METRIZATION OF WEIGHTED GRAPHS

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Abstract. We find a set of necessary and sufficient conditions under which the weight $w : E \to \mathbb{R}^+$ on the graph $G = (V, E)$ can be extended to a pseudometric $d : V \times V \to \mathbb{R}^+$. If these conditions hold and G is a connected graph, then the set $\mathfrak{M}_w$ of all such extensions is nonvoid and the shortest-path pseudometric d_w is the greatest element of $\mathfrak{M}_w$ with respect to the partial ordering $d_1 \leqslant d_2$ if and only if $d_1(u, v) \leqslant d_2(u, v)$ for all $u, v \in V$. It is shown that every nonvoid poset $(\mathfrak{M}_w, \leqslant)$ contains the least element $\rho_{0,w}$ if and only if G is a complete k-partite graph with $k \geqslant 2$ and in this case the explicit formula for computation of $\rho_{0,w}$ is obtained.

Key words: Weighted graph, Metric space, Embedding of graph, Shortest-path metric, Infinite graph, Complete k-partite graph.

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1. Introduction

Recall the basic definitions that we adopt here. A graph G is an ordered pair (V, E) consisting of a set $V = V(G)$ of *vertices* and a set $E = E(G)$ of *edges*. In this paper we study the *simple* graphs which are finite, $\text{card}(V) < \infty$, or infinite, $\text{card}(V) = \infty$. Since our graph G is simple we can identify $E(G)$ with a set of two-element subsets of $V(G)$, so that each edge is an unordered pair of distinct vertices. As usual we suppose that $V(G) \cap E(G) = \emptyset$. The edge $e = \{u, v\}$ is said to *join* u and v, and the vertices u and v are called *adjacent* in G. The graph G is *empty* if no two vertices are adjacent, i.e. if $E(G) = \emptyset$. We use the standard definitions of the *path*, the *cycle*, the *subgraph* and *supergraph*, see, for example, [1, p. 4, p. 40]. Note only that all paths and cycles are finite and simple graphs.

The following, basic for us, notion is a *weighted graph* (G, w), i.e., a graph $G = (V, E)$ together with a weight $w : E \to \mathbb{R}^+$ where $\mathbb{R}^+ = [0, \infty)$. If (G, w) is a weighted graph, then for each subgraph F of the graph G define

$$w(F) := \sum_{e \in E(F)} w(e). \tag{1.1}$$

The last sum may be equal $+\infty$ if F is infinite.

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Recall also that a *pseudometric* d on the set X is a function $d : X \times X \to \mathbb{R}^+$ such that $d(x,x) = 0$, $d(x,y) = d(y,x)$ and $d(x,y) \le d(x,z) + d(z,y)$ for all $x, y, z \in X$. The pseudometric d on X is a *metric* if, in addition,

$$(d(x,y) = 0) \Rightarrow (x = y)$$

for all $x, y \in X$. Using a pseudometric d on the set V of vertices of the graph $G = (V, E)$ one can simply define a weight $w : E \to \mathbb{R}^+$ by the rule

$$w(\{u,v\}) := d(u,v) \tag{1.2}$$

for all edges $\{u,v\} \in E(G)$. The correctness of this definition follows from the symmetry of d.

A legitimate question to raise in this point is whether there exists a pseudometric d such that the given weight $w : E \to \mathbb{R}^+$ is produced as in (1.2). If yes, then we say that w is a *metrizable weight*.

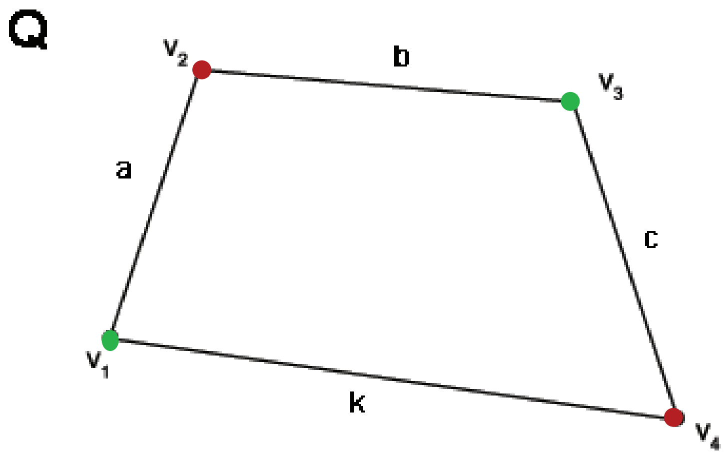

Figure 1. Here (Q, w) is a weighted quadrilateral with $V(Q) = \{v_1, v_2, v_3, v_4\}$, $E(Q) = \{\{v_1,v_2\}, \{v_2,v_3\}, \{v_3,v_4\}, \{v_4,v_1\}\}$ and $w(\{v_1,v_2\}) = a, \quad w(\{v_2,v_3\}) = b, \quad w(\{v_3,v_4\}) = c, \quad w(\{v_4,v_1\}) = k.$

The above formulated question seems to be converse for the question of embeddings of metrics into weighted graphs. (In the standard terminology one says about the *realization* of metric spaces by graphs.) This topic is rich and has many applications in various areas, such as psychology, phylogenetic analysis and recent applications from the field of computer science. Some results and references in this direction can be found in [2] and [5].

If (G, w) is a weighted graph with metrizable w, then we shall denote by $\mathfrak{M}_w$ the set of all pseudometrics $d : V \times V \to \mathbb{R}^+$ such that

$$d(v_i, v_j) = w(\{v_i, v_j\})$$

for all $\{v_i, v_j\} \in E(G)$.

The starting point of our considerations is the following Model Example.

Theorem 1.3 (Model Example)**.** *Let (Q, w) be a weighted graph depicted by Figure 1. The weight w on the graph Q is metrizable if and only if*

$$2 \max\{a, b, c, k\} \leqslant a + b + c + k. \tag{1.4}$$

If w is metrizable, then for each $d \in \mathfrak{M}_w$ we have the double inequalities

$$\max\{|b - c|, |a - k|\} \leqslant d(v_2, v_4) \leqslant \min\{b + c, a + k\}$$

and

$$\max\{|a - b|, |c - k|\} \leqslant d(v_1, v_3) \leqslant \min\{a + b, c + k\}. \tag{1.5}$$

Conversely, if p and q are real numbers such that

$$\max\{|b - c|, |a - k|\} \leqslant p \leqslant \min\{b + c, a + k\}$$

and

$$\max\{|a - b|, |c - k|\} \leqslant q \leqslant \min\{a + b, c + k\}, \tag{1.6}$$

then there is $d \in \mathfrak{M}_w$ with

$$d(v_2, v_4) = p \qquad , \qquad d(v_1, v_3) = q.$$

This theorem was proved in [4] and used there as a base to finding of extremally Ptolemeic and extremally non-Ptolemeic metric spaces. The results of the present paper generalize the Model Example to the case of arbitrary (finite or infinite) weighted graphs (G, w).

– Theorem 2.2 gives necessary and sufficient conditions under which a weight w is metrizable. The key point here is an extension of inequality (1.4) to an arbitrary cycle $C \subseteq G$.
– Proposition 3.3 claims that for connected G and metrizable w the shortest-path pseudometric d_w belongs to $\mathfrak{M}_w$ and that this pseudometric is the greatest element of $\mathfrak{M}_w$. The reader can observe that the right-side in double inequalities (1.5) and (1.6) are, in fact, $d_w(v_2, v_4)$ and $d_w(v_1, v_3)$.
– Theorem 4.3 shows that the least pseudometric in $\mathfrak{M}_w$, (see the left-side in (1.5), (1.6)) exists for each metrizable w if and only if G is a complete k-partite graph with $k \geqslant 2$.
– In Theorem 4.36 we show that for complete k-partite graphs G with $k \geqslant 2$ and with the cardinality of partitions $\leqslant 2$ we have the analog of the last part of the Model Example: a symmetric function $f : V \times V \to \mathbb{R}^+$ belongs to $\mathfrak{M}_w$ if and only if it "lies between" the greatest element of $\mathfrak{M}_w$ and the least one.

Moreover in Theorem 3.11 we describe the structure of connected graphs G which admit strictly positive metrizable weights w such that $\mathfrak{M}_w$ does not contain any metrics.

2. Embeddings of weighted graphs into pseudometric spaces

Let (G, w) be a weighted graph and let u, v be vertices belonging to a connected component of G. Let us denote $\mathcal{P}_{u,v} = \mathcal{P}_{u,v}(G)$ the set of all paths joining u and v in G. Write

$$d_w(u, v) := \inf\{w(F) : F \in \mathcal{P}_{u,v}\} \tag{2.1}$$

where $w(F)$ is the weight of the path F, see formula (1.1). It is well known for the connected graph G that the function d_w is a pseudometric on the set $V(G)$. This pseudometric will be termed as the *weighted shortest-path pseudometric*. It coincides with the usual path metric if $w(e) = 1$ for all $e \in E(G)$.

Theorem 2.2. *Let (G, w) be a weighted graph. The following statements are equivalent.*

(i) *The weight w is metrizable.*
(ii) *The equality*

$$w(\{u, v\}) = d_w(u, v) \tag{2.3}$$

holds for all $\{u, v\} \in E(G)$.
(iii) *For every cycle $C \subseteq G$ we have the inequality*

$$2 \max_{e \in E(C)} w(e) \leq w(C). \tag{2.4}$$

It seems to be interesting to have conditions under which the set $\mathfrak{M}_w$ contains some metrics of a special type. In particular: What are restrictions on the weight w guaranteeing the existence of ultrametrics (or pseudoultrametrics) in the set $\mathfrak{M}_w$?

