$ is a C^∞-function which is positive in the complement of L, that is when $r > 0$,

$$\begin{aligned} g(s, 0) &= 0, \\ g_r(s, 0) &:= \frac{\partial g}{\partial r}(s, 0) = 1, \end{aligned}$$

and whose partial derivatives of even order with respect to r vanishes at $r = 0$, i.e.

$$g_r^{(2k)}(s, 0) = \frac{\partial^{2k} g}{(\partial r)^{2k}}(s, 0) = 0, \quad k = 1, 2, \ldots.$$

The function g will be constructed so that the sectional curvatures of M satisfy

$$\operatorname{Sect} \le -1$$

and that it is possible to construct families $\{\varphi_a\}$ and $\{\psi_a\}$ in Theorem 2.2. We look for a function g that is of the form

$$g(s,r)=\frac{1}{2}\sinh\bigl(\sinh 2\varrho(s,r)\bigr), \tag{3.2}$$

where ϱ is a C^∞-function, with $\varrho(s,r)=r$ for $0\le r\le 3$ and $\varrho(s,r)\ge r$ for all $r\ge 0$.

It is crucial to note that all geodesic rays of $\mathbb{H}^3$ starting at L will remain geodesic rays also in M, and therefore the sphere at infinity, $M(\infty)$, of M and the cone topology of $\bar M$ can be identified with those of $\mathbb{H}^3$. Hence we can equip points in $M(\infty)\setminus\{L(+\infty),L(-\infty)\}$ with Fermi coordinates in an obvious way. Given $y\in M(\infty)\setminus\{L(+\infty),L(-\infty)\}$ we first represent y as the unique geodesic ray γ_y (in the equivalence class y) emanating from $L(0)$. Next we identify y with the endpoint (x_y^1,x_y^2) of γ_y exactly as in the case of $\mathbb{H}^3$. Then the Fermi coordinates of y are $s_y=-\log\sqrt{(x_y^1)^2+(x_y^2)^2}$, $r_y=+\infty$, and ϑ_y given by the polar coordinate representation of (x_y^1,x_y^2). We also assign $(+\infty,0,*)$ and $(-\infty,0,*)$ for Fermi coordinates of $L(+\infty)$ and $L(-\infty)$, respectively.

4. p-Laplacian of φ_a

The construction of the family $\{\varphi_a\}$ follows the idea of Borbély. We consider a family of vector fields

$$X^{(a)}=R+q_aS,\quad a\in\mathbb{R},$$

on $M\setminus L$, where, for each $a\in\mathbb{R}$, $q_a\colon M\to\mathbb{R}$ is a C^∞ function depending only on the r-coordinate of a point $(s,r,\vartheta)\in M\setminus L$ and $q_a|L=0$. From now on we usually omit the parameter a and abbreviate $X=X^{(a)}$ and write $q(r)=q_a(r)=q_a(s,r,\vartheta)$. Since X does not have the Θ-component and

$$\int_0^{r_0} q(r)\,dr<\infty$$

for every r_0, all integral curves of X can be extended to L. More precisely, a point $\alpha(t)$ of an integral curve α of X starting at $(s_0,r_0,\vartheta_0)\in M\setminus L$, i.e. $\alpha(0)=(s_0,r_0,\vartheta_0)$, will hit L at the point $L(s_0+s_0')$ as

$$t\searrow s_0':=-\int_0^{r_0} q(r)\,dr\,.$$

Thus we may and from now on will talk about integral curves of X starting at a point of L even though X is not defined on X. Since X does not have the Θ-component, the (Fermi) ϑ-coordinate remains constant along integral curves of X. Furthermore, integrals curves of X starting at $L(s)$ are rotationally symmetric around L; each of them is obtained from another by a suitable rotation around L. Denote by $\gamma_{a,s}$ any integral curve of $X^{(a)}$ starting at $L(s)$. Let $S_s=S_s^{(a)}$ be the surface that is obtained by rotating any $\gamma_{a,s}$ around L. It is worth observing that the surfaces $S_s^{(a)}$ for fixed a are obtained from each other by a Euclidean dilation with respect to x_0 in our upper half space model of M since q_a is independent of s. Note also that the relation between the (Fermi) s-coordinate of a point $(s,r,\vartheta)\in S_{s_0}^{(a)}$ and s_0 is given by

$$s=s_0+\int_0^r q_a(t)\,dt. \tag{4.1}$$

The functions φ_a will be constructed so that the surfaces $S_s^{(a)}$ are the level sets of φ_a. Thus $\varphi_a|S_s^{(a)}$ has a constant value $f(s)=f^{(a)}(s)$ depending only on a and s. It is convenient to choose

$$f(s)=\max\bigl\{0,\tanh\bigl(\delta(s-a)\bigr)\bigr\}, \tag{4.2}$$

with $\delta=\frac{1}{2(p-1)}$. Hence $\varphi_a|M\setminus M_a=0$, where M_a is the open set

$$M_a=\bigcup_{s>a}S_s^{(a)}.$$

The functions q_a will be constructed in such a way that they result in smooth functions φ_a in M_a. Therefore we may compute the p-Laplacian of φ_a,

$$\operatorname{div}\left(|\nabla\varphi_a|^{p-2}\nabla\varphi_a\right),$$

pointwise in M_a. For a fixed $a\in\mathbb{R}$ we write $\varphi=\varphi_a$ and

$$V=|\nabla\varphi|^{p-2}\nabla\varphi.$$

Since $S,\ R$, and Θ are commuting and q is independent of s, we have

$$[X,S]=[R,S]+[qS,S]=q[S,S]=0. \tag{4.3}$$

Furthermore, $X\varphi=0$ since φ is constant along integral curves of X. Hence

$$X(S\varphi)=X(S\varphi)-S(X\varphi)=[X,S]\varphi=0, \tag{4.4}$$

and therefore also $S\varphi$ is constant along integral curves of X. We obtain that

$$\langle\nabla\varphi,S\rangle_{(s',r,\vartheta)}=S\varphi(s',r,\vartheta)=S\varphi(s,0,*)=f'(s)>0 \tag{4.5}$$

for points $(s',r,\vartheta)\in S_s$, with $s>a$. For the other components of $\nabla\varphi$, we have

$$\langle\nabla\varphi,\Theta\rangle=0$$

and

$$\begin{aligned}\langle\nabla\varphi,R\rangle&=\langle\nabla\varphi,X\rangle-q\langle\nabla\varphi,S\rangle\\&=X\varphi-qS\varphi\\&=-qS\varphi.\end{aligned}$$

From now on we abbreviate $h(r)=\cosh r$. Sometimes we also write

$$v'_r=Rv,\ v'_s=Sv,\ v''_{rs}=S(Rv),\ \text{etc.}$$

for partial derivatives of a function v. Hence

$$\begin{aligned}\nabla\varphi&=\langle\nabla\varphi,S\rangle h^{-2}S-q\langle\nabla\varphi,S\rangle R\\&=\varphi'_s(h^{-2}S-qR),\\|\nabla\varphi|&=\varphi'_s\sqrt{h^{-2}+q^2},\end{aligned}$$

and

$$V=(\varphi'_s)^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1}(h^{-2}S-qR),$$

more precisely,

$$\nabla\varphi(s',r,\vartheta)=f'(s)\bigl(\cosh^{-2}rS-q(r)R\bigr), \tag{4.6}$$

$$|\nabla\varphi(s',r,\vartheta)|=f'(s)\sqrt{\cosh^{-2}r+q^2(r)}, \tag{4.7}$$

and

$$V_{(s',r,\vartheta)} = \big(f'(s)\big)^{p-1}\big(\cosh^{-2} r + q^2(r)\big)^{\frac{p}{2}-1}\big(\cosh^{-2} rS - q(r)R\big), \tag{4.8}$$

where

$$s' = s + \int_0^r q(t)\,dt.$$

Furthermore, by (4.4)

$$X\big(S(S\varphi)\big) = X\big(S(S\varphi)\big) - S\big(X(S\varphi)\big) = [X,S]\varphi = 0,$$

and therefore also $\varphi''_{ss} = S(S\varphi)$ is constant along integral curves of X. Thus we have

$$\varphi''_{ss}(s',r,\vartheta) = \varphi''_{ss}(s,0,*) = f''(s) < 0 \tag{4.9}$$

for points $(s',r,\vartheta) \in S_s$, with $s > a$. To calculate the p-Laplacian of φ we recall that

$$\mathcal{L}_V\omega = (\operatorname{div} V)\omega,$$

where $\mathcal{L}_V\omega$ is the Lie derivative of the (Riemannian) volume form ω with respect to the vector field V. By Cartan's magic formula, we have

$$\mathcal{L}_V\omega = d(V \lrcorner\, \omega) + V \lrcorner\, d\omega = d(V \lrcorner\, \omega),$$

where $V \lrcorner\, \omega$ is the contraction of ω by the vector field V. In $M \setminus L$ the volume form ω is given by

$$\omega = g(s,r)\cosh r\, ds \wedge dr \wedge d\vartheta.$$

Hence

$$V \lrcorner\, \omega = (\varphi'_s)^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1} gh(h^{-2}dr \wedge d\vartheta + q ds \wedge d\vartheta),$$

and consequently

$$\begin{aligned} d(V \lrcorner\, \omega) &= S\left((\varphi'_s)^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1} gh^{-1}\right) ds \wedge dr \wedge d\vartheta \\ &\quad - R\left((\varphi'_s)^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1} ghq\right) ds \wedge dr \wedge d\vartheta. \end{aligned}$$

