Large deviations of random walks under distribution whose tail is bounded by power functions

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Abstract

Let $X, X_1, X_2, \ldots$ be independent identically distributed random variables and $S_n = X_1 + \cdots + X_n$. Thus $\{S_n; n = 1, 2, \ldots\}$ is a random walk. Let F be the distribution function and μ the expectation of X. We investigate the large deviation event $\left\{\frac{S_n}{n} > a\right\}$, where $a > \mu$ is a constant. The basic assumption is that for large x, the tail probability $1 - F(x)$ is bounded from above and below by a power of x. Our main results say that under weak additional conditions, the large deviation is essentially caused by a single large random variable. We use this to derive asymptotic estimates for the probability $\mathbb{P}\left\{\frac{S_n}{n} > a\right\}$.

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1 Introduction

Let $X, X_1, X_2, \ldots$ be independent indentically distributed random variables with distribution function F and mean μ. Let $S_n = X_1 + \cdots + X_n$ for $n \in \mathbb{N}$ so that $\{S_n; n = 1, 2, \ldots\}$ is a random walk in $\mathbb{R}$. We are interested in the event

$$\text{LD} := \left\{\frac{S_n}{n} > a\right\},$$

where a is a constant larger than μ. Write $\overline{F}(x) = 1 - F(x)$ for $x \in \mathbb{R}$. We will work in a setting where for large x, $\overline{F}(x)$ is bounded both from above and below by a power of x. The assumption implies that X is heavy-tailed from the right.

A typical phenomenon in case of heavy tailed increments is that a large value of S_n corresponds to a large value of the maximum of $X_1, \ldots, X_n$. At least in smooth cases, this leads to the the asymptotic estimate

$$\mathbb{P}\left\{\frac{S_n}{n} > a\right\} \sim n\overline{F}(n(a-\mu)), \quad n \to \infty. \tag{1.1}$$

The estimate was first obtained in the case where $\overline{F}$ is regularly varying at infinity. We refer the reader to Nagaev (1979) for the early history. Later on, extensions are obtained for

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more general random walks. As examples, Cline and Hsing (1991) prove (1.1) by assuming that $\overline{F}$ is of extended regular variation, and Ng et al. (2004) derive (1.1) by assuming that $\overline{F}$ varies consistently. More recently, Denisov et al. (2008) study estimate (1.1) under subexponentiality assumption. For a wide survey on heavy tailed phenomena for random walks, we refer the reader to Borovkov and Borovkov (2008).

Our main objective is to study the increments of S_n conditionally, given that LD happens. More precisely, we want to find conditions under which the large deviation is caused by a single dominant random variable. To this end, we study the internal structure of the event LD in detail by deriving estimates for the probabilities of various subevents. Our conditions are general enough to give a rise for discussions of their necessity. A related paper is Nyrhinen (2009) where estimates are obtained for multivariate random walks in a similar but less general context. More specifically, we allow here $\overline{F}(x)$ to vary between two fixed powers of x while in Nyrhinen (2009), $\overline{F}$ can be described by one power of x.

Our study also leads to asymptotic estimates for large deviations probabilities. The estimates can be viewed as generalizations of (1.1). We also obtain natural bounds for the probability of LD in the case where $\overline{F}$ is of dominated variation. This clarifies a related result of Tang and Yan (2002).

The paper is organized as follows. In Section 2, we describe our basic conditions and the subevents of LD to be considered. Section 3 deals with the question of the large deviation with small increments only. In Section 4, the role of large increments is studied. A discussion of our conditions is given in Section 5. Main results are stated in Section 6 and special cases are considered in Section 7.

2 Setup

Denote

$$-\overline{\alpha} := \limsup_{x\to\infty} \frac{1}{\log x} \log \overline{F}(x), \tag{2.1}$$

$$-\underline{\alpha} := \liminf_{x\to\infty} \frac{1}{\log x} \log \overline{F}(x). \tag{2.2}$$

We will often assume that

$$-\overline{\alpha} < -1, \tag{UL}$$

$$-\underline{\alpha} > -\infty. \tag{LL}$$

We consider (UL) and (LL) as our basic conditions even if some results will be stated more generally. If the conditions hold then for any given $\eta > 0$, we have the bounds

$$x^{-\underline{\alpha}-\eta} \leq \overline{F}(x) \leq x^{-\overline{\alpha}+\eta}. \tag{2.3}$$

for large enough x. We use assumptions (UL) and (LL) to control the right tail of F. To ensure that the left tail does not carry too much weight we assume throughout that expectation $\mu = \mathbb{E}(X)$ exists and is finite.