Remark 2.5. *If C is a 3-cycle, then (2.4) turns to the symmetric form*

$$2 \max\{w(e_1), w(e_2), w(e_3)\} \leq w(e_1) + w(e_2) + w(e_3)$$

of the triangle inequality. Thus (2.4) can be considered as a "cyclic generalization" of this inequality.

Remark 2.6. *Theorem 2.2 evidently holds if G is the null graph, i.e. if $V(G) = \emptyset$. In this case the related metric space (V, d) is empty.*

Proof of Theorem 2.2. **(i)⇒(ii)** Suppose that there is a pseudometric ρ on V such that

$$w(\{u, v\}) = \rho(u, v)$$

for each $\{u,v\} \in E(G)$. Then for every sequence of points $v_1, \dots v_n,\ v_1 = u$ and $v_n = v,\ v_i \in V,\ i = 1, \dots, n$, the triangle inequality implies

$$w(\{u,v\}) = \rho(v_1, v_n) \le \sum_{i=1}^{n-1} \rho(v_i, v_{i+1}).$$

Consequently for paths $F \subseteq G$ joining u and v the inequality

$$w(\{u,v\}) \le w(F)$$

holds. Passing in the last inequality to the infimum over the set $\{w(F) : F \in \mathcal{P}_{u,v}\}$ we obtain

$$\rho(u,v) = w(\{u,v\}) \le d_w(u,v), \tag{2.7}$$

see (2.1). The converse inequality $w(\{u,v\}) \ge d_w(u,v)$ holds because the path $(u = v_1, v_2 = v)$ belongs to $\mathcal{P}_{u,v}$.

(ii)⇒(iii) Suppose statement (ii) holds. Let C be an arbitrary cycle in G and let $\{u,v\} \in E(C)$ be an edge for which

$$w(\{u,v\}) = \max_{e \in E(C)} w(e). \tag{2.8}$$

Deleting the edge $\{u,v\}$ from the cycle C we obtain the path $F := C \setminus \{u,v\}$ joining the vertices u and v. Using equalities (2.1), (2.3) and (2.8) we conclude that

$$\max_{e \in E(C)} w(e) = d_w(u,v) \le w(F). \tag{2.9}$$

Since $w(F) = w(C) - w(\{u,v\})$, (2.4) follows from (2.9).

(iii)⇒(i) Suppose (iii) is true. If G is a connected graph, then we can equip G by the weighted shortest-path pseudometric d_w, so it is sufficient to show that $d_w \in \mathfrak{M}_w$. Let $\{u,v\} \in E(G)$. In the case where there is no cycle $C \subseteq G$ such that $\{u,v\} \in E(C)$ the path $(u = v_1, v_2 = v)$ is the unique path joining u and v. Hence, in this case, equality (2.3) follows from (2.1). Let $P = (u = v_1, \dots, v_{k+1} = v)$ be an arbitrary k-path, $k \ge 2$, joining u and v. Then $C := (u = v_1, \dots, v_{k+1}, v_{k+2} = u)$ is a $k+1$-cycle with $\{u,v\} \in E(C)$. Hence by (2.4) we have

$$2w(\{u,v\}) \le 2 \max_{e \in E(C)} w(e) \le w(C) = w(P) + w(\{u,v\}).$$

This implies the inequality $w(\{u,v\}) \le w(P)$ for all $P \in \mathcal{P}_{u,v}$. Consequently $w(\{u,v\}) \le d_w(u,v)$. The converse inequality is trivial. Thus if G is connected, then $d_w \in \mathfrak{M}_w$.

Consider now the case of disconnected graph G. Let $\{G_i : i \in \mathcal{I}\}$ be the set of all components of G and let $\{v_i^* : i \in \mathcal{I}\}$ be the subset of $V(G)$ such that

$$v_i^* \in V(G_i)$$

for each $i \in \mathcal{I}$. We choose an index $i_0 \in \mathcal{I}$ and fix nonnegative constants $a_i,\ i \in \mathcal{I}$ such that $a_{i_0} = 0$. Let us define the function $\rho : V(G) \times V(G) \to \mathbb{R}^+$

as

$$\rho(u,v) = d_{w_i}(u,v) \tag{2.10}$$

if u and v lie in the same component G_i and as

$$\rho(u,v) = a_i + a_j + d_{w_i}(u,v_i^*) + d_{w_j}(v,v_j^*) \tag{2.11}$$

if $u \in G_i$ and $v \in G_j$ with $i \neq j$. Here w_i is the restriction of w on the set $E(G_i)$ and d_{w_i} is the weighted shortest-path pseudometric corresponding to the weight w_i. It is easy to see that

$$a_i = \rho(v_i^*, v_{i_0}^*) \tag{2.12}$$

for all $i \in \mathcal{I}$.

It follows directly from (2.10) and (2.11) that ρ is a pseudometric on $V(G)$ and $\rho \in \mathfrak{M}_w$. □

Remark 2.13. *To obtain the pseudometric ρ described by formulas* (2.10), (2.11) *we can consider the supergraph G^* of G such that $V(G^*) = V(G)$ and*

$$E(G^*) = E(G) \cup \{\{v_i^*, v_{i_0}^*\} : i \in \mathcal{I} \setminus \{i_0\}\},$$

see Fig. 2. Then G^* is a connected graph with the same set of cycles as in G and all edges $\{v_i^*, v_{i_0}^*\}$ are bridges of G^*. Now we can extend the weight $w : E(G) \to \mathbb{R}^+$ to a weight $w^* : E(G^*) \to \mathbb{R}^+$ by the rule:

$$w^*(\{u,v\}) := \begin{cases} w(\{u,v\}) & \text{if } \{u,v\} \in E(G) \\ a_i & \text{if } \{u,v\} = \{v_i^*, v_{i_0}^*\},\ i \in \mathcal{I} \setminus \{i_0\}. \end{cases}$$

It can be shown that the pseudometric ρ is simply the weighted shortest-path pseudometric with respect to the weight w^*.

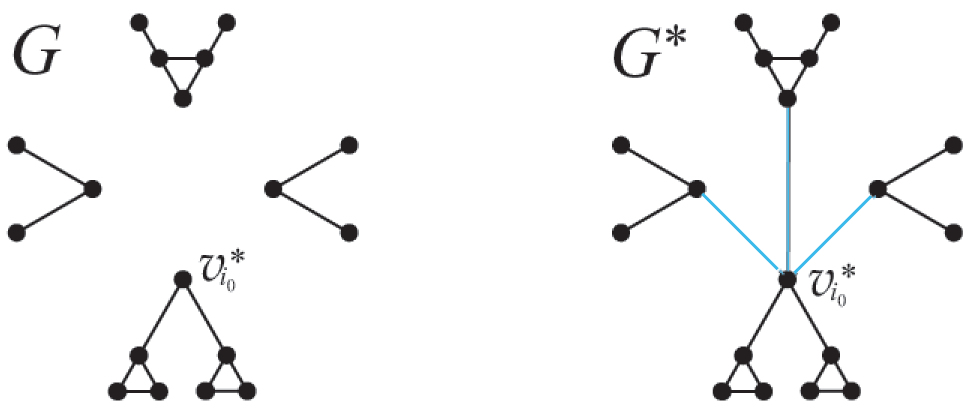

Figure 2. Inclusion of the disconnected G in the connected G^*. There are no new cycles in G^*.

Let w_1 and w be two weights with the same underlieing graph G. Suppose the weight w_1 is metrizable. What are condition under which the weight w is also metrizable?

To describe such type conditions we recall the definition of a *bridge*.

Definition 2.14. *Let G be a graph and let $e_0 \in E(G)$. For a connected G, e_0 is a bridge of G, if $G - e_0$ is a disconnected graph. If G is disconnected and G_0 is the connected component of G such that $e_0 \in E(G_0)$, then e_0 is a bridge of G, if $G_0 - e_0$ is disconnected.*

Above we denote by $G - e_0$ the *edge-deleted* subgraph of G, see, for example, [1, p. 40].

For weights w_1 and w_2 on $E(G)$ define a set $w_1 \Delta w_2 \subseteq E(G)$ as

$$w_1 \Delta w_2 = \{e \in E(G) : w_1(e) \neq w_2(e)\}.$$

Proposition 2.15. *Let (G, w_1) be a weighted graph with a metrizable w_1 and let E_1 be a subset of $E(G)$. The following statements are equivalent.*

(i) *All weights $w : E(G) \to \mathbb{R}^+$ with $w_1 \Delta w_2 \subseteq E_1$ are metrizable.*
(ii) *Each element $e \in E_1$ is a bridge of the graph G.*

Lemma 2.16. *An edge $e \in E(G)$ is a bridge if and only if e is not in $E(C)$ for any cycle $C \subseteq G$.*

This lemma is known for the finite graphs G, see [6, Theorem 2.3]. The proof for infinite G is completely analogous, so we omit it here.

Proof of Proposition 2.15. The implication (ii)⇒(i) follows directly from condition (iii) of Theorem 2.2 and Lemma 2.16. Conversely, if some $e_0 \in E_1$ is not bridge, then by Lemma 2.16 there is a cycle $C_0 \subseteq G$ such that $e_0 \in E(C_0)$. Let us define the weight $w_0 : E(G) \to \mathbb{R}$,

$$w_0(e) = \begin{cases} w_1(e) & \text{if} \quad e \neq e_0 \\ 1 + w(C_0) - w(e_0) & \text{if} \quad e = e_0. \end{cases}$$

Then we have $w_1 \Delta w_0 = \{e_0\}$ and

$$2w_0(e_0) = 2 + 2w_1(C_0) - 2w_1(e_0) > \\ (w_1(C_0) - w_1(e_0) + 1) + (w_1(C_0) - w_1(e_0)) = w_0(C_0).$$

It is clear, that $w_1 \Delta w_0 \subseteq E_1$ but, by Theorem 3.11, the weight w_0 is not metrizable. Thus the implication (i)⇒(ii) follows. □

Recall that *acyclic* graphs are usually called the *forests*. Lemma 2.16 implies that a graph G is a forest if and only if all $e \in E(G)$ are bridges of G. Hence as a particular case of Proposition 2.15 we obtain

Corollary 2.17. *The following conditions are equivalent for every graph G.*

(i) *G is a forest.*
(ii) *Every weight $w : E(G) \to \mathbb{R}^+$ is metrizable.*

Our final corollary shows that the property of a weight to be metrizable is local.