We obtain

$$gh \operatorname{div} V = S\left((\varphi'_s)^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1} gh^{-1}\right) - R\left((\varphi'_s)^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1} ghq\right) \tag{4.10}$$

in $M_a \setminus L$. Next we calculate the terms on the right hand side of (4.10) separately. Since h and q are independent of s, we have

$$\begin{aligned} &S\big((\varphi'_s)^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1} gh^{-1}\big) \\ &\qquad = (\varphi'_s)^{p-2}(h^{-2}+q^2)^{\frac{p}{2}-1} h^{-1}\big((p-1)g\varphi''_{ss} + g'_s\varphi'_s\big). \end{aligned} \tag{4.11}$$

Moreover, since $R = X - qS$ and $X(S\varphi) = 0$ (and $Sh = Sq = 0$), we get

$$\begin{aligned} &R\big((\varphi'_s)^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1} ghq\big) \\ &\qquad =(\varphi'_s)^{p-2}(h^{-2}+q^2)^{\frac{p}{2}-2}\Big[\varphi'_s(h^{-2}+q^2)(ghq'_r + gh'_r q + g'_r hq) \\ &\qquad\quad + (p-2)ghq\varphi'_s(-h^{-3}h'_r + qq'_r) - (p-1)ghq^2(h^{-2}+q^2)\varphi''_{ss}\Big] \end{aligned} \tag{4.12}$$

We are interested in conditions for functions g and q that imply that φ is a p-subsolution, that is $\operatorname{div} V \ge 0$. Since $\varphi = \varphi_a = 0$ in $M \setminus M_a$, it follows from (4.10) that φ is a p-subsolution in $M \setminus L$ if

$$S\Big((\varphi_s')^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1}gh^{-1}\Big) - R\Big((\varphi_s')^{p-1}(h^{-2}+q^2)^{\frac{p}{2}-1}ghq\Big) \ge 0.$$

By (4.5), (4.11), and (4.12), this is equivalent to

$$\frac{(p-1)\varphi_{ss}''(h^{-2}+q^2)}{\varphi_s'} + \frac{g_s'}{gh^2} - \frac{g_r'q}{g} - q_r' - \frac{h_r'q}{h} - \frac{(p-2)q(qq_r' - h^{-3}h_r')}{h^{-2}+q^2} \ge 0. \tag{4.13}$$

Following Borbély, we define a C^∞-function $\beta\colon M \to [0,\infty)$ (denoted by p in [9]) by

$$\beta(s,r) = \frac{g_s'(s,r)}{g_r'(s,r)h^2(r)}. \tag{4.14}$$

Note that β is independent of the (Fermi) coordinate ϑ and that $\beta(s,r) = 0$ for $0 \le r \le 2$ by (3.2). Inserting β into (4.13) we finally get that φ is a p-subsolution in $M \setminus L$ if

(4.15)

$$\frac{g_r'h(\beta-q)}{g} - (p-1)hq_r' - h_r'q + \frac{(p-2)(hq_r' + h_r'q)}{1+h^2q^2} + \frac{(p-1)\varphi_{ss}''(1+h^2q^2)}{\varphi_s'h} \ge 0.$$

Remark 4.1. It is worth noting already at this stage that all terms above except the first one will eventually be negative for large r, and therefore the first term should dominate the others. This requirement puts strong constraints on functions β, g, and q.

Remark 4.2. There are, of course, many ways to calculate the p-Laplacian of φ. For example, we could use the formula

$$\operatorname{div}|\nabla\varphi|^{p-2}\nabla\varphi = |\nabla\varphi|^{p-2}\left(\Delta\varphi + \frac{p-2}{2}\frac{\langle\nabla(|\nabla\varphi|^2),\nabla\varphi\rangle}{|\nabla\varphi|^2}\right)$$

and compute first the Laplacian as the trace of the Hessian

$$\begin{aligned}\Delta\varphi &= \operatorname{Hess}\varphi(R,R) + h^{-2}\operatorname{Hess}(S,S) + g^{-2}\operatorname{Hess}(\Theta,\Theta)\\ &= -\varphi_s'q_r' + \varphi_{ss}''q^2 + h^{-2}\varphi_{ss}'' - \frac{h_r'\varphi_s'q}{h} - \frac{g_r'\varphi_s'q}{g} + \frac{g_s'\varphi_s'}{gh^2}\\ &= \varphi_s'\left(\frac{g_r'(\beta-q)}{g} - q_r - \frac{h_rq}{h} + \frac{\varphi_{ss}''(h^{-2}+q^2)}{\varphi_s'}\right)\end{aligned}$$

and then

$$\frac{p-2}{2}\frac{\langle\nabla(|\nabla\varphi|^2),\nabla\varphi\rangle}{|\nabla\varphi|^2} = (p-2)\varphi_{ss}''(h^{-2}+q^2) - \frac{(p-2)\varphi_s'(q_r'q^2 - h_r'h^{-3}q)}{h^{-2}+q^2}.$$

Remark 4.3. Still another way to calculate the p-Laplacian of φ is to use the formula

$$\operatorname{div}|\nabla\varphi|^{p-2}\nabla\varphi = |\nabla\varphi|^{p-2}\big(\Delta\varphi + (p-2)\operatorname{Hess}(Y,Y)\big),$$

where $Y = \frac{\nabla\varphi}{|\nabla\varphi|}$, and first compute the Laplacian as the trace of the Hessian by using the basis $\{X,\Theta,Y\}$, cf. [9, pp. 233-234]. Then

$$\Delta\varphi = \frac{\operatorname{Hess}\varphi(X,X)}{\langle X,X\rangle} + \frac{\operatorname{Hess}\varphi(\Theta,\Theta)}{\langle\Theta,\Theta\rangle} + \operatorname{Hess}\varphi(Y,Y),$$

where

$$\frac{\operatorname{Hess}\varphi(X,X)}{\langle X,X\rangle} = \frac{-\varphi_s'\left(hq_r' + 2h_r'q + h^3h_r'q^3\right)}{h(1+h^2q^2)},$$
$$\frac{\operatorname{Hess}\varphi(\Theta,\Theta)}{\langle \Theta,\Theta\rangle} = \frac{\varphi_s' g_r' h(\beta - q)}{g},$$

and

$$\operatorname{Hess}(Y,Y) = \varphi_{ss}''(h^{-2}+q^2) - \frac{\varphi_s'(q_r'q^2 - h_r'h^{-3}q)}{h^{-2}+q^2}.$$

It should be noted that there is a misprint in the formula [9, (3.7)] for $\operatorname{Hess}\varphi(Y,Y)$; using Borbély's notation the term

$$\frac{f''}{hf'\sqrt{q^2+1/h}}$$

on the right hand side of [9, (3.7)] should be replaced by

$$\frac{f''\sqrt{q^2+1/h}}{f'}.$$

However, it turns out that this is a harmless misprint.

5. Conditions for functions q_a

In this section we study sufficient conditions on functions q_a that imply the correct boundary behavior of functions φ_a, i.e. conditions (a) and (b'), as well as the condition (c) in Theorem 2.3. Since $M(\infty)$ can be identified with $\mathbb{H}^3(\infty)$, the condition (b') written in Fermi coordinates reads as

$$\lim_{r\to\infty} \varphi_a(s,r,\vartheta) = 0 \tag{5.1}$$

for all $a, s \in \mathbb{R},\ \vartheta \in [0, 2\pi[$. This condition is clearly satisfied if there exists a cone neighborhood of $(s, +\infty, \vartheta)$ that intersects with no level sets $S_s^{(a)}$ for $s \ge a$, since in that case φ_a would vanish identically in that neighborhood. Since the surfaces $S_s^{(a)}$, with $s \ge a$, are obtained from $S_a^{(a)}$ by a Euclidean dilation $x \mapsto \lambda x,\ \lambda \le 1$, with respect to x_0 in our upper half space model of M, it is enough to find a cone neighborhood of $(s, +\infty, \vartheta)$ that does not intersect with $S_a^{(a)}$. This, in turn, is satisfied, if for all fixed $a \in \mathbb{R}$ and for every cone neighborhood U of x_0, all integral curves $\gamma_{a,s},\ s \ge a$, eventually enter at and stay in U. Since $X^{(a)} = R + q_a S$, integral curves $\gamma_{a,s}$ are given in Fermi coordinates as

$$\gamma_{a,s}(r) = \Big(s + \int_0^r q_a(t)\,dt, r, \vartheta\Big).$$

Thus we require that

$$\int_0^\infty q_a(t)\,dt = \infty \tag{5.2}$$

for all $a \in \mathbb{R}$. It is easy to see that (5.2) is also a necessary condition for (b') to hold. Recall that

$$\varphi_a(x) = f_a(s') = \max\{\tanh\delta(s-a), 0\}$$

for points $x = (s, r, \vartheta) \in M$, where

$$s' = s + \int_0^r q_a(t)\,dt.$$

Hence $\lim_{a\to-\infty} \varphi_a(x) = 1$ if

$$\lim_{a\to-\infty} \Bigl(\int_0^r q_a(t)\,dt - a \Bigr) = \infty.$$

This holds if

$$\int_0^r q_a(t)\,dt \le b_r < \infty \tag{5.3}$$

independently of $a \in \mathbb{R}$. Let us collect these to the following lemma.