We identify subevents of LD in which 0, 1 or several of the random variables X_i are large. Let ε and δ be positive real numbers. Write

$$\begin{aligned}
\mathrm{LD}_{0,\delta} &:= \left\{ \frac{S_n}{n} > a; X_i \le n^{1-\delta}, i = 1, 2, \dots, n \right\}, \\
\mathrm{LD}_0^{\varepsilon} &:= \left\{ \frac{S_n}{n} > a; X_i \le \varepsilon n, i = 1, 2, \dots, n \right\}, \\
\mathrm{LD}_k^{\varepsilon} &:= \left\{ \frac{S_n}{n} > a; X_i \le \varepsilon n \text{ for all } i = 1, 2, \dots, n \right. \\
&\qquad \left. \text{except for exactly } k \text{ indices for which } X_i > \varepsilon n \right\}, \\
\mathrm{LD}_s^{\varepsilon} &:= \bigcup_{k \ge 2} \mathrm{LD}_k^{\varepsilon}, \\
\overline{\mathrm{LD}}_k^{\varepsilon,\xi} &:= \left\{ \frac{S_n}{n} > a; X_i \le \varepsilon n \text{ for all } i = 1, 2, \dots, n \right. \\
&\qquad \left. \text{except for exactly } k \text{ indices for which } X_i > \frac{n}{k}(a - \mu - \xi) \right\}.
\end{aligned}$$

In event $\mathrm{LD}_{0,\delta}$, none of the random variables is large. $\mathrm{LD}_0^{\varepsilon}$ is similar but we allow faster growth in the upper bound for X_i. We consider a random variable large if it is above $n\varepsilon$. In $\mathrm{LD}_k^{\varepsilon}$, exactly k of the random variables X_i are large. We bundle events with 2 or more large random variable into $\mathrm{LD}_s^{\varepsilon}$. Note that LD is the disjoint union of $\mathrm{LD}_0^{\varepsilon}$, $\mathrm{LD}_1^{\varepsilon}$ and $\mathrm{LD}_s^{\varepsilon}$.

Our main interest is in the events $\overline{\mathrm{LD}}_k^{\varepsilon,\xi}$. Here we consider ε and ξ as being close to zero. Most of the random variables are small i.e. at most εn. The difference to exceptional k variables is at least $(\frac{1}{k}(a - \mu - \xi) - \varepsilon)n$. It is assumed throughout that ξ is less than $a - \mu$ and ε is smaller than $\frac{a-\mu-\xi}{k}$.

When we need to take limits along subsequences of natural numbers we put the index on the left side of the event, e.g.

$$_{n_i}\mathrm{LD} = \left\{ \frac{S_{n_i}}{n_i} > a \right\}.$$

3 The event with no large random variables

Condition (UL) is enough to prove that the probability of $\mathrm{LD}_0^{\varepsilon}$ goes to zero faster than any given power if we take ε small enough.

Lemma 3.1 *Assume that μ is finite and (UL) holds. We have*

$$\lim_{\varepsilon \to 0+} \limsup_{n \to \infty} (\log n)^{-1} \log \mathbb{P}(\mathrm{LD}_0^{\varepsilon}) = -\infty. \tag{3.1}$$

Proof Let $\delta > 0$ be fixed. We use a related result where the random variables X_i are bounded from above by $n^{1-\delta}$ instead of εn. In Nyrhinen (2009) it is proved that if μ is finite and (UL) holds then

$$\lim_{n \to \infty} (\log n)^{-1} \log \mathbb{P}(\mathrm{LD}_{0,\delta}) = -\infty. \tag{3.2}$$

This is done using Tšebysev's inequality together with a technique described in e.g. Nagaev (1979).