Corollary 2.18. *Let (G, w_G) be a weighted graph. The weight w_G is metrizable if and only if the restrictions $w_H = w_G|_{E(H)}$ are metrizable for all finite subgraphs H of the graph G.*

3. Maximality of the weighted shortest-path pseudometric

Let G be a graph and let w be a metrizable weight on $E(G)$. Recall that $\mathfrak{M}_w$ is the set of all pseudometrics ρ on $V(G)$ satisfying the restriction

$$\rho(u, v) = w(\{u, v\}) \tag{3.1}$$

for each $\{u, v\} \in E(G)$. Let us introduce the ordering relation $\leqslant$ on the set $\mathfrak{M}_w$ as

$$(\rho_1 \leqslant \rho_2) \quad \text{if and only if} \quad (\rho_1(u, v) \leqslant \rho_2(u, v)) \tag{3.2}$$

for all $u, v \in E(G)$. A reasonable question to ask is whether it is possible to find the greatest and least elements of the partially ordered set $(\mathfrak{M}_w, \leqslant)$.

In the present section we show that the shortest-path pseudometric d_w is the greatest element in $(\mathfrak{M}_w, \leqslant)$ for a connected G and apply this result to the search of metrics in $\mathfrak{M}_w$. The existence of the least element of the poset $(\mathfrak{M}_w, \leqslant)$ will be discussed in Section 4.

Proposition 3.3. *Let (G, w) be a nonempty weighted graph with a metrizable weight w. If G is connected then the weighted shortest-path pseudometric d_w belongs to $\mathfrak{M}_w$ and this pseudometric is the greatest element of the poset $(\mathfrak{M}_w, \leqslant)$, i.e., the inequality*

$$\rho \leqslant d_w \tag{3.4}$$

holds for each $\rho \in \mathfrak{M}_w$. Conversely, if the poset $(\mathfrak{M}_w, \leqslant)$ contains the greatest element, then G is connected.

Proof. In fact, for connected G, the membership relation $d_w \in \mathfrak{M}_w$ was justified in the third part of the proof of Theorem 2.2. To prove (3.4) see (2.7).

If G is disconnected and some vertices u, v lie in distinct components, $u \in G_i,\ v \in G_j$, then letting $a_i, a_j \to +\infty$ in (2.11) we obtain

$$\sup\{\operatorname{diam}_\rho(A) : \rho \in \mathfrak{M}_w\} = \infty$$

for the two-element set $A = \{u, v\}$. Thus the poset $(\mathfrak{M}_w, \leqslant)$ does not contain the greatest element. □

Remark 3.5. *If G is a disconnected graph and u, v belong to distinct connected components of G, then according to* (2.1) *we can put*

$$d_w(u, v) = +\infty$$

as for the infimum over the empty set. Under this agreement, the weighted shortest-path pseudometric is also "the greatest element" of $\mathfrak{M}_w$ for the disconnected graphs G.

Recall that connected acyclic graphs are called the trees, so that each tree is a connected forest. The last proposition and Corollary 2.17 imply

Corollary 3.6. *A graph G is a tree if and only if each weight $w : E(G) \to \mathbb{R}^+$ is metrizable and the inequality*

$$\sup\{\operatorname{diam}_\rho(A) : \rho \in \mathfrak{M}_w\} < \infty$$

holds for every finite $A \subseteq V(G)$.

Let (G, w) be a weighted graph with a metrizable w. Then each $\rho \in \mathfrak{M}_w$ is a *pseudometric* satisfying (3.1). We ask under what conditions does a **metric** $\rho \in \mathfrak{M}_w$ exist. To this end it is necessary for the weight $w : E(G) \to \mathbb{R}^+$ to be *strictly positive* in the sense that $w(e) > 0$ for all $e \in E(G)$. This trivial condition is also sufficient for the graphs with the vertices of *finite degrees*. Here, as usual, by the degree of a vertex v we understand the cardinal number of edges incident with v. More generally we have

Corollary 3.7. *Let (G, w) be a weighted graph such that each connected component of G contains at most one vertex of infinite degree. If the weight w is metrizable and strictly positive, then there is a metric $\rho \in \mathfrak{M}_w$.*

Proof. Suppose G is connected and w is strictly positive and metrizable. Let $\{u, v\} \notin E(G)$. Without loss of generality we may suppose that the edges of G which are incident with u form the finite set $\{e_1, \ldots, e_n\}$. Then the inequalities

$$w(F) > \min_{1 \leqslant i \leqslant n} w(e_i) > 0$$

holds for each path $F \in \mathcal{P}_{u,v}$. Thus $d_w(u, v) > 0$ for every pair of distinct $u, v \in V(G)$. It still remains to note that $d_w \in \mathfrak{M}_w$ by Proposition 3.3.

For the case of disconnected G we can obtain the desirable metric $\rho \in \mathfrak{M}_w$ using (2.10) and (2.11) with strictly positive a_i, a_j. □

Remark 3.8. *The main point of the previous proof is the following: If there is a metric $\rho \in \mathfrak{M}_w$, then the weighted shortest-path pseudometric is also a metric.*

The following example shows that the conclusion of Corollary 3.7 is, generally speaking, false for connected graphs G, containing at least two vertices of infinite degree.

Example 3.9. *Let (G, w) be the infinite weighted graph depicted by Fig. 3 where $\varepsilon_n = w(\{v_n, u_1\}) = w(\{v_n, u_2\})$, are real numbers such that*

$$\lim_{n \to \infty} \varepsilon_n = 0$$

and $\varepsilon_n > \varepsilon_{n+1} > 0$ for each $n \in \mathbb{N}$. Each cycle C of G is a quadrilateral of the form u_1, v_n, u_2, v_m, u_1 with $n \neq m$. Since C has two distinct edges

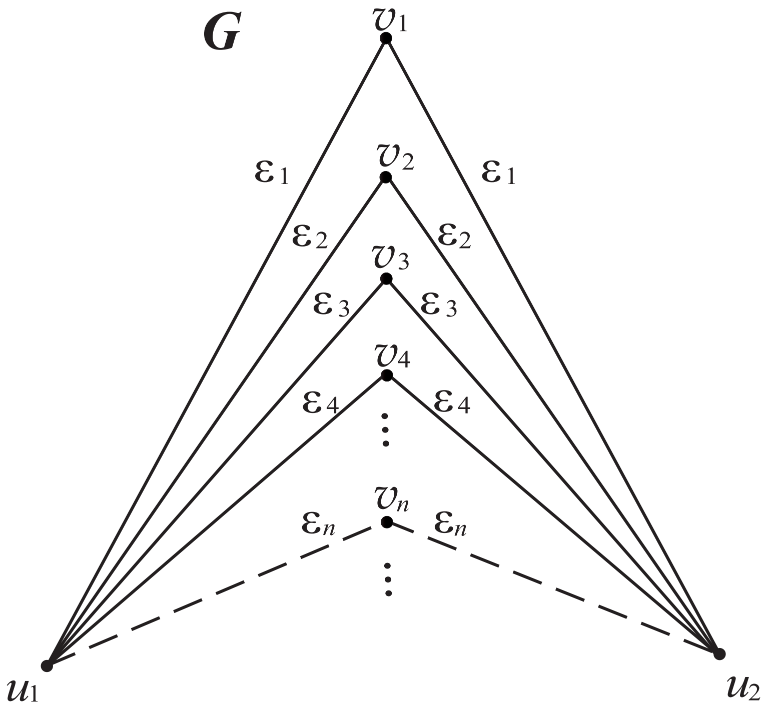

Figure 3. The infinite G with metrizable w having no metrics in $\mathfrak{M}_w$.

of the maximal weight, inequality (2.4) *holds. By Theorem 2.2 it means that w is metrizable. Letting $n, m \to \infty$ and using formula* (2.1) *we obtain $d_w(u_1, u_2) = 0$. Consequently d_w is a pseudometric so, in accordance with Remark 3.8, there are no metrics in $\mathfrak{M}_w$.*

To describe the characteristic structural properties of graphs G having metrics in $\mathfrak{M}_w$ for each metrizable $w : H(G) \to \mathbb{R}^+$ we recall some definitions.

Given an infinite sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets, we call the *upper limit* of this sequence, $\limsup_{n\to\infty} A_n$, the set of all elements a such that $a \in A_n$ holds for an infinity of values of the index n. We have

$$\limsup_{n\to\infty} A_n = \bigcap_{k=1}^{\infty} \left(\bigcup_{n=1}^{\infty} A_{n+k} \right). \tag{3.10}$$

Let G be a graph. A set F of vertices of G is *independent* if every two vertices in F are nonadjacent.

Theorem 3.11. *Let $G = (V, E)$ be an infinite connected graph. The following two conditions are equivalent.*

(i) *There is a strictly positive metrizable weight w such that each $\rho \in \mathfrak{M}_w$ is not metric but pseudometric only.*

(ii) *There are two vertices $u^*, v^* \in V(G)$ and a sequence $\tilde{F}$ of paths F_j, $j \in \mathbb{N}$, joining u^* and v^* such that the upper limit of the sequence $\{V(F_j)\}_{j\in\mathbb{N}}$ is an independent set.*

Remark 3.12. *It is clear that the relations*

$$u^*, v^* \in \limsup_{n\to\infty} V(F_n)$$

hold for each sequence $\tilde{F} = \{F_j\}_{j\in\mathbb{N}}$ joining u^* and v^*. Hence vertices u^* and v^* are nonadjacent if condition (ii) of Theorem 3.11 holds.