Lemma 5.1. *If the functions q_a satisfy* (5.2) *for all $a \in \mathbb{R}$ and* (5.3) *independently of $a \in \mathbb{R}$, then the family of functions φ_a has the properties (b') and (c) in Theorem 2.3.*

Note that the condition (a) in Theorem 2.3 is just a matter of convention. It is also clear that φ_a is continuous in M and smooth in $\{(s, r, \vartheta) \colon s > a, r > 0\}$. Smoothness in $L \cap M_a$ will be assured later by a particular choice of q_a for small r.

6. Curvature conditions for g

In this section we seek conditions on the function g in the definition (3.1) of the Riemannian metric of M that result in the sectional curvature upper bound -1. We calculate the sectional curvatures of M by using the Cartan formalism with an orthonormal frame $E_1 = R,\ E_2 = \frac{1}{h}S,\ E_3 = \frac{1}{g}\Theta$ and the corresponding Cartan forms, i.e. the dual coframe, $\alpha^1 = dr,\ \alpha^2 = h\,ds$, and $\alpha^3 = g\,d\vartheta$. The connection 1-forms ω_i^j, determined by $\nabla_Y E_i = \omega_i^j E_j$, satisfy $\omega_i^j = -\omega_j^i$ and Cartan's first structural equations

$$d\alpha^j = \alpha^i \wedge \omega_i^j.$$

Since $d\alpha^1 = 0,\ d\alpha^2 = \frac{h_r'}{h}\alpha^1 \wedge \alpha^2$, and $d\alpha^3 = \frac{g_r'}{g}\alpha^1 \wedge \alpha^3 + \frac{g_s'}{gh}\alpha^2 \wedge \alpha^3$ we get the connection matrix

$$(\omega_i^j) = \begin{pmatrix} 0 & -\frac{h_r'}{h}\alpha^2 & -\frac{g_r'}{g}\alpha^3 \\ \frac{h_r'}{h}\alpha^2 & 0 & -\frac{g_s'}{gh}\alpha^3 \\ \frac{g_r'}{g}\alpha^3 & \frac{g_s'}{gh}\alpha^3 & 0 \end{pmatrix}.$$

Furthermore, the curvature 2-forms Ω_i^j, defined by $\Omega_i^j(V, W) = \alpha^j\bigl(R(V, W)E_i\bigr)$, satisfy Cartan's second structural equations

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j.$$

Hence

$$\begin{aligned}
\Omega_1^1 &= \Omega_2^2 = \Omega_3^3 = 0,\\
\Omega_2^1 &= -\Omega_1^2 = -\frac{h''_{rr}}{h}\alpha^1\wedge\alpha^2,\\
\Omega_3^1 &= -\Omega_1^3 = -\frac{g''_{rr}}{g}\alpha^1\wedge\alpha^3 + \left(-\frac{g''_{rs}}{gh} + \frac{g'_s h'_r}{gh^2}\right)\alpha^2\wedge\alpha^3,\\
\Omega_2^3 &= -\Omega_3^2\\
&= \left(\frac{g''_{sr}}{gh} - \frac{g'_s h'_r}{gh^2}\right)\alpha^1\wedge\alpha^3 + \left(\frac{g''_{ss}}{gh^2} + \frac{g'_r h'_r}{gh}\right)\alpha^2\wedge\alpha^3.
\end{aligned}$$

Since $R(V,W)E_i = \Omega_i^j(V,W)E_j$, we get

$$\begin{aligned}
\langle R(E_1,E_2)E_2,E_1\rangle &= -\frac{h''_{rr}}{h},\\
\langle R(E_1,E_3)E_3,E_1\rangle &= -\frac{g''_{rr}}{g},
\end{aligned}$$

and

$$\langle R(E_2,E_3)E_3,E_2\rangle = -\frac{g''_{ss}}{gh^2} - \frac{g'_r h'_r}{gh}.$$

Thus for arbitrary vectors $u = \sum_{i=1}^3 u_iE_i$ and $v = \sum_{i=1}^3 v_iE_i$ in a tangent space T_xM, we have

$$\begin{aligned}
\langle R(u,v)v,u\rangle = {}& A(u_1v_2-u_2v_1)^2 + B(u_1v_3-u_3v_1)^2 + C(u_2v_3-u_3v_2)^2\\
&+ 2D(u_1v_3-u_3v_1)(u_2v_3-u_3v_2),
\end{aligned}\tag{6.1}$$

where

$$\begin{aligned}
A &= -\frac{h''_{rr}}{h},\quad B = -\frac{g''_{rr}}{g},\\
C &= -\frac{g''_{ss}}{gh^2} - \frac{g'_r h'_r}{gh},
\end{aligned}$$

and

$$D = -\frac{g''_{rs}}{gh} + \frac{g'_s h'_r}{gh^2}.$$

On the other hand,

$$|u\wedge v|^2 = (u_1v_2-u_2v_1)^2 + (u_1v_3-u_3v_1)^2 + (u_2v_3-u_3v_2)^2,$$

and therefore all sectional curvatures of M have upper bound -1 if and only if, for every $x\in M$ and $u,v\in T_xM$,

$$\langle R(u,v)v,u\rangle \le -|u\wedge v|^2$$

which by (6.1) is equivalent to

$$\begin{aligned}
(A+1)(u_1v_2-u_2v_1)^2 + (B+1)(u_1v_3-u_3v_1)^2 + (C+1)(u_2v_3-u_3v_2)^2&\\
+ 2D(u_1v_3-u_3v_1)(u_2v_3-u_3v_2) \le 0&
\end{aligned}\tag{6.2}$$

We conclude that all sectional curvatures of M are at most -1 if and only if the following four inequalities hold:

$$\frac{h''_{rr}}{h} \geq 1, \tag{6.3}$$

$$\frac{g''_{rr}}{g} \geq 1, \tag{6.4}$$

$$\frac{g''_{ss}}{gh^2} + \frac{g'_r h'_r}{gh} \geq 1, \tag{6.5}$$

$$\left(-\frac{g''_{rs}}{gh} + \frac{g'_s h'_r}{gh^2}\right)^2 \leq \left(\frac{g''_{rr}}{g} - 1\right)\left(\frac{g''_{ss}}{gh^2} + \frac{g'_r h'_r}{gh} - 1\right) \tag{6.6}$$

The first condition (6.3) holds as an equality since $h(r) = \cosh r$. Thus it suffices to construct g such that conditions (6.4) and (6.6) hold. Recall from (3.2) that we look for g of the form

$$g = \tfrac{1}{2}\sinh\Phi,$$

where $\Phi(s,r) = \sinh 2\varrho(s,r)$, with $\varrho(s,r) = r$ for $0 \leq r \leq 3$. Then

$$g'_r = \tfrac{1}{2}\Phi'_r \cosh\Phi = \varrho'_r \cosh(2\varrho)\cosh(\sinh 2\varrho)$$

and

$$g'_s = \tfrac{1}{2}\Phi'_s \cosh\Phi = \varrho'_s \cosh(2\varrho)\cosh(\sinh 2\varrho).$$

By (4.14), g satisfies the partial differential equation

$$g'_s = \beta h^2 g'_r. \tag{6.7}$$

We observe that both Φ and ϱ satisfy the same partial differential equation, i.e.

$$\Phi'_s = \beta h^2 \Phi'_r, \tag{6.8}$$

and

$$\varrho'_s = \beta h^2 \varrho'_r. \tag{6.9}$$

Note that $\nabla\varrho = \varrho'_r(\beta S + R)$, and therefore $\nabla\varrho \perp (\beta h^2 R - S)$. Hence ϱ is constant along any integral curve of the vector field

$$Z = \beta h^2 R - S. \tag{6.10}$$

Now the idea is to construct an unbounded domain $\Omega \subset M$ of the form

$$\Omega = \{(s,r,\vartheta) \in M \colon r < 3\} \cup \{(s,r,\vartheta) \in M \colon s < -\ell(r)\} \tag{6.11}$$

such that all integral curves of Z will eventually enter at Ω, and then construct β so that it vanishes identically in Ω, and finally fix the "initial condition"

$$\varrho(s,r) = r \tag{6.12}$$

for all $(s,r,\vartheta) \in \Omega$. Note that $(s,r,\vartheta) \in \Omega$ for all $s \leq s'$ if $(s',r,\vartheta) \in \Omega$. Consequently, once an integral curve of Z enters at Ω, it will then stay in Ω forever. The function ℓ that appears in (6.11) is closely related to β and will, together with β, be determined in the next section. Then g, and hence the Riemannian structure of M, will be completely determined by constructing the functions β and ℓ.

Remark 6.1. Note that by the choice (6.12), ϱ coincides with the function f in [7, (2.11)].