From the above result, we can prove our claim. Fix large $\gamma > 0$, and take small enough $\delta \in (0,1)$ such that $(1-\delta)(\overline{\alpha}-\delta)-1 > 0$. Then for every $k \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}\left\{X_i > n^{1-\delta} \text{ for at least } k \text{ indices } i \in \{1,\dots,n\}\right\} \\
&\leq \binom{n}{k}\left(n^{1-\delta}\right)^{(-\overline{\alpha}+\delta)k} \\
&\leq n^{-k((1-\delta)(\overline{\alpha}-\delta)-1)}
\end{aligned}$$

for large n, where we have used the upper bound from (2.3). Thus we can find $K = K(\gamma,\delta)$ such that

$$\begin{aligned}
\mathbb{P}(\mathrm{LD}_0^\varepsilon) = \mathbb{P}&\left(\mathrm{LD}_0^\varepsilon \cap \{X_i > n^{1-\delta} \text{ for at most } K \text{ indices } i \in \{1,\dots,n\}\}\right) \\
&+ O(n^{-\gamma}).
\end{aligned} \tag{3.3}$$

Choose $\varepsilon > 0$ small enough so that $a - K\varepsilon > \mu$. Then for every $k \leq K$,

$$\begin{aligned}
&\mathbb{P}\left(\mathrm{LD}_0^\varepsilon \cap \{X_i > n^{1-\delta} \text{ for exactly } k \text{ indices } i \in \{1,\dots,n\}\}\right) \\
&= \binom{n}{k}\mathbb{P}\left\{\frac{S_n}{n} > a, X_1 \leq n^{1-\delta}, \dots, X_{n-k} \leq n^{1-\delta},\right. \\
&\qquad \left. X_{n-k+1} \in (n^{1-\delta}, n\varepsilon], \dots, X_n \in (n^{1-\delta}, n\varepsilon]\right\} \\
&\leq n^k \mathbb{P}\left\{S_{n-k} > n(a-K\varepsilon), X_1 \leq n^{1-\delta}, \dots, X_{n-k} \leq n^{1-\delta}\right\}.
\end{aligned}$$

By (3.2) this is less than $n^{-\gamma}$ for large n and therefore

$$\mathbb{P}\left(\mathrm{LD}_0^\varepsilon \cap \{X_i > n^{1-\delta} \text{ for at most } K \text{ indices } i \in \{1,\dots,n\}\}\right) \leq K n^{-\gamma}. \tag{3.4}$$

By combining estimates (3.3) and (3.4) we conclude that for sufficiently small $\varepsilon > 0$

$$\limsup_{n\to\infty} (\log n)^{-1} \log \mathbb{P}(\mathrm{LD}_0^\varepsilon) \leq -\gamma.$$

This implies (3.1) because the probability of the event $\mathrm{LD}_0^\varepsilon$ is increasing in ε. □

We have yet to compare the probability of $\mathrm{LD}_0^\varepsilon$ to LD. Later we will use assumption (LL) to make sure that $\mathbb{P}(\mathrm{LD}_0^\varepsilon)$ is essentially smaller than $\mathbb{P}(\mathrm{LD})$.

4 Is it one or is it several?

We start with an estimate for having exactly k large random variables.

Lemma 4.1 *Assume that μ is finite and (UL) holds. Let ε and ξ be such that*

$$0 < \xi < a - \mu \quad \textit{and} \quad 0 < \varepsilon < \frac{a-\mu-\xi}{k}.$$

Then

$$(1+o(1))\binom{n}{k}\overline{F}\left(\frac{n}{k}(a-\mu+\xi)\right)^k \leq \mathbb{P}(\overline{\mathrm{LD}}_k^{\varepsilon,\xi}) \leq \binom{n}{k}\overline{F}\left(\frac{n}{k}(a-\mu-\xi)\right)^k. \tag{4.1}$$

If also (LL) *holds, then for any given $\eta > 0$*

$$\mathbb{P}(\mathrm{LD}) \geq n^{1-\underline{\alpha}-\eta} \tag{4.2}$$

for large n.

Proof First note that the second inequality of formula (4.1) is immediate from the definition of $\overline{\mathrm{LD}}_k^{\varepsilon,\xi}$.

The probability that all random variables in the sequence $X_1, X_2, \ldots$ are less than or equal to εn tends to 1 as n tends to ∞. On the other hand, the law of large numbers guarantees that the sum S_{n-k} is close to $(n-k)\mu$ with probability tending to 1. Therefore, the event we need to concern us with is the one in which one or more random variables are large.