Using (3.10) we can reformulate the last part of (*ii*) in the form:

(ii_1) *for every $e^0 \in E(G)$ there are $u^0 \in e^0$ and $i = i(e^0)$ such that*

$$u^0 \notin \bigcup_{k=1}^{\infty} V(F_{i+k}). \tag{3.13}$$

Lemma 3.14. *Let G be an infinite connected graph, let u^* and v^* be two distinct nonadjacent vertices of G and let $\tilde{F} = \{F_j\}_{j\in\mathbb{N}}$ be a sequence of paths joining u^* and v^* such that (ii_1) holds. Then there is a subsequence $\{F_{j_k}\}_{k\in\mathbb{N}}$ of $\tilde{F}$ such that:*

(ii_2) *the equality*

$$E(F_{j_l}) \cap E(F_{j_k}) = \emptyset \tag{3.15}$$

holds whenever $l \neq k$;

(ii_3) *if a cycle C is contained in the graph $\bigcup_{k\in\mathbb{N}} F_{j_k}$,*

$$C \subseteq \bigcup_{k\in\mathbb{N}} F_{j_k}, \tag{3.16}$$

and

$$k_0 = k_0(C) := \min\{k \in \mathbb{N} : E(C) \cap E(F_{j_k}) \neq \emptyset\}, \tag{3.17}$$

then C and $F_{j_{k_0}}$ have at least two common edges.

Proof. For every $e \in E(G)$ define a set

$$N(e) := \{j \in \mathbb{N} : E(F_j) \ni e\}.$$

Condition (ii_1) implies that $N(e)$ is finite for each $e \in E(G)$. Now we construct a subsequence $\{F_{j_k}\}_{k\in\mathbb{N}}$ by induction. Write $j_1 := 1$ and

$$M_1 := \bigcup_{e\in F_{j_1}} N(e).$$

Since all $N(e)$ are finite, the set $M_1 \subseteq \mathbb{N}$ is also finite. Let j_2 be the least natural number in the set $\mathbb{N} \setminus M_1$. Write

$$M_2 := \bigcup_{e\in F_{j_2}} N(e), \quad j_3 = \min\{m : m \in \mathbb{N} \setminus (M_1 \cup M_2)\};$$

$$M_3 := \bigcup_{e \in F_{j_3}} N(e), \quad j_4 = \min\{m : m \in \mathbb{N} \setminus (M_1 \cup M_2 \cup M_3)\}$$

and so on. It is plain to show that (3.15) holds for distinct j_k and j_e. Thus the subsequence $\{F_{j_k}\}_{k\in\mathbb{N}}$ satisfies (ii_2). To construct a subsequence of $\tilde{F}$ which satisfies simultaneously (ii_2) and (ii_3), note that condition (ii_1) remains valid when one passes from the sequence $\tilde{F}$ to any of its subsequences. Hence, without loss of generality, we may assume that $\{F_{j_k}\}_{k\in\mathbb{N}} = \tilde{F}$.

To define a new subsequence $\{F_{j_k}\}_{k\in\mathbb{N}}$ we again use the induction. Put $j_1 := 1$. Suppose j_k are defined for $k = 1, \ldots, n-1$. By (ii$_1$) for every $e \in F_{j_{n-1}}$ there are $i(e) \in \mathbb{N}$ and $u \in e$ such that

$$u \notin \bigcup_{k=1}^{\infty} V(F_{i(e)+k}). \tag{3.18}$$

Define

$$j_n := 1 + \max_{e \in F_{j_{n-1}}} i(e). \tag{3.19}$$

Note that $j_n > j_{n-1}$.

Suppose that a cycle C is contained in $\bigcup_{k\in\mathbb{N}} F_{j_k}$ where $\{F_{j_k}\}_{k\in\mathbb{N}}$ is above constructed subsequence of $\tilde{F}$ and $k_0 = k_0(C)$ is defined by (3.17) but $F_{j_{k_0}}$ contains the unique edge $e = \{u, v\}$ from $E(C)$. Let $e_1, e_2 \in E(C)$ be the distinct edges which are adjacent to e. The uniqueness of e, (3.16) and (3.17) imply the relations

$$e_1 \in \bigcup_{k>k_0} E(F_{j_k}) \qquad \text{and} \qquad e_2 \in \bigcup_{k>k_0} E(F_{j_k}). \tag{3.20}$$

If $e_1 = \{u_1, v_1\}$ and $e_2 = \{u_2, v_2\}$, then

$$u \in \{u_1, v_1, u_2, v_2\} \qquad \text{and} \qquad v \in \{u_1, v_1, u_2, v_2\}$$

so that from (3.20) we obtain

$$u \in \bigcup_{k>k_0} V(F_{j_k}) \qquad \text{and} \qquad v \in \bigcup_{k>k_0} V(F_{j_k})$$

contrary to (3.18) and (3.19). □

Proof of Theorem 3.11. **(i)⇒(ii)** Let w be a strictly positive metrizable weight such that each $\rho \in \mathfrak{M}_w$ is a pseudometric only. Hence d_w is not metric, so for some distinct $u^*, v^* \in V(G)$ we have

$$d_w(u^*, v^*) = 0. \tag{3.21}$$

From the definition of d_w it follows at once that there is a sequence $\tilde{F} = \{F_i\}_{i\in\mathbb{N}}$, $F_i \in \mathcal{P}_{u^*,v^*}$, $i \in \mathbb{N}$, such that

$$\lim_{i\to\infty} w(F_i) = 0. \tag{3.22}$$

Since all F_i are finite, we may suppose, taking a subsequence of $\tilde{F}$ if it is necessary, that

$$\min_{e \in E(F_j)} w(e) > \sum_{i=1}^{\infty} w(F_{i+j}) \tag{3.23}$$

for all $j \in \mathbb{N}$. We claim that condition (ii_1) is fulfilled by $\tilde{F}$.

Assume there is $e^0 = \{u^0, v^0\} \in E(G)$ such that

$$u^0, v^0 \in \bigcup_{k=1}^{\infty} V(F_{i+k})$$

for each $i \in \mathbb{N}$. Since all F_i are paths joining u^* and v^*, there is an u^0v^0-walk in the graph

$$\tilde{G}_i := \bigcup_{k=1}^{\infty} F_{i+k}. \tag{3.24}$$

It is well known that if there is an xy-walk in a graph, then there is also a path joining x and y in the same graph, see, for example [1, p. 82]. Let P_i be a path joining u^0 and v^0 in $\tilde{G}_i$. The weight w is metrizable. Consequently we may use Theorem 2.2. Equalities (2.1)–(2.3) imply

$$w(\{u^0, v^0\}) \leq w(P_i) \leq \sum_{k=1}^{\infty} w(F_{i+k}).$$

Letting $i \to \infty$ and using (3.22), (3.23) we obtain

$$w(\{u^0, v^0\}) \leq \lim_{i \to \infty} \sum_{k=1}^{\infty} w(F_{i+k}) = 0,$$

contrary to the condition: $w(e) > 0$ for all $e \in E(G)$.

(ii)⇒(i) Let G be a graph satisfying condition (ii). In view of Lemma 3.14 we may suppose that (ii_2) and (ii_3) are also fulfilled with $\{F_{j_k}\}_{k \in \mathbb{N}} = \tilde{F}$ where $\tilde{F}$ is the sequence of paths appearing in (ii). To verify condition (i) it suffices, by Proposition 3.3, to find a metrizable weight $w : V(G) \to \mathbb{R}^+$ such that $w(e) > 0$ for all $e \in E(G)$ and

$$d_w(u^*, v^*) = 0$$

for some distinct vertices u^* and v^*.

In the rest of the proof we will construct the desired weight w.

Let us consider the graph

$$\tilde{G} = \bigcup_{i \in \mathbb{N}} F_i,$$

cf. (3.24). Denote by $m(F_i)$, $i \in \mathbb{N}$, the number of edges of F_i. Let $\{\varepsilon_i\}_{i \in \mathbb{N}}$ be a decreasing sequence of positive real numbers such that $\lim_{i \to \infty} \varepsilon_i = 0$

and that the sequence $\{\frac{\varepsilon_i}{m(F_i)}\}_{i\in\mathbb{N}}$ is also decreasing. Define a weight $w_1 : E(\tilde{G}) \to \mathbb{R}^+$ as

$$w_1(e) := \frac{\varepsilon_i}{m(F_i)}, \quad \text{if } e \in E(F_i). \tag{3.25}$$

This definition is correct because, by (ii$_2$), the edge sets $E(F_i)$ and $E(F_j)$ are disjoint for distinct i and j.

Let us note that the weight w_1 is metrizable. It follows from Theorem 2.2 because (ii$_3$), (3.25) and the decrease of the sequence $\{\frac{\varepsilon_i}{m(F_i)}\}_{i\in\mathbb{N}}$ imply inequality (2.4) for every cycle C in $\tilde{G}$. (As has been stated above, (2.4) holds if there are two distinct edges e_1, e_2 in C such that $w(e_1) = w(e_2) = \max_{e\in E(C)} w(e)$. To find these e_1 and e_2 we can use (ii$_3$).)