While constructing β we have to keep in mind the condition (4.15) for φ_a to be a p-subsolution together with Remark 4.1 and (5.2). This leads to the first requirement that

$$\int_0^\infty \beta(s,r)\,dr = \infty \tag{6.13}$$

for all $s \in \mathbb{R}$. For the construction of g we require that

$$\int_{r_0}^\infty \frac{dr}{\beta(s,r)\cosh^2 r} = \infty \tag{6.14}$$

for all $r_0 > 0$ and $s \in \mathbb{R}$. To obtain the curvature conditions (6.4) and (6.6) we will also require that

$$0 \le \beta \le \frac{1}{1000}, \quad |\beta'_r| \le \frac{1}{1000}, \quad 0 \le \beta'_s \le \frac{1}{1000}, \quad \beta\beta'_r h^3 \le \frac{h'_r}{1000}, \tag{6.15}$$

and that βh^2 is a convex non-decreasing function in the variable r, that is

$$(\beta h^2)'_r \ge 0 \quad \text{and} \quad (\beta h^2)''_{rr} \ge 0. \tag{6.16}$$

Recall from (6.9) that ϱ satisfies the partial differential equation

$$\varrho'_s = \beta h^2 \varrho'_r,$$

and hence we may apply the proof of [9, Lemma 2.2] to the function ϱ. Since $\varrho'_r(s,r) \equiv 1$ and $\varrho''_{rr}(s,r) \equiv 0$ in Ω, we get

$$\varrho'_r \ge 1, \tag{6.17}$$

$$\varrho''_{rr} \ge 0, \tag{6.18}$$

and

$$\varrho(s,r) \ge r \tag{6.19}$$

in M. Furthermore, since $\Phi = \sinh 2\varrho$, $\Phi'_r = 2\varrho'_r \cosh 2\varrho$, and $\Phi''_{rr} = 4(\varrho'_r)^2 \sinh 2\varrho + 2\varrho''_{rr}\cosh 2\varrho$, it follows that

$$\Phi(s,r) = \sinh 2\varrho(s,r) \ge \sinh 2r, \tag{6.20}$$

$$\Phi'_r(s,r) \ge 2\cosh 2r, \tag{6.21}$$

and

$$\Phi''_{rr}(s,r) \ge 4\sinh 2r. \tag{6.22}$$

in M. In order to scrutinize conditions (6.4) and (6.6) we first compute

$$\begin{aligned}
g''_{rr} &= \tfrac12 \Phi''_{rr}\cosh\Phi + \tfrac12 (\Phi'_r)^2 \sinh\Phi,\\
g''_{rs} &= g''_{sr} = (\beta h^2 g'_r)'_r = 2\beta h h'_r g'_r + \beta'_r h^2 g'_r + \beta h^2 g''_{rr}\\
&= \beta h h'_r \Phi'_r \cosh\Phi + \tfrac12 \beta'_r h^2 \Phi'_r \cosh\Phi + \tfrac12 \beta h^2 \Phi''_{rr}\cosh\Phi + \tfrac12 \beta h^2 (\Phi'_r)^2 \sinh\Phi,
\end{aligned}$$

and

$$\begin{aligned}
g''_{ss} &= (\beta h^2 g'_r)'_s = \beta'_s h^2 g'_r + \beta h^2 g''_{rs}\\
&= \tfrac12 \beta'_s h^2 \Phi'_r \cosh\Phi + \beta^2 h^3 h'_r \Phi'_r \cosh\Phi + \tfrac12 \beta\beta'_r h^4 \Phi'_r \cosh\Phi + \tfrac12 \beta^2 h^4 \Phi''_{rr}\cosh\Phi\\
&\quad + \tfrac12 \beta^2 h^4 (\Phi'_r)^2 \sinh\Phi.
\end{aligned}$$

Condition (6.4) holds since

$$
\begin{aligned}
\frac{g''_{rr}}{g} &= \Phi''_{rr}\coth\Phi + (\Phi'_r)^2 \\
&= 2\varrho''_{rr}\cosh 2\varrho \coth(\sinh 2\varrho) + 4(\varrho'_r)^2 \sinh 2\varrho \coth(\sinh 2\varrho) \\
&\quad + 4(\varrho'_r)^2\cosh^2 2\varrho \\
&\geq 4\max(1, \sinh 2r) + 4\cosh^2 2r.
\end{aligned}
\tag{6.23}
$$

Let us denote by LHS and RHS the left hand side and the right hand side of (6.6), respectively. For LHS we first have

$$
\frac{g''_{rs}}{gh} - \frac{g'_s h'_r}{gh^2} = \beta h(\Phi'_r)^2 + \beta h'_r\Phi'_r\coth\Phi + \beta'_r h\Phi'_r\coth\Phi + \beta h\Phi''_{rr}\coth\Phi.
$$

Hence

$$
\begin{aligned}
\text{LHS} &= \Big(-\frac{g''_{rs}}{gh} + \frac{g'_s h'_r}{gh^2}\Big)^2 \\
&= \beta^2h^2(\Phi'_r)^4 + 2\beta^2hh'_r(\Phi'_r)^3\coth\Phi + 2\beta\beta'_rh^2(\Phi'_r)^3\coth\Phi \\
&\quad + 2\beta^2h^2(\Phi'_r)^2\Phi''_{rr}\coth\Phi + \beta^2(h'_r)^2(\Phi'_r)^2\coth^2\Phi + 2\beta\beta'_rhh'_r(\Phi'_r)^2\coth^2\Phi \\
&\quad + 2\beta^2hh'_r\Phi'_r\Phi''_{rr}\coth^2\Phi + (\beta'_r)^2h^2(\Phi'_r)^2\coth^2\Phi + 2\beta\beta'_rh^2\Phi'_r\Phi''_{rr}\coth^2\Phi \\
&\quad + \beta^2h^2(\Phi''_{rr})^2\coth^2\Phi.
\end{aligned}
$$

For RHS we compute

$$
\begin{aligned}
\frac{g''_{ss}}{gh^2} + \frac{g'_r h'_r}{gh} &= \frac{g''_{ss}}{gh^2} + \frac{g'_r}{g}\tanh r \\
&= \beta'_s\Phi'_r\coth\Phi + \beta(\beta h^2)'_r\Phi'_r\coth\Phi + \beta^2h^2\Phi''_{rr}\coth\Phi + \beta^2h^2(\Phi'_r)^2 \\
&\quad + \Phi'_r\tanh r\coth\Phi.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
\text{RHS} &= \left(\frac{g''_{rr}}{g} - 1\right)\left(\frac{g''_{ss}}{gh^2} + \frac{g'_r h'_r}{gh} - 1\right) \\
&= \beta'_s\Phi'_r\Phi''_{rr}\coth^2\Phi + 2\beta^2hh'_r\Phi'_r\Phi''_{rr}\coth^2\Phi + \beta\beta'_rh^2\Phi'_r\Phi''_{rr}\coth^2\Phi \\
&\quad + \beta^2h^2(\Phi''_{rr})^2\coth^2\Phi + \beta^2h^2(\Phi'_r)^2\Phi''_{rr}\coth\Phi + \Phi'_r\Phi''_{rr}\tanh r\coth^2\Phi \\
&\quad + \beta'_s(\Phi'_r)^3\coth\Phi + 2\beta^2hh'_r(\Phi'_r)^3\coth\Phi + \beta\beta'_rh^2(\Phi'_r)^3\coth\Phi \\
&\quad + \beta^2h^2(\Phi'_r)^2\Phi''_{rr}\coth\Phi + \beta^2h^2(\Phi'_r)^4 + (\Phi'_r)^3\tanh r\coth\Phi \\
&\quad - \beta'_s\Phi'_r\coth\Phi - 2\beta^2hh'_r\Phi'_r\coth\Phi - \beta\beta'_rh^2\Phi'_r\coth\Phi \\
&\quad - \beta^2h^2\Phi''_{rr}\coth\Phi - \beta^2h^2(\Phi'_r)^2 - \Phi'_r\tanh r\coth\Phi \\
&\quad - \Phi''_{rr}\coth\Phi - (\Phi'_r)^2 + 1.
\end{aligned}
$$

Hence we get after simplifying

$$\begin{aligned}
&\mathrm{RHS}-\mathrm{LHS}\\
&=\beta_s'\Phi_r'\coth\Phi\big((\Phi_r')^2+\Phi_{rr}''\coth\Phi-1\big)-\beta\beta_r'h^2\Phi_r'\coth\Phi-\beta\beta_r'h^2(\Phi_r')^3\coth\Phi\\
&-2\beta\beta_r'hh_r'(\Phi_r')^2\coth^2\Phi-\beta\beta_r'h^2\Phi_r'\Phi_{rr}''\coth^2\Phi\\
&+\Phi_r'\tanh r\coth\Phi\big((\Phi_r')^2+\Phi_{rr}''\coth\Phi-1-2\beta^2h^2\big)\\
&-(\Phi_r')^2\big(1+\beta^2h^2+\beta^2(h_r')^2\coth^2\Phi+(\beta_r')^2h^2\coth^2\Phi\big)-\Phi_{rr}''(1+\beta^2h^2)\coth\Phi+1.
\end{aligned}$$

To prove that $\mathrm{RHS}-\mathrm{LHS}\geq 0$, we first estimate

$$\Phi_r'\coth\Phi=\frac{g_r'}{g}=2\varrho_r'\cosh 2\varrho\coth(\sinh 2\varrho)\geq 2\cosh 2r. \tag{6.24}$$

Since

$$\begin{aligned}
\frac{\Phi_{rr}''}{\Phi_r'}+\Phi_r'\tanh\Phi&=\frac{\varrho_{rr}''}{\varrho_r'}+2\varrho_r'\tanh 2\varrho+2\varrho_r'\cosh 2\varrho\tanh\Phi\\
&\geq 2\tanh 2r+2\cosh 2r\tanh(\sinh 2r),
\end{aligned}$$

we get another useful estimate

$$\Phi_{rr}''\coth\Phi+(\Phi_r')^2\geq 2\big(\tanh 2r+\cosh 2r\tanh(\sinh 2r)\big)\Phi_r'\coth\Phi. \tag{6.25}$$