To make this precise we investigate a subevent of $\overline{\mathrm{LD}}_k^{\varepsilon,\xi}$. Namely, we have

$$\begin{aligned} \overline{\mathrm{LD}}_k^{\varepsilon,\xi} \supseteq \{S_{n-k} > n(\mu-\xi); X_i \leq \varepsilon n,\ i = 1, \ldots, n-k; \\ X_{n-k+1}, \ldots, X_n > \frac{n}{k}(a-\mu+\xi)\Big\}, \end{aligned} \tag{4.3}$$

which we see as an intersection of three events. The third is independent of the first two. Starting with the second set, take γ such that $-\overline{\alpha}+1+\gamma < 0$. Then

$$\begin{aligned} &\mathbb{P}\{X_i \leq \varepsilon n,\ i = 1, \ldots, n-k\} \\ &= F(\varepsilon n)^{n-k} = e^{(n-k)\log F(\varepsilon n)} \\ &\geq e^{n \log F(\varepsilon n)} = e^{n \log(1-\overline{F}(\varepsilon n))} && (4.4) \\ &= e^{-n(1+o(1))\overline{F}(\varepsilon n)} && (4.5) \\ &\geq e^{-n(1+o(1))(\varepsilon n)^{-\overline{\alpha}+\gamma}} = e^{-(1+o(1))\varepsilon^{-\overline{\alpha}+\gamma} n^{-\overline{\alpha}+\gamma+1}} && (4.6) \\ &\to_{n\to\infty} 1. \end{aligned}$$

In (4.4) we use the fact that $\log F(\varepsilon n) \leq 0$. Equality (4.5) comes from the approximation

$$\log x = (1+o(1))x,$$

where $o(1) \to 0$ as $x \to 1$. In (4.6), we have used condition (UL).

Turning to the first set of the intersection, first observe that

$$\mathbb{P}\{S_{n-k} > n(\mu-\xi)\} \geq \mathbb{P}\left\{S_{n-k} > (n-k)(\mu - \frac{1}{2}\xi)\right\}$$

for large n. The law of large numbers implies that $\mathbb{P}\left\{S_{n-k} > (n-k)(\mu-\frac{1}{2}\xi)\right\}$ tends to one as n tends to infinity.

For the last set of the intersection we have

$$\mathbb{P}\left\{X_{n-k+1}, \ldots, X_n > \frac{n}{k}(a-\mu+\xi)\right\} = \overline{F}\left(\frac{n}{k}(a-\mu+\xi)\right)^k.$$

Combining these, using the assumption that X_i are independent, and noting that the large random variables in (4.3) can be any k from the set of $\{X_1, \ldots, X_n\}$ gives the first estimate in (4.1).

Turning to (4.2), we use (4.1) and (2.3) with parameter $\frac{\eta}{2}$ to get

$$\begin{aligned} \mathbb{P}(\mathrm{LD}) &\geq \mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi}) \geq (1+o(1))n(a-\mu+\xi)^{-\underline{\alpha}-\frac{\eta}{2}} n^{-\underline{\alpha}-\frac{\eta}{2}} && (4.7) \\ &\geq n^{1-\underline{\alpha}-\eta} && (4.8) \end{aligned}$$

for large n. □

The second part of the lemma is important to establish a scale to compare subevents of the large deviation. This reveals the purpose of Lemma 3.1. If we assume (LL) then $\mathrm{LD}_0^\varepsilon$ plays no essential role in causing LD.

Our goal is is to make $\overline{\mathrm{LD}}_1^{\varepsilon,\xi}$ the dominant event. To this end, we need a condition under which the probability of $\mathrm{LD}_s^\varepsilon$ is essentially smaller than the probability of $\overline{\mathrm{LD}}_1^{\varepsilon,\xi}$. We choose the following:

$$\forall\kappa > 0 : \frac{n\overline{F}(\kappa n)^2}{\overline{F}(n)} \to_{n\to\infty} 0. \tag{CS}$$