Let d_{w_1} be the weighted shortest-path pseudometric generated by the weight w_1. The condition $\lim_{i\to\infty} \varepsilon_i = 0$ implies $d_{w_1}(u^*, v^*) = 0$. Indeed, we have

$$d_{w_1}(u^*, v^*) \le \inf_{i\in\mathbb{N}} w_1(F_i) \le \overline{\lim_{i\to\infty}} \sum_{e\in E(F_i)} w_1(e)$$
$$= \lim_{i\to\infty} m(F_i)\frac{\varepsilon_i}{m(F_i)} = \lim_{i\to\infty} \varepsilon_i = 0.$$

Let $e^0 = \{u^0, v^0\} \in E(G)$ with $u^0, v^0 \in V(\tilde{G})$. We will use (ii$_2$) to prove the inequality

$$d_{w_1}(u^0, v^0) > 0. \tag{3.26}$$

Condition (ii$_1$) implies at least one from the relations

$$u^0 \notin \bigcup_{k=1}^{\infty} V(F_{i+k}) \qquad , \qquad v^0 \notin \bigcup_{k=1}^{\infty} V(F_{i+k})$$

for sufficiently large i. Suppose, for instance, that there is $i_0 = i_0(e_0) \in \mathbb{N}$ such that

$$u^0 \notin V(F_i) \quad \text{if } i > i_0. \tag{3.27}$$

Let F be an arbitrary path in $\tilde{G}$ joining u^0 and v^0 and let $e \in E(F)$ be the edge incident with the end u^0. From (3.27) follows

$$e \in \bigcup_{i=1}^{i_0} E(F_i).$$

Using this relation, (3.25) and the decrease of the sequence $\{\frac{\varepsilon_i}{m(F_i)}\}_{i\in\mathbb{N}}$ we obtain

$$d_{w_1}(u^0, v^0) \ge \frac{\varepsilon_{i_0}}{m(F_{i_0})} > 0,$$

so that (3.26) holds.

Write

$$V' := V(G) \setminus V(\tilde{G}).$$

If $V' = \emptyset$, then the desirable strictly positive weight $w : E(G) \to \mathbb{R}^+$ can be obtained as

$$w(\{u,v\}) := d_{w_1}(u,v), \quad \{u,v\} \in E(G)$$

because as was shown above $d_{w_1}(u,v) > 0$ for each $\{u,v\} \in E(G)$. The weight w is metrizable because it is a "restriction" of the pseudometric d_{w_1}. It is easy to show also that

$$d_{w_1}(u,v) = d_w(u,v) \tag{3.28}$$

for all $u, v \in V(G)$. Indeed, since $d_{w_1} \in \mathfrak{M}_w$ the inequality

$$d_{w_1}(u,v) \le d_w(u,v)$$

follows from Proposition 3.3. To prove the converse inequality note that

$$\mathcal{P}_{u,v}(\tilde{G}) \subseteq \mathcal{P}_{u,v}(G)$$

because $\tilde{G}$ is a subgraph of G. Consequently,

$$\begin{aligned} d_w(u,v) = \inf\{w(F) : F \in \mathcal{P}_{u,v}(G)\} &\le \inf\{w(F) : F \in \mathcal{P}_{u,v}(\tilde{G})\} \\ &= \inf\{w_1(F) : F \in \mathcal{P}_{u,v}(\tilde{G})\} = d_{w_1}(u,v). \end{aligned} \tag{3.29}$$

Equality (3.28) implies, in particular, that $d_w(u^*, v^*) = 0$.

Let us consider now the case where

$$V' = V(G) \setminus V(\tilde{G}) \ne \emptyset.$$

Let v' be a fixed point of the set V'. Define a distance function d' on the set $V(G)$ as $d'(v,v) = 0$ for all $v \in V(G)$ and as

$$d'(u,v) := \begin{cases} 1 & \text{if } u = v^*,\ v = v' \\ 1 & \text{if } u, v \in V',\ u \ne v \\ d_{w_1}(u,v) & \text{if } u, v \in V(\tilde{G}) \\ d_{w_1}(u,v^*) + 1 & \text{if } u \in V(\tilde{G}),\ v = v' \\ 2 & \text{if } u = v^*,\ v \in V',\ v \ne v' \\ d_{w_1}(u,v^*) + 2 & \text{if } u \in V(\tilde{G}),\ v \in V',\ u \ne u^*,\ v \ne v'. \end{cases} \tag{3.30}$$

Note that the past three lines in the right side of (3.30) can be rewritten in the form:

$$d'(u,v) = d'(u,v^*) + d'(v^*,v') + d'(v',v) \tag{3.31}$$

if $u \in V(\tilde{G})$ and $v \in V'$. The last equality and (3.30) imply that d' is a pseudometric. Writing

$$w(\{u,v\}) = d'(u,v)$$

for all $\{u,v\} \in E(G)$ we obtain the weighted graph (G,w) with $d' \in \mathfrak{M}_w$. The weight w is strictly positive since, by (3.30), $d'(u,v) \ge 1$ if $u \ne v$ and $\{u,v\} \cap V' \ne \emptyset$ and, by (3.26), $d'(u,v) > 0$ if $u \ne v$ and $u, v \in V(\tilde{G})$, and $\{u,v\} \in E(G)$.

Proposition 3.3 implies that

$$d_w(u,v) = w(\{u,v\})$$

for each $\{u, v\} \in E(G)$. To complete the proof, it suffices to observe that $d_w(u^*, v^*) = 0$. To see this we can use (3.29) with $u = u^*$ and $v = v^*$. □

For the case of disconnected graphs G we have the following

Proposition 3.32. *Let (G, w) be a disconnected weighted graph with the strictly positive metrizable w. Then there is a pseudometric $\rho \in \mathfrak{M}_w$ which is not metric. Moreover let G_i be connected components of G and let w_i be the restrictions of the weight w on the sets $E(G_i)$. Then the following statements are equivalent.*

(i) *There exists a metric in $\mathfrak{M}_w$.*
(ii) *The shortest-path pseudometrics d_{w_i} are metrics for all G_i.*

Proof. Set in (2.11) $a_i = 0$ for some $i \neq i_0$. Then, by (2.12), ρ is not a metric. If all d_{w_i} are metrics, then to obtain a metric in $\mathfrak{M}_w$ it is sufficient to put $a_i > 0$ for all $i \neq i_0$. □

4. The least element in $\mathfrak{M}_w$

We wish characterize the structure of the graphs G for which the set $\mathfrak{M}_w$ contains the least pseudometric $\rho_{0,w}$ as soon as w is metrizable. To this end, we recall the definition of k-partite graph.

Definition 4.1. *Let G be a simple graph and let k be a cardinal number. The graph G is k-partite if the vertex set $V(G)$ can be partitioned into k nonvoid disjoint subsets, or parts, in such a way that no edge has both ends in the same part. A k-partite graph is complete if any two vertices in different parts are adjacent.*

We shall say that G is a complete multipartite graph if there is a cardinal number $k \geqslant 1$ such that G is complete k-partite, cf. [3, p. 14]

Remark 4.2. *It is easy to prove that each nonempty complete k-partite graph G is connected if $k \geqslant 2$. Each 1-partite graph G is an empty graph.*

Theorem 4.3. *The following conditions are equivalent for each nonempty graph G.*

(i) *For every metrizable weight $w : E(G) \to \mathbb{R}^+$ the poset $(\mathfrak{M}_w, \leqslant)$ contains the least pseudometric $\rho_{0,w}$, i.e., the inequality*

$$\rho_{0,w}(u, v) \leqslant \rho(u, v) \tag{4.4}$$

holds for all $\rho \in \mathfrak{M}_w$ and all $u, v \in V(G)$;

(ii) *G is a complete k-partite graph with $k \geqslant 2$.*

If condition (ii) holds and w is a metrizable weight, then for each pair of distinct nonadjacent vertices u, v we have

$$d_w(u, v) = \inf_{\substack{\alpha \neq \alpha_0,\\ \alpha \in \mathcal{I}}} \inf_{p \in X_\alpha} \Big(w(\{u, p\}) + w(\{p, v\}) \Big), \tag{4.5}$$

and

$$\rho_{0,w}(u,v) = \sup_{\substack{\alpha\neq\alpha_0,\\ \alpha\in\mathcal{I}}} \sup_{p\in X_\alpha} |w(\{u,p\}) - w(\{p,v\})| \tag{4.6}$$

where $\{X_\alpha : \alpha \in \mathcal{I}\}$ *is a partition of* G *and* $\alpha_0 \in \mathcal{I}$ *is the index such that* $u, v \in X_{\alpha_0}$.

Remark 4.7. *Formulas* (4.5) *and* (4.6) *give the generalization of double inequality* (1.5) *for an arbitrary complete* k*-partite graph, with* $k \geqslant 2$. *The quadrilateral* Q *depicted by Figure 1 is evidently a complete bipartite graph.*

Lemma 4.8. *Let* G *be a connected nonempty graph. The following conditions are equivalent*

(i) *For each metrizable* $w : E(G) \to \mathbb{R}^+$ *the poset* $(\mathfrak{M}_w, \leqslant)$ *contains the least pseudometric* $\rho_{0,w}$.

(ii) *For every two distinct nonadjacent vertices* u *and* v *and each* $p \in V(G)$ *with* $u \neq p \neq v$ *we have either*

$$\{u, p\} \in E(G) \ \&\ \{v, p\} \in E(G)$$

or

$$\{u, p\} \notin E(G) \ \&\ \{v, p\} \notin E(G).$$

Proof. **(i)⇒(ii)** Suppose condition (ii) does not hold. Let v_1, v_2, v_3 be distinct vertices of G such that v_1 and v_2 are nonadjacent and v_2 and v_3 are also nonadjacent but $\{v_3, v_1\} \in E(G)$. Define the weight w as $w(e) = 1$ for all $e \in E(G)$. Let ρ_1 and ρ_2 be the distance functions on $V(G)$ such that:

$\rho_1(v_2, v_1) = \rho_1(v_1, v_2) = \rho_1(u, u) = 0$ for all $u \in V(G)$ and $\rho_1(u, v) = 1$ otherwise;

$\rho_2(v_3, v_1) = \rho_2(v_2, v_3) = \rho_2(u, u) = 0$ for all $u \in V(G)$ and $\rho_2(u, v) = 1$ otherwise.