Let us return to prove that $\mathrm{RHS}-\mathrm{LHS}\geq 0$. The first term

$$\beta_s'\Phi_r'\coth\Phi\ \big((\Phi_r')^2+\Phi_{rr}''\coth\Phi-1\big)$$

can be omitted since it is non-negative by (6.13) and (6.23). If $0\leq r\leq 3$, $g(s,r)=\frac{1}{2}\sinh(\sinh 2r)$ and thus $g_s'(s,r)=g_{sr}''(s,r)=0$. Consequently, $\mathrm{LHS}=0$ and $\mathrm{RHS}\geq 0$. Suppose next that $r\geq 3$. By (6.23),

$$\begin{aligned}
\tfrac{1}{4}\big((\Phi_r')^2+\Phi_{rr}''\coth\Phi\big)&\geq\max(1,\sinh 2r)+\cosh^2 2r\\
&\geq 1+2\beta^2h^2,
\end{aligned}$$

and therefore

$$\begin{aligned}
&\Phi_r'\tanh r\coth\Phi\big((\Phi_r')^2+\Phi_{rr}''\coth\Phi-1-2\beta^2h^2\big)\\
&\geq\tfrac{3}{4}\Phi_r'\tanh r\coth\Phi\big((\Phi_r')^2+\Phi_{rr}''\coth\Phi\big).
\end{aligned}\tag{6.26}$$

On the other hand,

$$\begin{aligned}
&\tfrac{3}{16}\Phi_r'\tanh r\coth\Phi\big((\Phi_r')^2+\Phi_{rr}''\coth\Phi\big)\\
&\geq\tfrac{3}{8}(\Phi_r')^2\tanh r\coth^2\Phi\big(\tanh 2r+\cosh 2r\tanh(\sinh 2r)\big)\\
&\geq(\Phi_r')^2\big(1+\beta^2h^2+\beta^2(h_r')^2\coth^2\Phi+(\beta_r')^2h^2\coth^2\Phi\big)
\end{aligned}\tag{6.27}$$

by (6.25) and (6.15). Furthermore, by using (6.21) and (6.15), we get

$$\begin{aligned}
&\tfrac{3}{16}\Phi_r'\tanh r\coth\Phi\big((\Phi_r')^2+\Phi_{rr}''\coth\Phi\big)\\
&\geq\tfrac{3}{8}\tanh r\cosh 2r\big(4\cosh^2 2r+\Phi_{rr}''\coth\Phi\big)\\
&\geq\Phi_{rr}''\coth\Phi\big(1+\beta^2h^2\big).
\end{aligned}\tag{6.28}$$

Now (6.26), (6.27), and (6.28) imply that

$$\begin{aligned}
&\Phi_r'\tanh r\coth\Phi\big((\Phi_r')^2+\Phi_{rr}''\coth\Phi-1-2\beta^2h^2\big)\\
&-(\Phi_r')^2\big(1+\beta^2h^2+\beta^2(h_r')^2\coth^2\Phi+(\beta_r')^2h^2\coth^2\Phi\big)-\Phi_{rr}''(1+\beta^2h^2)\coth\Phi+1\\
&\geq\tfrac{3}{8}\Phi_r'\tanh r\coth\Phi\big((\Phi_r')^2+\Phi_{rr}''\coth\Phi\big)+1.
\end{aligned}$$

Hence it is enough to prove that

$$
\begin{aligned}
&\tfrac{3}{8}\Phi'_r \tanh r \coth\Phi \left((\Phi'_r)^2 + \Phi''_{rr}\coth\Phi\right) + 1\\
&- \beta\beta'_r h^2 \Phi'_r \coth\Phi - \beta\beta'_r h^2 (\Phi'_r)^3 \coth\Phi - 2\beta\beta'_r h h'_r (\Phi'_r)^2 \coth^2\Phi \qquad (6.29)\\
&- \beta\beta'_r h^2 \Phi'_r \Phi''_{rr} \coth^2\Phi \ge 0.
\end{aligned}
$$

Since $\beta\beta_r h^3 \le h'_r/1000'$ by the assumption (6.15), we have

$$
\begin{aligned}
&- \beta\beta'_r h^2 \Phi'_r \coth\Phi - \beta\beta'_r h^2 (\Phi'_r)^3 \coth\Phi - 2\beta\beta'_r h h'_r (\Phi'_r)^2 \coth^2\Phi\\
&- \beta\beta'_r h^2 \Phi'_r \Phi''_{rr} \coth^2\Phi \qquad (6.30)\\
&\ge -\tfrac{1}{1000}\Phi'_r \tanh r \coth\Phi \left(1 + (\Phi'_r)^2 + 2\Phi'_r \tanh r \coth\Phi + \Phi''_{rr}\coth\Phi\right).
\end{aligned}
$$

By (6.23) and (6.25), we obtain

$$
\begin{aligned}
\tfrac{3}{32}\Phi'_r \tanh r \coth\Phi \left((\Phi'_r)^2 + \Phi''_{rr}\coth\Phi\right) &\ge \tfrac{3}{8}\Phi'_r \tanh r \coth\Phi\\
&\ge \tfrac{1}{1000}\Phi'_r \tanh r \coth\Phi,\\
\tfrac{3}{32}\Phi'_r \tanh r \coth\Phi \left((\Phi'_r)^2 + \Phi''_{rr}\coth\Phi\right) &\ge \tfrac{3}{32}(\Phi'_r)^3 \tanh r \coth\Phi\\
&\ge \tfrac{1}{1000}(\Phi'_r)^3 \tanh r \coth\Phi,\\
\tfrac{3}{32}\Phi'_r \tanh r \coth\Phi \left((\Phi'_r)^2 + \Phi''_{rr}\coth\Phi\right)&\\
&\ge \tfrac{3}{16}(\Phi'_r)^2 \tanh r \coth^2\Phi \,(\tanh 2r + \tanh(\sinh 2r)\cosh 2r)\\
&\ge \tfrac{1}{500}(\Phi'_r)^2 \tanh^2 r \coth^2\Phi,
\end{aligned}
$$

and

$$
\begin{aligned}
\tfrac{3}{32}\Phi'_r \tanh r \coth\Phi \left((\Phi'_r)^2 + \Phi''_{rr}\coth\Phi\right) &\ge \tfrac{3}{32}\Phi'_r \Phi''_{rr} \tanh r \coth^2\Phi\\
&\ge \tfrac{1}{1000}\Phi'_r \Phi''_{rr} \tanh r \coth^2\Phi
\end{aligned}
$$

that together with (6.30) verify (6.29).

7. Construction of g

In this section we briefly recall from [9] and [7] the constructions of functions β and ℓ and hence complete the construction of the Riemannian metric. We proceed as in [7] which gives a slight modification of the original construction due to Borbély [9].

First we define a C^∞-smooth function $\beta_0\colon [0,\infty) \to [0,\infty)$ inductively on intervals $[r_n, r_{n+1}]$, with $n \in \mathbb{N}$ and $r_{n+1} - r_n > 3$. In fact, we first define a piecewise smooth approximation of β_0 which we then smoothe in neighborhoods of r_n. To simplify the notation we denote both the piecewise smooth function and the final smooth function by the same symbol β_0. On $[0, r_1]$ we let β_0 be a smooth non-decreasing function such that it vanishes identically on $[0,3]$, it takes the constant (positive) value $\beta_0(5)$ on $[5, r_1]$, and it is a positive slowly increasing function on the interval $(3,5]$ so that (6.15) and (6.16) hold. Here r_1 is large enough such that $\beta_0(r_1)\cosh^2 r_1 = \beta_0(5)\cosh^2 r_1 > 1$. We further require that β_0 is non-increasing on $[r_1,\infty)$, with $\lim_{r\to\infty}\beta_0(r) = 0$, whereas $\beta_0 h^2$ is an increasing strictly convex function. More precisely, on the interval $[r_1, r_2]$, and, in general, on intervals $[r_{2n-1}, r_{2n}]$, we define β_0 such that $\beta_0 h^2$ will be the C^1-continuation of $(\beta_0 h^2)|[0, r_{2n-1}]$ satisfying the differential equation

$$
(\beta_0 h^2)''_{rr} = \frac{1}{2\beta_0 h^2}.
$$

We smoothe β_0 on intervals $[r_{2n-1}, r_{2n-1}+1]$ to a C^∞-function, still denoted by β_0, such that (6.15) and (6.16), with strict inequalities, hold and that $(\beta_0 h^2)''_{rr} > 1/(4\beta_0 h^2)$. By [9, Lemma 2.4], β_0 will be decreasing on $[r_{2n-1}, r_{2n}]$, and by [9, Lemma 2.3], the upper interval bound r_{2n} can be chosen such that

$$\int_{r_{2n-1}}^{r_{2n}-1} \frac{dr}{\beta_0(r)\cosh^2 r} > 1$$

for every n. On the interval $[r_2, r_3]$ and, in general, on intervals $[r_{2n}, r_{2n+1}]$ we let $\beta_0(r) = c_n/r$ for some well-chosen constants c_n. Again we smoothe β_0 on intervals $[r_{2n}-1, r_{2n}]$ such that (6.15) and (6.16), with strict inequalities, hold and that

$$(\beta_0 h^2)''_{rr} > \frac{\varepsilon}{\beta_0 h^2}, \tag{7.1}$$

where $0 < \varepsilon < 1/4$ is small enough and independent of n but depends on the choice of $\beta_0|[0,5]$. Given r_2 and, in general, r_{2n} we choose r_3 and, respectively, r_{2n+1} sufficiently large so that

$$\int_{r_{2n}}^{r_{2n+1}} \beta_0(r)\, dr > 1$$

for every n.