Note that it is equivalent to take the limit along real numbers:

$$\frac{x\overline{F}(\kappa x)^2}{\overline{F}(x)} \leq \frac{2\lfloor x\rfloor \overline{F}\left(\frac{\kappa}{2}2\lfloor x\rfloor\right)^2}{\overline{F}(2\lfloor x\rfloor)} \to_{x\to\infty} 0, \tag{4.9}$$

where $\lfloor x\rfloor$ is the integer that satisfies $\lfloor x\rfloor \leq x < \lfloor x\rfloor + 1$. Further, if (CS) holds and b is an arbitrary positive real number then

$$\forall\kappa > 0 : \frac{x\overline{F}(\kappa x)^2}{\overline{F}(bx)} \to_{x\to\infty} 0. \tag{4.10}$$

This is seen by writing

$$\frac{x\overline{F}(\kappa x)^2}{\overline{F}(bx)} = \frac{1}{b}\frac{bx\overline{F}(\frac{\kappa}{b}bx)^2}{\overline{F}(bx)}.$$

The intuition behind condition (CS) is to compare directly the probability of having two large random variables of size κn with having one large random variables of size n. The requirement that the limit is 0 says that the probability of having two large random variables is neglible in comparison to having one.

Lemma 4.2 *Assume that μ is finite,* (UL) *and* (CS). *For any* $0 < \varepsilon < a - \mu$

$$\mathbb{P}(\mathrm{LD}_s^\varepsilon) = o(1)\mathbb{P}(\mathrm{LD}_1^\varepsilon).$$

Proof Choose $b > 0$. By (4.10),

$$\begin{aligned}\mathbb{P}(\mathrm{LD}_s^\varepsilon) &\leq \mathbb{P}\left(X_i > n\varepsilon, X_j > n\varepsilon \text{ for some } 1 \leq i < j \leq n\right)\\ &\leq n^2\overline{F}(n\varepsilon)^2 = \mathrm{o}(1)n\overline{F}(bn).\end{aligned}$$

Lemma 4.1 gives us the lower bound

$$\mathbb{P}(\mathrm{LD}_1^\varepsilon) \geq \mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi}) \geq (1 + o(1))n\overline{F}((a - \mu + \xi)n)$$

for some small $\xi > 0$. Combining the results and setting $b = a - \mu + \xi$ gives

$$\frac{\mathbb{P}(\mathrm{LD}_s^\varepsilon)}{\mathbb{P}(\mathrm{LD}_1^\varepsilon)} \leq \frac{o(1)n\overline{F}((a - \mu + \xi)n)}{(1 + o(1))n\overline{F}((a - \mu + \xi)n)} = o(1).$$

□

The next step is to show that $\overline{\mathrm{LD}}_1^{\varepsilon,\xi}$ is a dominant part of $\mathrm{LD}_1^\varepsilon$.

Lemma 4.3 *Assume that μ is finite,* (UL), (LL), *and* (CS). *Let ξ and ε be such that* $0 < \xi < a - \mu$ *and* $0 < \varepsilon < a - \mu - \xi$. *Then*

$$\mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi}) = (1 + \mathrm{o}(1))\mathbb{P}(\mathrm{LD}_1^\varepsilon).$$

Proof Let us see what is between $\overline{\mathrm{LD}}_1^{\varepsilon,\xi}$ and $\mathrm{LD}_1^{\varepsilon}$:

$$\begin{aligned}
&\mathbb{P}(\mathrm{LD}_1^{\varepsilon} \setminus \overline{\mathrm{LD}}_1^{\varepsilon,\xi}) \\
&\le n\mathbb{P}\left\{X_n \in (\varepsilon n, (a-\mu-\xi)n], S_{n-1} > an - (a-\mu-\xi)n\right\} \\
&\le n\overline{F}(n\varepsilon)\mathbb{P}\{S_{n-1} > n(\mu+\xi)\}.
\end{aligned} \tag{4.11}$$

For the last probability, we have

$$\begin{aligned}
&\mathbb{P}\{S_{n-1} > n(\mu+\xi)\} \\
&= \mathbb{P}\{S_{n-1} > n(\mu+\xi); X_i > \varepsilon' n \text{ for some } i \in \{1,\dots,n-1\}\} \\
&\quad + \mathbb{P}\{S_{n-1} > n(\mu+\xi); X_i \le \varepsilon' n \text{ for all } i \in \{1,\dots,n-1\}\} \\
&\le n\overline{F}(\varepsilon' n) + \mathbb{P}\{S_{n-1} > n(\mu+\xi); X_i \le \varepsilon' n, \text{ for all } i \in \{1,\dots,n-1\}\},
\end{aligned}$$