It is easy to see that every triangle in $(V(G), \rho_1)$ or in $(V(G), \rho_2)$ is an isosceles triangle having the third side shorter or equal to the common length of the other two sides. Hence ρ_1 and ρ_2 are pseudometrics and even pseudoultrametrics. Furthermore ρ_1 and ρ_2 belong to $\mathfrak{M}_w$. Suppose that there is the least pseudometric $\rho_{0,w}$ in $\mathfrak{M}_w$. Then we obtain the contradiction

$$\begin{aligned} 1 = \rho_{0,w}(v_2, v_3) &\leq \rho_{0,w}(v_2, v_1) + \rho_{0,w}(v_1, v_3) \\ &\leq (\rho_1 \wedge \rho_2)(v_2, v_1) + (\rho_1 \wedge \rho_2)(v_1, v_3) = 0 + 0 = 0. \end{aligned}$$

(ii)⇒(i) Suppose condition (ii) holds. Let w be a metrizable weight. For each pair u, v of vertices of G write:

$$\rho_0(u, v) = 0 \quad \text{if } u = v; \qquad \rho_0(u, v) = w\{u, v\} \quad \text{if } \{u, v\} \in E(G);$$

and

$$\rho_0(u, v) := \sup_{P\in\mathcal{P}_{u,v}} \max_{e\in P} (2w(e) - w(P))_+ \tag{4.9}$$

if $\{u, v\} \notin E(G)$ and $u \neq v$. Here we use the notation

$$t_+ := \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0. \end{cases}$$

We claim that ρ_0 is the least element of $(\mathfrak{M}_w, \leqslant)$. To show this it suffices to prove the triangle inequality for ρ_0. Indeed, in this case ρ_0 is a pseudometric and $\rho_0 \in \mathfrak{M}_w$ by the definition of ρ_0. Moreover if ρ is an arbitrary pseudometric from $\mathfrak{M}_w$, then (2.4) implies:

$$2w(e) \leq w(P) + \rho(u, v)$$

for all distinct $u, v \in V(G)$, all $P \in \mathcal{P}_{u,v}$ and all $e \in P$. Consequently we have

$$2w(e) - w(P) \leq \rho(u, v),$$

$$(2w(e) - w(P))_+ \leq (\rho(u, v))_+ = \rho(u, v),$$

$$\sup_{P \in \mathcal{P}_{u,v}} \max_{e \in P} (2w(e) - w(P))_+ \leq \rho(u, v). \tag{4.10}$$

The last inequality and (4.9) imply (4.4) with $\rho_{0,w} = \rho_0$. Thus ρ_0 is the least pseudometric in $\mathfrak{M}_w$.

Let us turn to the triangle inequality for ρ_0. Let x, y, z be some distinct vertices of G. Since w is metrizable, the definition of ρ_0 implies this inequality if $\{x, y\}, \{y, z\}$ and $\{z, x\}$ belong to $E(G)$. Let $\{x, y\} \notin E(G)$. In accordance with condition (ii), either both $\{y, z\}$ and $\{z, x\}$ are edges of G or both $\{y, z\}$ and $\{z, x\}$ are not edges of G.

Suppose $\{y, z\}, \{z, x\} \in E(G)$. The three-point sequence $P_1 := (x, z, y)$ is a path joining x and y. Consequently by (4.9) we obtain

$$\rho_0(x, y) \geq \max_{e \in P_1} (2w(e) - w(P_1))_+ = |w(\{x, z\}) - w(\{z, y\})|.$$

Thus

$$\rho_0(x, y) + \min(\rho_0(x, z), \rho_0(z, y)) \geq \max(\rho_0(x, z), \rho_0(z, y)). \tag{4.11}$$

To prove the inequality

$$\rho_0(x, y) \leq \rho_0(x, z) + \rho_0(z, y)$$

it is sufficient to show

$$\max_{e \in P} (2w(e) - w(P))_+ \leq \rho_0(x, z) + \rho_0(z, y) \tag{4.12}$$

for each path P joining x and y. This inequality is trivial if its left part equals zero. In the opposite case, (4.12) can be rewritten in the form

$$2 \max_{e \in P} w(e) \leq w(P) + w(\{x, z\}) + w(\{z, y\}).$$

Applying (2.4) we see that the last inequality holds, so (4.12) follows. (Note that inequality (2.4) holds for each closed walk in G if it holds for each cycle in G.)

It is slightly more difficult to prove the triangle inequality for ρ_0 when

$$\{y, z\} \notin E(G), \quad \{z, x\} \notin E(G) \quad \text{and} \quad \{z, y\} \notin E(G). \tag{4.13}$$

To this end, we establish first the following lemma.

Lemma 4.14. *Let (G, w) be a connected, weighted graph with a metrizable w, let condition* (ii) *of Lemma 4.8 hold and let x, y be distinct nonadjacent vertices of G. Then, for every $P \in \mathcal{P}_{x,y}$ there is $v \in V(G)$ with $\{v, x\}, \{v, y\} \in E(G)$ and such that*

$$\max_{e \in P}(2w(e) - w(P))_+ \le |w(\{x, v\}) - w(\{v, y\})|. \tag{4.15}$$

Proof. Let $P = (x, v_1, \dots, v_n, y)$ be a path joining x and y in G. We claim that there is a path (x, v, y) in G such that (4.15) holds. It is trivial if $n = 1$ or if the left part in (4.15) is zero. So we may suppose that $n \ge 2$ ($v_1 \ne v_2$) and

$$2 \max_{e \in P} w(e) > w(P). \tag{4.16}$$

Condition (ii) of Lemma 4.8 implies

$$\{v_1, y\} \in E(G) \quad \text{and} \quad \{x, v_n\} \in E(G), \tag{4.17}$$

see Figure 4. For convenience we write

$$M := \max_{e \in P} w(e).$$

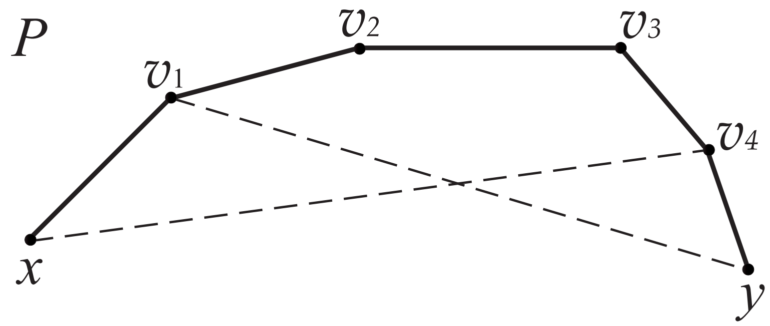

Figure 4. The path P joining x and y with $n = 4$ (heavily drawn lines), and two additional edges $\{x, v_4\}, \{y, v_1\}$ (dotted lines).

Let us prove (4.15). If $M = w(\{x, v_1\})$, then

$$2M - w(P) = w(\{x, v_1\}) - \left(\sum_{i=1}^{n-1} w(\{v_i, v_{i+1}\}) + w(\{v_n, y\}) \right)$$
$$\le w(\{x, v_1\}) - w(\{v_1, y\})$$

because w is metrizable and so we have the "triangle inequality"

$$w(\{v_1, y\}) \leq \sum_{i=1}^{n-1} w(\{v_i, v_{i+1}\}) + w(\{v_n, y\}).$$

Hence the path (x, v_1, y) satisfies (4.15) with $v = v_1$. Similarly if $M = (\{v_n, y\})$, then the desired path is (x, v_n, y).

Suppose now that

$$w(\{v_1, y\}) \geq w(\{x, v_1\}) \quad \text{and} \quad M = \max_{1 \leq i \leq n-1} w(\{v_i, v_{i+1}\}). \tag{4.18}$$

Since w is metrizable, applying (2.4) to the cycle $(v_1, v_2, \ldots, y, v_1)$ we obtain

$$w(\{v_1, y\}) + w(\{v_n, y\}) + \sum_{i=1}^{n-1} w(\{v_i, v_{i+1}\}) \geq 2M.$$

Consequently

$$w(\{v_1, y\}) - w(\{x, v_1\}) \geq 2M - w(P).$$

Thus (x, v_1, y) satisfies (4.15) with $v = v_1$ if (4.18) holds. Similarly (4.15) holds with $v = v_n$ if

$$w(\{v_n, x\}) \geq w(\{y, v_n\}) \quad \text{and} \quad M = \max_{1 \leq i \leq n-1} w(\{v_i, v_{i+1}\}). \tag{4.19}$$

It still remains to find (x, v, y) satisfying (4.15) if

$$\begin{gathered} w(\{v_1, y\}) \leq w(\{x, v_1\}), \quad w(\{v_n, x\}) \leq w(\{y, v_n\}) \\ \text{and } M = \max_{1 \leq i \leq n-1} w(\{v_i, v_{i+1}\}). \end{gathered} \tag{4.20}$$

Let us consider the new path $F = (x, u_1, \ldots, u_n, y)$ such that $u_1 = v_n$, $u_2 = v_{n-1}, \ldots, u_n = v_1$, see Fig. 5. Condition (4.20) implies that