Next we define β by setting

$$\beta(s,r) = \xi(s + \ell(r))\beta_0(r),$$

where $\xi\colon \mathbb{R} \to [0,1]$ is a smooth non-decreasing function such that $\xi|(-\infty, 0] = 0$, $\xi|[4,\infty) = 1$, and that $\xi', |\xi''| < 1/2$, and $\xi'' + \xi > 0$ on $(0,4)$. The smooth function ℓ is constructed so that $\ell(r) = 0$ for $r \in [0,3]$ and

$$\ell' = \frac{\varepsilon}{\beta_0 h^2} \tag{7.2}$$

on the interval $[5,\infty)$, with the same ε as in (7.1). Finally, the two pieces are connected smoothly such that

$$\ell'' \geq \frac{-\ell'(\beta_0 h^2)'_r}{\beta_0 h^2} \quad \text{and} \quad 0 \leq \ell' \leq \frac{\varepsilon}{\beta_0 h^2}$$

for all $r > 0$. By (7.2) and (6.14),

$$\ell(r) = \ell(5) + \int_5^r \ell'(t)\, dt \to \infty$$

as $r \to \infty$. Then $\beta(s,r) = \xi(s + \ell(r))\beta_0(r)$ satisfies the conditions (6.13)–(6.16); see [9] for the details.

Next we complete the construction of g. Recall from (6.11) and (6.12) that

$$\Omega = \{(s,r,\vartheta) \in M : r < 3\} \cup \{s,r,\vartheta) \in M : s < -\ell(r)\}$$

and hence $\beta \equiv 0$ and $g(s,r) = \frac{1}{2}\sinh(\sinh 2r)$ in $\bar\Omega$ and $\beta > 0$ in $M \setminus \bar\Omega$. Notice that integral curves of $W = R - \ell' S$ starting at points in $\partial\Omega \cap \{(s,r,\vartheta) \in M : r > 3\}$ will stay in $\partial\Omega$. Since

$$\frac{1}{\beta h^2} - \ell' \geq \frac{1}{\beta_0 h^2} - \frac{\varepsilon}{\beta_0 h^2} = \frac{1-\varepsilon}{\beta_0 h^2}$$

we conclude from (6.14) that all integral curves of $Z = \beta h^2 R - S$ starting at points in $M \setminus \bar{\Omega}$ will eventually enter at Ω and stay in there. This is seen by comparing the s-components of the integral curves of $W = R - \ell' S$ and

$$\tilde{Z} = \left(\beta h^2\right)^{-1} Z = R - \left(\beta h^2\right)^{-1} S$$

(in $M \setminus \bar{\Omega}$) starting at $(s', r_0, \vartheta_0) \in \partial\Omega$, with $r_0 > 3$, and $(s_0, r_0, \vartheta_0) \in M \setminus \bar{\Omega}$, respectively, see [9, p. 229]). As observed earlier, ϱ and g are constant along any integral curves of Z (or $\tilde{Z}$). This completes the construction of g and the Riemannian metric of M.

8. Construction of p-subsolutions φ_a

In this section we construct the functions $q_a\colon [0,\infty) \to \mathbb{R},\ a \in \mathbb{R}$, so that the resulting functions φ_a satisfy the conditions in Theorem 2.3. For each fixed $a \in \mathbb{R}$, we first define $q = q_a$ piecewise on intervals $[0, T_0],\ [T_0, T_1],\ [T_1, T_2],\ [T_2, T_3]$, and $[T_3, \infty)$, where $T_0, \dots, T_3$ depend on a and p, and then finally smoothe q in neighborhoods of $T_i,\ i = 0, 1, 2, 3$. We denote both the piecewisely constructed functions and the final smooth functions by the same symbol q.

Recall from (4.2) that

$$f(s) = \max\bigl\{0, \tanh\bigl(\delta(s-a)\bigr)\bigr\},$$

with $\delta = \frac{1}{2(p-1)}$. Furthermore, by (4.5) and (4.9), we have

$$\varphi'_s(s', r, \vartheta) = f'(s)$$

and

$$\varphi''_{ss}(s', r, \vartheta) = f''(s),$$

where

$$s' = s + \int_0^r q(t)\, dt.$$

Hence

$$\frac{(p-1)\varphi''_{ss}(1 + h^2 q^2)}{\varphi'_s h} \geq -\frac{1 + h^2 q^2}{h} \tag{8.1}$$

in $M_a = \cup_{s > a} S_s^{(a)}$.

It is straightforward to check that integral curves of vector fields $R - \tanh r\, S,\ r > 0$, are horizontal (Euclidean) lines, i.e. the x^3-coordinate remains constant along an integral curve. Hence we define $q(r) = q_a(r) = -\tanh r$ for $r \in [0, T_0]$, where T_0 will be chosen later. Then the surfaces $S_s^{(a)}$ coincide with horizontal Euclidean planes $x^3 \equiv e^{-s}$ near L. Consequently, the functions φ_a are smooth in M_a. We notice that $q'_r = -\cosh^{-2} r$ and $1 + h^2 q^2 = \cosh^2 r$. Since $\beta \geq 0$, we obtain, by using (8.1), the following estimate from below for the left hand side of (4.15)

$$\begin{aligned} &\frac{g'_r h(\beta - q)}{g} - (p-1)h q'_r - h'_r q + \frac{(p-2)(h q'_r + h'_r q)}{1 + h^2 q^2} + \frac{(p-1)\varphi''_{ss}(1 + h^2 q^2)}{\varphi'_s h} \\ &\geq\ \coth(\sinh 2\varrho)\sinh(2\varrho)\coth(2\varrho) 2\varrho\varrho'_r \frac{\sinh r}{\varrho} \geq\ \varrho'_r \frac{\sinh r}{\varrho} > 0 \end{aligned} \tag{8.2}$$

in $M_a \cap \{(s', r, \vartheta)\colon 0 < r < T_0\}$. Since $\varrho = r$ and hence $\varrho'_r = 1$ for $0 \leq r \leq 3$, we have $\varrho'_r \varrho^{-1} \sinh r \to 1$ as $r \to 0$.

For $r \in [T_0, T_1]$, we define

$$q(r) = q_a(r) = \frac{-\cosh T_0 \sinh r}{\cosh^2 r}.$$

Then

$$\begin{aligned} q_r'(r) &= \cosh T_0 (\sinh^2 r - 1)\cosh^{-3} r, \\ 1 + h^2 q^2 &= 1 + \cosh^2 T_0 \tanh^2 r, \\ h q_r' &= \cosh T_0 (\tanh^2 r - \cosh^{-2} r), \end{aligned}$$

and

$$h_r' q = -\cosh T_0 \tanh^2 r.$$

Again since $\beta \geq 0$, we get from (8.1) that

$$\begin{aligned} &\frac{g_r' h(\beta - q)}{g} - (p-1) h q_r' - h_r' q + \frac{(p-2)(h q_r' + h_r' q)}{1 + h^2 q^2} + \frac{(p-1)\varphi_{ss}''(1 + h^2 q^2)}{\varphi_s' h} \\ \geq\ & \cosh T_0 \Big(2\varrho_r' \coth(\sinh 2\varrho) \cosh(2\varrho) \tanh r - p \tanh^2 r + \tanh^2 r + (p-1)\cosh^{-2} r \\ &\qquad + \tanh^2 r - \frac{p-2}{\cosh^2 r + \cosh^2 T_0 \sinh^2 r} - \frac{1 + \cosh^2 T_0 \tanh^2 r}{\cosh T_0 \cosh r} \Big) \\ &> \cosh T_0 \Big(\cosh 2r - (p-2)\tanh^2 r - \frac{p-2}{\cosh^2 r + \cosh^2 T_0 \sinh^2 r} - 2 \Big) > 0 \end{aligned} \tag{8.3}$$

in $M_a \cap \{(s', r, \vartheta) \colon T_0 < r < T_1\}$, where $T_0 = T_0(p)$ is large enough. Here we used estimates

$$\frac{1 + \cosh^2 T_0 \tanh^2 r}{\cosh T_0 \cosh r} \leq 2$$

and

$$2\varrho_r' \coth(\sinh 2\varrho) \cosh(2\varrho) \tanh r \geq \cosh 2r$$

for $r \geq T_0$, with T_0 large enough.