where $\varepsilon' > 0$ is arbitrary. From assumption (LL) we know that $n\overline{F}(\varepsilon' n) \ge n^{-\underline{\alpha}}$ for large n. Lemma 3.1 gives us $\varepsilon' \in (0, \varepsilon)$ such that

$$\begin{aligned}
&\mathbb{P}\{S_{n-1} > n(\mu+\xi); X_i \le \varepsilon' n, \text{ for all } i \in \{1,\dots,n-1\}\} \\
&\le n^{-\underline{\alpha}-1} = o(1)n\overline{F}(\varepsilon' n)
\end{aligned}$$

for large n.

Continuing from (4.11) we fix b and estimate

$$\begin{aligned}
&\mathbb{P}(\mathrm{LD}_1^{\varepsilon} \setminus \overline{\mathrm{LD}}_1^{\varepsilon,\xi}) \\
&\le n\overline{F}(\varepsilon n)(1+o(1))n\overline{F}(\varepsilon' n) \\
&\le n^2\overline{F}(\varepsilon' n)^2(1+o(1)) \\
&\le o(1)n\overline{F}(bn),
\end{aligned} \tag{4.12}$$

where in (4.12) we have used (4.10). Setting $b = a - \mu + \xi$ in this estimate we have

$$\begin{aligned}
&\frac{\mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi})}{\mathbb{P}(\mathrm{LD}_1^{\varepsilon})} \\
&= \frac{\mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi}) + \mathbb{P}(\mathrm{LD}_1^{\varepsilon} \setminus \overline{\mathrm{LD}}_1^{\varepsilon,\xi})}{\mathbb{P}(\mathrm{LD}_1^{\varepsilon})} - \frac{\mathbb{P}(\mathrm{LD}_1^{\varepsilon} \setminus \overline{\mathrm{LD}}_1^{\varepsilon,\xi})}{\mathbb{P}(\mathrm{LD}_1^{\varepsilon})} && (4.13) \\
&\ge 1 - \frac{o(1)n\overline{F}((a-\mu+\xi)n)}{(1+o(1))n\overline{F}((a-\mu+\xi)n)} && (4.14) \\
&= 1 - o(1),
\end{aligned}$$

where in (4.13) recall that $\overline{\mathrm{LD}}_1^{\varepsilon,\xi} \subset \mathrm{LD}_1^{\varepsilon}$, and in (4.14) we have applied Lemma 4.1. □

5 Discussion of conditions

We often assumed in the first part of the paper that μ is finite, and that (UL) and (LL) hold. These assumptions are general and appear in many other papers. Condition (CS) seems to be new and deserves more attention. The following results give motivation for the condition.

The first observation is that $\frac{x\overline{F}(\kappa x)^2}{\overline{F}(x)}$ at least oscillates close to zero. From this point of view, condition (CS) means that these oscillations stabilize in the limit.

Lemma 5.1 *Assume* (UL)*. Then for any* $\kappa > 0$

$$\liminf_{x\to\infty} \frac{x\overline{F}(\kappa x)^2}{\overline{F}(x)} = 0. \tag{5.1}$$

Proof Let $\eta > 0$. By (UL), we can find a sequence (a_i) tending to infinity such that

$$\overline{F}(a_i) \geq a_i^{-\overline{\alpha}-\eta}$$

for large i. By (2.3),

$$\frac{a_i\overline{F}(\kappa a_i)^2}{\overline{F}(a_i)} \leq \frac{a_i(\kappa a_i)^{-2\overline{\alpha}+2\eta}}{a_i^{-\overline{\alpha}-\eta}} = \kappa^{-2\overline{\alpha}+2\eta} a_i^{-\overline{\alpha}+1+3\eta} \tag{5.2}$$

for large i. Take η small enough so that $-\overline{\alpha}+1+3\eta < 0$. Then the right hand side of (5.2) goes to 0 as $i \to \infty$, which implies (5.1). $\Box$.

The main implication of Lemmata 4.2 and 4.3 for our purposes is that under the conditions of lemma 4.3,

$$\mathbb{P}(\mathrm{LD}_s^\varepsilon) = o(1)\mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi}).$$

The next observation is a partial converse to this estimate.