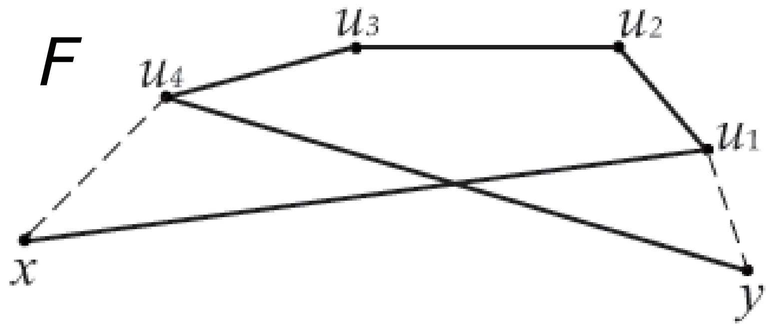

Figure 5. The new path F is a modification of the old path P.

$$M = \max_{1 \leq i \leq n-1} w(\{u_i, u_{i+1}\}) = \max_{e \in F} w(e)$$

and, moreover, that $w(F) \leq w(P)$. Hence it suffices to prove the inequality

$$\max_{e \in F} (2w(e) - w(F))_+ \leq |w(\{x, v\}) - w(\{v, y\})|$$

for a 2-path (x, v, y) in G. We can make it in a way analogous to that was used under consideration of restriction (4.18) if

$$w(\{u_1, y\}) \geq w(\{x, u_1\}).$$

To complete the proof, it suffices to observe that the last inequality can be rewritten as

$$w(\{v_n, y\}) \geq w(\{x, v_n\})$$

which follows from (4.20). □

Continuation of the proof of Lemma 4.8. It still remains to prove the inequality

$$\rho_0(x, y) \leq \rho_0(x, z) + \rho_0(z, y) \tag{4.21}$$

if x, y, z are distinct vertices such that

$$\{x, y\} \notin E(G), \quad \{x, z\} \notin E(G) \quad \text{and} \quad \{z, y\} \notin E(G). \tag{4.22}$$

It follows from Lemma 4.14 that

$$\rho_0(x, y) = \sup_v |w(\{x, v\}) - w(\{v, y\})| \tag{4.23}$$

where the supremum is taken over the set of all vertices v such that

$$\{x, v\}, \{v, y\} \in E(G). \tag{4.24}$$

Condition (ii), relations (4.22) and relations (4.24) give the membership relation

$$\{z, v\} \in E(G).$$

Thus the weight function w is defined at the "point" $\{z, v\}$. Hence

$$\begin{aligned}|w(\{x, v\}) - w(\{v, y\})| &\leq |w(\{x, v\}) - w(\{v, z\})| + |w(\{v, z\}) - w(\{v, y\})| \\ &\leq \rho_0(x, z) + \rho_0(z, y).\end{aligned}$$

These inequalities and (4.23) imply (4.21). □

Recall that a subgraph H of a graph G is *induced* if $E(H)$ consists of all edges of G which have both ends in $V(H)$.

Proposition 4.25. *A nonnull graph is complete multipartite if and only if it has no induced subgraphs depicted by Figure 6.*

Proof. Suppose G is a complete k-partite graph. If $k = 1$, then all subgraphs of G are empty. Let $k \geqslant 2$. If u and v are vertices of G such that $\{u, v\} \notin E(G)$, then there exists a part V_1 in the partition of $V(G)$ such that

$$u \in V_1 \quad \text{and} \quad v \in V_1.$$

If p is a vertex of G and $u \neq p \neq v$, then either $p \in V_1$ or there is a part $V_2 \neq V_1$ such that $p \in V_2$. Using Definition 4.1 we obtain that $\{p, v\} \notin E(G)$ and $\{p, u\} \notin E(G)$ if $p \in V_1$ or, in the opposite case $p \in V_2$, that $\{p, v\} \in E(G)$ and $\{p, u\} \in E(G)$.

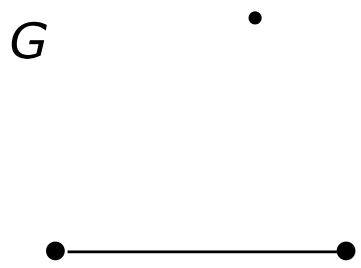

Figure 6. The graph G is not complete k-partite for any $k \geqslant 1$.

Assume now that G has no induced subgraphs depicted by Figure 6. Let us define a relation $\asymp$ on the set $V(G)$ as

$$(u \asymp v) \Leftrightarrow (\{u, v\} \notin E(G)). \tag{4.26}$$

Relation $\asymp$ is evidently symmetric. Since simple graphs contain no loops, we have $\{u, u\} \notin E(G)$ for each $u \in V(G)$. Consequently $\asymp$ is reflexive. Moreover if $\{u, v\} \notin E(G)$ and $\{v, p\} \notin E(G)$, then we obtain $\{u, p\} \notin E(G)$. Thus $\asymp$ is transitive, so this is an equivalence relation. The set $V(G)$ is partitioned by the relation $\asymp$ on the disjoint parts V_i, $i \in \mathcal{I}$, where $\mathcal{I}$ is an index set. It follows directly from (4.26) that no edge of G has both ends in the same part. Hence G is a k-partite graph with $k = \operatorname{card} \mathcal{I}$. Finally note that $\{u, v\} \in E(G)$ if and only if the relation $u \asymp v$ does not hold. Consequently G is a complete k-partite graph. □

This proposition implies

Lemma 4.27. *Let G be a nonempty graph. Condition (ii) of Lemma 4.8 holds if and only if G is a complete k-partite graph with $k \geqslant 2$.*

It still remains to prove the next lemma.

Lemma 4.28. *Let G be a nonempty graph. If condition (i) of Theorem 4.3 holds, then G is connected.*

Proof. Let $w : E(G) \to \mathbb{R}^+$ be a weight such that the equality

$$w(e) = 1 \tag{4.29}$$

holds for all $e \in E(G)$. It is clear that w is metrizable. Let $\{u_1, v_1\}$ be an edge of G. If G is disconnected, then there are two connected components

G_1 and G_2 of G such that

$$u_1 \in V(G_1), \quad v_1 \in V(G_1) \quad \text{and} \quad V(G_1) \cap V(G_2) = \emptyset.$$

Let p be a vertex of G_2. Using formulas (2.10), (2.11) with zero constants a_i we can find some pseudometrics $\rho_1, \rho_2 \in \mathfrak{M}_w$ for which

$$\rho_1(u_1, p) = 0 \qquad \text{and} \qquad \rho_2(v_1, p) = 0. \tag{4.30}$$

If condition (i) of Theorem 4.3 holds, then for the least pseudometric $\rho_{0,w}$ in $\mathfrak{M}_w$ we have the inequalities

$$\rho_{0,w}(u_1, p) \leqslant \rho_1(u_1, p) \quad \text{and} \quad \rho_{0,w}(v_1, p) \leqslant \rho_2(v_1, p).$$

These inequalities, the triangle inequality and (4.30) imply

$$\rho_{0,w}(u_1, v_1) \leqslant \rho_1(u_1, p) + \rho_2(v_1, p) = 0.$$

Since $\rho_{0,w} \in \mathfrak{M}_w$, it implies $w(e) = 0$ for $e = \{u_1, v_1\}$, contrary (4.29). □

Proof of Theorem 4.3. Suppose that condition (i) of the theorem holds. By Lemma 4.28, G is a connected graph and so we can use Lemma 4.8. Applying this lemma we obtain the equivalence of its condition (ii) with condition (i) of Theorem 4.3. By Lemma 4.27 condition (ii) of Lemma 4.8 implies condition (ii) of the Theorem 4.3.

Conversely, suppose that condition (ii) of Theorem 4.3 holds. Using Lemma 4.27 we see that condition (ii) of Lemma 4.8 holds. Moreover condition (ii) of Theorem 4.3 implies that G is connected, see Remark 4.2. Hence by Lemma 4.8 we obtain condition (i) of the theorem. Thus we have the implication (ii)⇒(i) in Theorem 4.3.

Assume now that G is complete k-partite graph with $k \geqslant 2$ and w is metrizable. Let u and v be some distinct nonadjacent vertices of G. Then we have $u, v \in X_{\alpha_0}$ for some $\alpha_0 \in \mathcal{I}$. We must to prove equalities (4.5) and (4.6). For every vertex $p \notin X_{\alpha_0}$ the sequence (u, p, v) is a path joining u and v. Consequently the inequality

$$d_w(u, v) \leqslant \inf_{\substack{\alpha \neq \alpha_0,\\ \alpha \in \mathcal{I}}} \inf_{p \in X_\alpha} |w(\{u, p\}) + w(\{p, v\})| \tag{4.31}$$

follows from (2.1). To prove the converse inequality it is sufficient to show that for every $F \in \mathcal{P}_{u,v}$ there is $p \in X_\alpha$, $\alpha \neq \alpha_0$, such that

$$w(F) \geqslant w(\{u, v\}) + w(\{p, v\}). \tag{4.32}$$

Since u and v are nonadjacent, the length (the number of edges) of F is more or equal 2 for every $F \in \mathcal{P}_{u,v}$. If the length of F is 2, then the "inner" vertex of F does not belong to X_{α_0} so we have (4.31). Let $(u = v_0, v_1, \ldots, v_n = v)$ belong to $\mathcal{P}_{u,v}$ and $n \geqslant 3$. If $v_1 \in X_\alpha$, then $\alpha \neq \alpha_0$ and $\{u, v\} \in E(G)$ because G is a complete k-partite graph. Since w is metrizable, statement (ii) of Theorem 2.2 implies

$$w(\{v_1, v\}) \leqslant w(F') \tag{4.33}$$

where F' is the part $(v_1, \dots, v_n)$. It is clear that

$$w(F) = w(\{u, v_1\}) + w(F').$$

Consequently (4.33) implies (4.32) with $p = v_2$. Equality (4.5) follows.