For $r \in [T_1, T_2]$, we let $q = q_a$ be a C^∞ continuation of $q|[0, T_1]$ such that

$$\frac{-\cosh T_0 \sinh r}{\cosh^2 r} \leq q \leq 0$$

and

$$0 < \left(\frac{-\cosh T_0 \sinh r}{\cosh^2 r} \right)_r' < q_r' < \frac{\cosh T_0}{\cosh r}.$$

Thus

$$\begin{aligned} 0 < \frac{(\sinh^2 r - 1)\cosh T_0}{\cosh^2 r} &\leq h q_r' \leq \cosh T_0, \\ -\tanh^2 r \cosh T_0 &\leq h_r' q \leq 0, \end{aligned}$$

and

$$-\frac{\cosh T_0}{\cosh^2 r} \leq h q_r' + h_r' q \leq \cosh T_0.$$

Furthermore,

$$1 \leq 1 + h^2 q^2 \leq 1 + \cosh^2 T_0 \tanh^2 r.$$

We choose $T_1 = T_1(a, T_0) > T_0$ large enough so that $s' + \ell(r) \geq 4$ for all $s' \geq a - \log\cosh T_0 - 1$ and $r \geq T_1$, which then implies that for all $s \geq a$ and $r \in [T_1, T_2]$ the point (s', r, ϑ) on any integral curve $\gamma_{a,s}$ of $X^{(a)}$, with

$$s' = s + \int_0^r q(t)\,dt \geq a - \log\cosh T_0 - 1,$$

lies in the set where $\beta(s', r) = \beta_0(r)$. Furthermore, we also require that T_1 is so large that $\beta_0(r)\cosh^2 r \geq 1$ for all $r \geq T_1$. Then in $M_a \cap \{(s', r, \vartheta)\colon T_1 < r < T_2\}$, with T_1 large enough, we have

$$\begin{aligned}
&\frac{g'_r h(\beta - q)}{g} - (p-1)hq'_r - h'_r q + \frac{(p-2)(hq'_r + h'_r q)}{1 + h^2q^2} + \frac{(p-1)\varphi''_{ss}(1 + h^2q^2)}{\varphi'_s h} \\
&\qquad \geq 2(\beta_0 - q)\varrho'_r \coth(\sinh 2\varrho)\cosh(2\varrho)\cosh r - (p-1)\cosh T_0 \\
&\qquad\qquad - \max\bigl((p-2)\cosh^{-2} r, 1\bigr)\cosh T_0 - \frac{1 + \cosh^2 T_0 \tanh^2 r}{\cosh r} \\
&\qquad \geq \cosh 2r \cosh^{-1} r - (p-1)\cosh T_0 - \max\bigl((p-2)\cosh^{-2} r, 1\bigr)\cosh T_0 \\
&\qquad\qquad - \frac{1 + \cosh^2 T_0 \tanh^2 r}{\cosh r} > 0.
\end{aligned} \tag{8.4}$$

Here we used estimates $\beta_0 - q \geq \cosh^{-2} r$ and $2\varrho'_r \coth(\sinh 2\varrho)\cosh(2\varrho) \geq \cosh 2r$ for $r \geq T_1$. The upper interval bound T_2 is determined by $q(T_2) = 0$. Such T_2 exists since q grows strictly faster than

$$\frac{-\cosh T_0 \sinh r}{\cosh^2 r}$$

which tends to zero as $r \to \infty$. Since

$$\int_t^\infty \beta_0(r)\,dr = \infty \quad \text{and} \quad \int_t^\infty \frac{dr}{\beta_0(r)\cosh^2 r} = \infty$$

for every $t > 3$, $\beta_0(r) - 1/\cosh r$ changes its sign infinitely often, more precisely, there are arbitrary large values of r, with $\beta_0(r) - 1/\cosh r = 0$. We let $T_3 > T_2$ be such a zero of $\beta_0 - 1/\cosh$ specified later. For $r \in [T_2, T_3]$ we let $q(r) = 0$. Then

$$\begin{aligned}
&\frac{g'_r h(\beta - q)}{g} - (p-1)hq'_r - h'_r q + \frac{(p-2)(hq'_r + h'_r q)}{1 + h^2q^2} + \frac{(p-1)\varphi''_{ss}(1 + h^2q^2)}{\varphi'_s h} \\
&\qquad \geq \frac{\cosh 2r - 1}{\cosh r} > 0
\end{aligned} \tag{8.5}$$

in $M_a \cap \{(s', r, \vartheta)\colon T_2 < r < T_3\}$.

For $r \geq T_3$ we define $q(r) = \beta_0(r) - 1/\cosh r$. Then $\beta - q = \beta_0 - q = 1/\cosh r$ and

$$\begin{aligned}
&\frac{g'_r h(\beta - q)}{g} - (p-1)hq'_r - h'_r q + \frac{(p-2)(hq'_r + h'_r q)}{1 + h^2q^2} + \frac{(p-1)\varphi''_{ss}(1 + h^2q^2)}{\varphi'_s h} \\
&\quad = 2\varrho'_r \coth(\sinh 2\varrho)\cosh(2\varrho) - (p-1)\bigl(\beta'_0(r)\cosh r + \tanh r\bigr) - \beta_0(r)\sinh r \\
&\qquad\qquad + \tanh r + \frac{(p-2)\bigl(\beta'_0(r)\cosh r + \beta_0(r)\sinh r\bigr)}{2 + \beta_0^2(r)\cosh^2 r - 2\beta_0(r)\cosh r} \\
&\qquad \geq \cosh 2r - (p-1)\bigl(\beta'_0(r)\cosh r + \tanh r\bigr) - \beta_0(r)\sinh r + \tanh r \\
&\qquad\qquad + \frac{(p-2)\bigl(\beta'_0(r)\cosh r + \beta_0(r)\sinh r\bigr)}{2 + \beta_0^2(r)\cosh^2 r - 2\beta_0(r)\cosh r} > 0
\end{aligned} \tag{8.6}$$

in $M_a \cap \{(s', r, \vartheta)\colon r > T_3\}$ if T_3 is large enough. Finally, since the estimates in (8.2)-(8.6) involve q and q'_r but not higher order derivatives of q, it is clear that q can be smoothen in neighborhoods of T_i such that

$$\frac{g'_r h(\beta - q)}{g} - (p-1)hq'_r - h'_r q + \frac{(p-2)(hq'_r + h'_r q)}{1+h^2q^2} + \frac{(p-1)\varphi''_{ss}(1+h^2q^2)}{\varphi'_s h} > 0$$

in M_a. Hence φ_a is a positive p-subsolution in M_a, continuous in M, with $\varphi_a = 0$ in $M \setminus M_a$, and therefore φ_a is a p-subsolution in whole M.

Finally,

$$\int_0^\infty q_a(t)\,dt = \infty$$

for all $a \in \mathbb{R}$ since $q_a(t) = \beta_0(t) - 1/\cosh t$ for $t \geq T_3$,

$$\int_{T_3} \beta_0(t)\,dt = \infty,$$

and

$$\int_{T_3}^\infty \frac{dt}{\cosh t} \leq \int_0^\infty \frac{dt}{\cosh t} = \pi/2.$$

Furthermore,

$$\int_0^r q_a(t)\,dt \leq \int_0^r \beta_0(t)\,dt + \int_0^r \frac{dt}{\cosh t} =: b_r < \infty$$

independently of $a \in \mathbb{R}$. Hence the family $\{\varphi_a\}$ has the desired properties.

9. Construction of p-supersolutions ψ_a

To construct the family of continuous p-supersolutions $\psi_a,\ a \in \mathbb{R}$, as in Theorem 2.3 we first record the following theorem from e.g. [11, Theorem 4.3]:

Theorem 9.1. *Let N be an n-dimensional Cartan-Hadamard manifold with sectional curvatures ≤ -1. Let $\Omega \subset N$ be a domain with C^∞-smooth boundary such that $\bar{\Omega}$ is convex. Then the distance function $\rho\colon N \setminus \bar{\Omega} \to (0,\infty)$,*

$$\rho(x) = \operatorname{dist}(x, \bar{\Omega}),$$

is C^∞ and

$$\Delta\rho \geq (n-1)\tanh\rho \tag{9.1}$$

in $N \setminus \bar{\Omega}$.

Suppose then that $\bar{\Omega} \subset N$ is a convex set and $\rho = \operatorname{dist}(\cdot, \bar{\Omega})$ is a distance function as in Theorem 9.1. Define a continuous function $v\colon N \to [0,1)$ by setting $v = 0$ in $\bar{\Omega}$ and $v(x) = \tanh\bigl(c\rho(x)\bigr)$ for $x \in N \setminus \bar{\Omega}$, where $c = c(p)$ is a positive constant depending only on p. Then in $N \setminus \bar{\Omega}$ we have

$$\nabla v = \frac{c}{\cosh^2(c\rho)}\nabla\rho$$

and

$$|\nabla v| = \frac{c}{\cosh^2(c\rho)},$$

and therefore

$$\begin{aligned}\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) &= c^{p-1}\operatorname{div}\left(\cosh^{2-2p}(c\rho)\nabla\rho\right)\\ &\ge c^{p-1}(n-1)\cosh(c\rho)\tanh\rho - 2c^p(p-1)\cosh^{1-2p}(c\rho)\sinh(c\rho)\\ &= \frac{c^{p-1}}{\cosh^{2p-2}(c\rho)}\left((n-1)\tanh\rho - 2c(p-1)\tanh(c\rho)\right)\end{aligned}$$

by (9.1). Choosing $c=\frac{1}{2(p-1)}$ if $p\ge 2$ and $c=1/2$ if $1<p<2$ yields

$$\operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right)\ge 0$$

in $N\setminus\bar\Omega$. Hence the function $\psi = 1-v$ is a continuous positive p-supersolution in the whole manifold N, $\psi=1$ in $\bar\Omega$, and $\psi(x)\to 0$ as $\operatorname{dist}(x,\bar\Omega)\to\infty$.

Thus to construct the family $\{\psi_a\},\ a\in\mathbb{R}$, it is enough to find appropriate convex sets. This is done in [9] as follows. Denote by α_a any integral curve of $-\nabla_\Theta\Theta = gg'_r(R+\beta S)$ starting at $L(a)$; see the discussion at the beginning of Section 4 for terminology. Furthermore, denote by P_a the surface obtained by rotating α_a around L and let V_a be the component of $M\setminus P_a$ containing points $L(s)$, with $s>a$. Observe that P_a is also obtained by rotating integral curves of $R+\beta S$ starting at $L(a)$ around L. It is proven in [9, p. 235] that $\bar V_a$ is convex for every $a\in\mathbb{R}$. Next we observe that, for each fixed $a\in\mathbb{R}$, the set $M_a=\{x\in M\colon \varphi_a(x)>0\}$ is contained in $\bar V_{a-b}$ for some $b=b(a,p)$. This is seen by comparing the (Fermi) s-coordinates of points (s'',r,ϑ) and (s',r,ϑ) on integral curves α_{a-b} and $\gamma_{a,s},\ s\ge a$, respectively. More precisely, $s'\ge s''$ for all such points (s'',r,ϑ) and (s',r,ϑ) if $b=b(a,p)$ is large enough since $\beta_0(r)-q_a(r)=1/\cosh r$ for $r\ge T_3=T_3(a,p)$ and $\int_0^\infty 1/\cosh r\,dr=\pi/2<\infty$. Finally, for each $a\in\mathbb{R}$, let $\psi_a=1-v_a$, where $v_a=\tanh(c\rho_a)$, where $\rho_a=\operatorname{dist}(\cdot,\bar V_{a-b})$ and $c=c(p)$ are as above. Then, by the discussion above, ψ_a is a continuous positive p-supersolution in M, $0\le\varphi_a\le\psi_a\le 1$, $\psi_a=1$ in $\bar V_{a-b}$, and $\lim_{y\to x}\psi_a(y)=0$ for all $y\in M(\infty)\setminus\{x_0\}$.

In conclusion, the families $\{\varphi_a\}$ and $\{\psi_a\}$ satisfy the conditions in Theorem 2.3, and thus Theorems 1.5, 2.1, 2.2, and 2.3 are proven.

9.1. **Final remarks.** In [7], Arnaudon, Thalmaier, and Ulsamer were able to characterize completely bounded harmonic functions on Borbély's example M by a careful study of the asymptotic behavior of Brownian motion. They proved a stochastic representation formula for bounded harmonic functions on M. For this representation the point $x_0\in M(\infty)$ has to be blown up to $\mathbb{R}\times\mathbb{S}^1$. In particular, there is a one-to-one correspondence between the set of bounded harmonic functions on M and the set of (equivalence classes of) bounded measurable functions on $\mathbb{R}\times\mathbb{S}^1$. Hence the space of bounded harmonic functions on M is as rich as in the case of pinched negative sectional curvature $-b^2\le \operatorname{Sect}\le -a^2<0$. It is an interesting question whether bounded p-harmonic functions on M could be characterized by describing their boundary behavior at the single point $x_0\in M(\infty)$.

We close this paper by a remark on the sharpness of the curvature lower bounds in Theorem 1.4. Recall from Section 6 that the sectional curvature of the 2-plane spanned by (orthonormal) vectors $R, g^{-1}\Theta\in T_{(s,r,\vartheta)}M$ is given by

$$\operatorname{Sect}_{(s,r,\vartheta)}(R,g^{-1}\Theta) = -\frac{g''_{rr}(s,r)}{g(s,r)}.$$

Furthermore, by (6.23)

$$-\frac{g''_{rr}(s,r)}{g(s,r)} \leq -4(\varrho'_r(s,r))^2 \cosh^2 2\varrho(s,r) \leq -4\cosh^2 2\varrho(s,r).$$

As we noticed in Remark 6.1, ϱ coincides with the function f in [7, (2.11)]. Hence, by [7, Lemma 5.3] there exist constants $\gamma > 1$ and $r_0 > 0$ such that

$$\varrho(s,r) \geq (\beta_0(r)\cosh^2 r)^\gamma$$

for all $r \geq r_0$ and $s \geq 0$. Therefore

$$\mathrm{Sect}_{(0,r,\vartheta)}(R, g^{-1}\Theta) \leq -4\cosh^2\big(2(\beta_0(r)\cosh^2 r)^\gamma\big)$$

for $r \geq r_0$. On the other hand, there exists a sequence $r_i \nearrow \infty$ such that $\beta_0(r_i) \leq r_i^{-2}$ since otherwise

$$\int_0^\infty \beta_0(r)\,dr < \infty$$

contradicting with (6.13). Hence

$$\mathrm{Sect}_{(0,r_i,\vartheta)}(R, g^{-1}\Theta) \leq -\exp\big(\tfrac{1}{2}\exp(2r_i)\big)$$

for a sequence $r_i \nearrow \infty$. Finally note that $d\big((0,r,\vartheta), L(0)\big) = r$. Thus there is a large gap between the curvature lower bounds in Theorem 1.4 and even an estimate from above for the curvature lower bound in our example.

References

[1] Ancona, A. Negatively curved manifolds, elliptic operators, and the Martin boundary. *Ann. of Math. (2) 125*, 3 (1987), 495–536.

[2] Ancona, A. Positive harmonic functions and hyperbolicity. In *Potential theory—surveys and problems (Prague, 1987)*, vol. 1344 of *Lecture Notes in Math.* Springer, Berlin, 1988, pp. 1–23.

[3] Ancona, A. Théorie du potentiel sur les graphes et les variétés. In *École d'été de Probabilités de Saint-Flour XVIII—1988*, vol. 1427 of *Lecture Notes in Math.* Springer, Berlin, 1990, pp. 1–112.

[4] Ancona, A. Convexity at infinity and Brownian motion on manifolds with unbounded negative curvature. *Rev. Mat. Iberoamericana 10*, 1 (1994), 189–220.

[5] Anderson, M. T. The Dirichlet problem at infinity for manifolds of negative curvature. *J. Differential Geom. 18*, 4 (1983), 701–721 (1984).

[6] Anderson, M. T., and Schoen, R. Positive harmonic functions on complete manifolds of negative curvature. *Ann. of Math. (2) 121*, 3 (1985), 429–461.

[7] Arnaudon, M., Thalmaier, A., and Ulsamer, S. Existence of non-trivial harmonic functions on Cartan-Hadamard manifolds of unbounded curvature. *Math. Z. 263*, 2 (2009), 369–409.

[8] Borbély, A. A note on the Dirichlet problem at infinity for manifolds of negative curvature. *Proc. Amer. Math. Soc. 114*, 3 (1992), 865–872.

[9] Borbély, A. The nonsolvability of the Dirichlet problem on negatively curved manifolds. *Differential Geom. Appl. 8*, 3 (1998), 217–237.

[10] Cheng, S. Y. The Dirichlet problem at infinity for non-positively curved manifolds. *Comm. Anal. Geom. 1*, 1 (1993), 101–112.

[11] Choi, H. I. Asymptotic Dirichlet problems for harmonic functions on Riemannian manifolds. *Trans. Amer. Math. Soc. 281*, 2 (1984), 691–716.

[12] Eberlein, P., and O'Neill, B. Visibility manifolds. *Pacific J. Math. 46* (1973), 45–109.

[13] Greene, R. E., and Wu, H. *Function theory on manifolds which possess a pole*, vol. 699 of *Lecture Notes in Mathematics.* Springer, Berlin, 1979.

[14] Heinonen, J., Kilpeläinen, T., and Martio, O. *Nonlinear potential theory of degenerate elliptic equations.* The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[15] Holopainen, I. Asymptotic Dirichlet problem for the p-Laplacian on Cartan-Hadamard manifolds. *Proc. Amer. Math. Soc. 130*, 11 (2002), 3393–3400 (electronic).
[16] Holopainen, I., Lang, U., and Vähäkangas, A. Dirichlet problem at infinity on Gromov hyperbolic metric measure spaces. *Math. Ann. 339*, 1 (2007), 101–134.
[17] Holopainen, I., and Vähäkangas, A. Asymptotic Dirichlet problem on negatively curved spaces. *J. Anal. 15* (2007), 63–110.
[18] Hsu, E. P. Brownian motion and Dirichlet problems at infinity. *Ann. Probab. 31*, 3 (2003), 1305–1319.
[19] Hsu, P., and Kendall, W. S. Limiting angle of Brownian motion in certain two-dimensional Cartan-Hadamard manifolds. *Ann. Fac. Sci. Toulouse Math. (6) 1*, 2 (1992), 169–186.
[20] Hsu, P., and March, P. The limiting angle of certain Riemannian Brownian motions. *Comm. Pure Appl. Math. 38*, 6 (1985), 755–768.
[21] Kumura, H. The Dirichlet problem at infinity on Hadamard manifolds. *Nagoya Math. J. 138* (1995), 1–18.
[22] Le, H. Limiting angle of Brownian motion on certain manifolds. *Probab. Theory Related Fields 106*, 1 (1996), 137–149.
[23] Pansu, P. Cohomologie L^p des variétés à courbure négative, cas du degré 1. *Rend. Sem. Mat. Univ. Politec. Torino*, Special Issue (1989), 95–120 (1990). Conference on Partial Differential Equations and Geometry (Torino, 1988).
[24] Sullivan, D. The Dirichlet problem at infinity for a negatively curved manifold. *J. Differential Geom. 18*, 4 (1983), 723–732 (1984).
[25] Vähäkangas, A. Dirichlet problem on unbounded domains and at infinity. Reports in Mathematics, Preprint 499, Department of Mathematics and Statistics, University of Helsinki.
[26] Vähäkangas, A. Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions. *Potential Anal. 27*, 1 (2007), 27–44.

Department of Mathematics and Statistics, P.O. Box 68, FIN-00014 University of Helsinki, Finland.
E-mail address: ilkka.holopainen@helsinki.fi