To prove (4.6) we now return to lemmas 4.8, 4.14, 4.27. By the assumption G is a complete k-partite graph. Hence, by Lemma 4.27, condition (ii) of Lemma 4.8 holds. This condition implies that

$$\rho_{0,w}(u, v) = \sup_{F \in \mathcal{P}_{u,v}} \max_{e \in F} (2w(e) - w(F))_+ \tag{4.34}$$

see (4.9) and (4.10). Using Lemma 4.14 we obtain that for every $F \in \mathcal{P}_{u,v}$ there is $p \in V(G)$ with $\{u, p\}, \{p, v\} \in E(G)$ and such that

$$\max_{e \in F} (2w(e) - w(F))_+ \leqslant |w(\{u, p\}) - w(\{p, v\})|.$$

Consequently we have

$$\rho_{0,w}(u, v) \leqslant \sup_{\substack{\alpha \neq \alpha_0, \\ \alpha \in \mathcal{I}}} \sup_{p \in X_\alpha} |w(\{u, p\}) - w(\{p, v\})|$$

for every two distinct nonadjacent vertices u, v. The converse inequality follows from (4.34). Indeed, for every path F of the form (u, p, v) we have

$$|w(\{u, p\}) - w(\{p, v\})| = \max_{e \in F} (2w(e) - w(F))_+.$$

□

Recall that the star is a complete bipartite graph G with a bipartition (X, Y),

$$V(G) = X \cup Y, \qquad X \cap Y = \emptyset$$

such that $\operatorname{card} X = 1$ or $\operatorname{card} Y = 1$.

Corollary 4.35. *The following conditions are equivalent for each nonempty graph G.*

(i) *Every weight $w : E(G) \to \mathbb{R}^+$ is metrizable and the poset $(\mathfrak{M}_w, \leqslant)$ contains the least pseudometric $\rho_{0,w}$.*

(ii) *G is a star.*

Proof. Let condition (i) hold. Then, by Theorem 4.3, G is complete k-partite with $k \geqslant 2$ and by Corollary 2.17 G is acyclic. Each k-partite graph with $k \geqslant 3$ contains a 3-cycle (triangle). Hence we have $k = 2$, i.e. G is bipartite. If (X, Y) is a bipartion of G with

$$\operatorname{card} X \geqslant 2 \qquad \text{and} \qquad \operatorname{card} Y \geqslant 2,$$

then we can find some vertices

$$x_1, x_3 \in X \qquad \text{and} \qquad x_2, x_4 \in Y.$$

Since G is a complete bipartite graph, G contains the quadrilateral Q, see Fig. 1. Consequently we have $\operatorname{card} X = 1$ or $\operatorname{card} Y = 1$. Thus G is a star.

Conversely suppose G is a star. Then G is acyclic, so using Corollary 2.17 we obtain that every weight w is metrizable. Since stars are complete bipartite graphs, Theorem 4.3 implies the existence of the least pseudometric $\rho_{0,w} \in \mathfrak{M}_w$. □

Theorem 4.36. *The following conditions (i) and (ii) are equivalent for each nonempty graph G.*

(i) *For each metrizable weight $w : E(G) \to \mathbb{R}^+$ the set $\mathfrak{M}_w$ contains the least pseudometric $\rho_{0,w}$ and this set contains also all symmetric functions $f : V(G) \times V(G) \to \mathbb{R}^+$ which lie between $\rho_{0,w}$ and the shortest-path pseudometric d_w, i.e., which satisfy the double inequality*

$$\rho_{0,w}(u,v) \leqslant f(u,v) \leqslant d_w(u,v) \tag{4.37}$$

for all $u, v \in V(G)$.

(ii) *G is a complete k-partite graph with a partition $\{X_\alpha : \alpha \in \mathcal{I}\}$ such that* $\operatorname{card} \mathcal{I} = k \geqslant 2$ *and* $\operatorname{card} X_\alpha \leqslant 2$ *for each part X_α.*

Proof. **(i)⇒(ii)** Suppose that (i) holds. By Theorem 4.3 G is a complete k-partite graph with $k \geqslant 2$. Assume that there is a part X_{α_0} such that $\operatorname{card}(X_{\alpha_0}) \geqslant 3$. Let v_1, v_2, v_3 be some pairwise distinct elements of X_{α_0} and let w be the weight such that $w(e) = 1$ for each $e \in E(G)$. Define functions ρ_1 and ρ_2 on the set $V(G) \times V(G)$ as

$$\rho_1(u,v) = \begin{cases} 0 & \text{if } u = v \\ 1 & \text{if } u \neq v \end{cases} \quad \text{and} \quad \rho_2(u,v) = \begin{cases} 1 & \text{if } \{u,v\} \in E(G) \\ 0 & \text{if } \{u,v\} \notin E(G) \end{cases}.$$

It is clear that $\rho_1 \in \mathfrak{M}_w$. To prove that $\rho_2 \in \mathfrak{M}_w$ it is sufficient to verify the triangle inequality

$$\rho_2(u,v) \leqslant \rho_2(u,s) + \rho_2(v,s) \tag{4.38}$$

for all $u, v, s \in V(G)$. If (4.38) does not hold, then $\rho_2(u,v) = 1$ and $\rho_2(u,s) = \rho_2(v,s) = 0$. Consequently we have that

$$\{u,v\} \in E(G) \text{ and } \{u,s\} \notin E(G) \text{ and } \{v,s\} \notin E(G). \tag{4.39}$$

The relation $\{u,s\} \notin E(G)$ and $\{v,s\} \notin E(G)$ imply that u, s belong to a part X_u, similarly v, s belong to a part X_v. Since $s \in X_u \cap X_v$ we obtain that $X_u = X_v$, hence $\{u,v\} \notin E(G)$ contrary to the first membership relation in (4.39). Thus (4.38) holds for all $u, v, s \in V(G)$, so $\rho_2 \in \mathfrak{M}_w$. The function $f : V(G) \times V(G) \to \mathbb{R}^+$ defined as

$$f(v_1,v_2) = f(v_2,v_1) = 1, \quad f(v_1,v_3) = f(v_3,v_1) = f(v_3,v_2) = f(v_2,v_3) = 0$$

and

$$f(u,v) = \rho_1(u,v)$$

for $(u,v) \in (V(G)\times V(G)) \setminus \{(v_1,v_2),(v_2,v_1),(v_2,v_3),(v_3,v_2),(v_1,v_3),(v_3,v_2)\}$ satisfies the double inequality

$$\rho_2(u,v) \leqslant f(u,v) \leqslant \rho_1(u,v)$$

that implies (4.37). Hence by (i) we must have

$$f(v_1,v_2) \leqslant f(v_1,v_3) + f(v_3,v_2)$$

that contradicts the definition of the function f. Thus the inequality card $X_\alpha \leqslant 2$ holds for each part X_α.

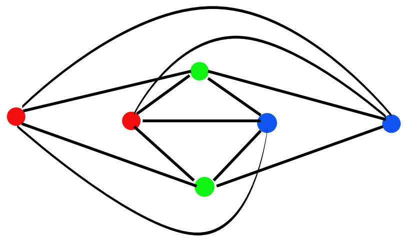

Figure 7. A complete 3-partite graph G satisfies condition (ii) of Theorem 4.36.

(ii)⇒(i) Suppose that condition (ii) holds. Since $k \geqslant 2$ and G is a complete k-partite graph, Theorem 4.3 provides the existence of the least pseudometric $\rho_{0,w}$ for each metrizable weight w. Let $f : V(G) \times V(G) \to \mathbb{R}^+$ be a symmetric function such that (4.37) holds for all $u, v \in V(G)$. The double inequality (4.37) implies that f is nonnegative and $f(u,v) = w(\{u,v\})$ for all $\{u,v\} \in E(V)$ and $f(u,u) = 0$ for all $u \in V(G)$. Consequently to prove that $f \in \mathfrak{M}_w$ it is sufficient to obtain the triangle inequality

$$f(u,v) \leqslant f(u,p) + f(p,v) \tag{4.40}$$

for all $u, v, p \in V(G)$. We may assume u, v and p are pairwise disjoint, otherwise (4.40) is trivial. Since card $X_\alpha \leqslant 2$ for each part X_α, at most one pair from the vertices u, v and p are nonadjacent. If $\{u,v\} \notin E(G)$, then using (4.37) we obtain

$$\begin{aligned} f(u,v) \leqslant d_w(u,v) &\leqslant d_w(u,p) + d_w(p,v) \\ &= w(\{u,p\}) + w(\{p,v\}) \leqslant f(u,p) + f(p,v). \end{aligned}$$

Similarly if $\{u,p\} \notin E(G)$ or $\{p,v\} \notin E(G)$, then we have

$$f(u,v) \leqslant \rho_{0,w}(u,v) \leqslant \rho_{0,w}(u,p) + \rho_{0,w}(p,v) = f(u,p) + f(p,v).$$

Inequality (4.40) follows and we obtain condition (i). □

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References

[1] J. A. Bondy, U. S. R. Murty, *Graphs Theory.* Graduate Texts in Mathematics, Vol. 244, Springer, Berlin, 2008.
[2] M. M. Deza, M. Laurent, *Geometry of Cuts and Metrics.* Springer, Berlin, 1997.
[3] R. Diestel, *Craphs Theory. Third edition*, Graduate Texts in Mathematics, Vol. 173, Springer, Berlin, 2005.
[4] O. Dovgoshey, E. Petrov, *Ptolemeic spaces* // Siberian Math. J., **52**, N2, (2011), 222–229.
[5] A. Lesser, *Optimal and Hereditarily Optimal Realization of Metric Spaces* // Uppsala Dissertations in Mathematics, **52**, 2007, Uppsala, 70 pp.
[6] T. Harju, *Lecture Notes on graph theory.* University of Turku, 1994–2011, http://users.utu.fi/harju/graphtheory.pdf

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