Lemma 5.2 *Assume that* μ *is finite,* (UL)*, and that*

$$\limsup_{x\to\infty} \frac{x\overline{F}(\kappa x)^2}{\overline{F}(x)} > 0$$

for some $\kappa > \frac{1}{2}$*. Then for sufficiently small* $\xi > 0$ *and* $0 < \varepsilon < a - \mu - \xi$*, there exist a constant* $c > 0$ *and a sequence of natural numbers* (n_i) *tending to infinity such that*

$$\mathbb{P}\left({}_{n_i}\mathrm{LD}_s^\varepsilon\right) \geq c\mathbb{P}({}_{n_i}\overline{\mathrm{LD}}_1^{\varepsilon,\xi})(1+o(1)).$$

Proof Choose $\kappa > \frac{1}{2}$ and $l \in (0,\infty)$, and a sequence (b_i) tending to infinity such that

$$\frac{b_i\overline{F}(\kappa b_i)^2}{\overline{F}(b_i)} \geq l \tag{5.3}$$

for every i. We choose a sequence of natural numbers (n_i) for which

$$\frac{n_i(a-\mu+\xi)}{2\kappa} \leq b_i < n_i(a-\mu-\xi)$$

for large i. This is possible for small ξ because $\kappa > \frac{1}{2}$. Starting from the estimate given by Lemma 4.1 we get

$$\begin{aligned}
\mathbb{P}({}_{n_i}\overline{\mathrm{LD}}_2^{\varepsilon,\xi}) &\geq \binom{n_i}{2}\overline{F}\Big(\frac{n_i}{2}(a-\mu+\xi)\Big)^2(1+o(1)) \\
&\geq \binom{n_i}{2}\overline{F}\left(\kappa b_i\right)^2(1+o(1)) \\
&\geq \binom{n_i}{2}\frac{l}{b_i}\overline{F}(b_i)(1+o(1)) \qquad (5.4)\\
&\geq cn_i\overline{F}(n_i(a-\mu-\xi))(1+o(1)), \qquad (5.5)\\
&\geq c\mathbb{P}({}_{n_i}\overline{\mathrm{LD}}_1^{\varepsilon,\xi})(1+o(1)) \qquad (5.6)
\end{aligned}$$

where in (5.4) we have used (5.3), in (5.5) we have taken $c = \frac{l}{4}\frac{1}{a-\mu-\xi}$, and in (5.6) we have used Lemma 3.1 with $k=1$. The claim follows since $\overline{\mathrm{LD}}_2^{\varepsilon,\xi} \subseteq \mathrm{LD}_s^\varepsilon$. $\Box$

6 What do the lemmata tell us?

Now we can give conditions under which the large deviation is caused by a single random variable.

Theorem 6.1 *Assume that μ is finite,* (UL), (LL) *and* (CS). *Let ξ and ε be such that $0 < \xi < a - \mu$ and $0 < \varepsilon < a - \mu - \xi$. Then*

$$\lim_{n\to\infty} \mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi}|\mathrm{LD}) = 1.$$

Proof From the definition of $\overline{\mathrm{LD}}_1^{\varepsilon,\xi}$ we see that its probability increases as ε increases. It is therefore sufficient to prove the claim for ε close to zero. Lemma 4.1 gives the estimate $\mathbb{P}(\mathrm{LD}) \geq n^{-\underline{\alpha}}$ for large n. According to Lemma 3.1, $\mathbb{P}(\mathrm{LD}_0^\varepsilon) \leq n^{-\underline{\alpha}-1}$ for ε small enough and large n. These lead to

$$\mathbb{P}(\mathrm{LD}_0^\varepsilon) = o(1)\mathbb{P}(\mathrm{LD}).$$

By Lemma 4.2,

$$\mathbb{P}(\mathrm{LD}_s^\varepsilon) = o(1)\mathbb{P}(\mathrm{LD}).$$

Using these estimates, we conclude that

$$\begin{aligned}
&\mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi}|\mathrm{LD}) \\
&= \frac{\mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi})}{\mathbb{P}(\mathrm{LD})} = \frac{(1+o(1))\mathbb{P}(\mathrm{LD}_1^\varepsilon)}{\mathbb{P}(\mathrm{LD})} && (6.1)\\
&= \frac{1}{\mathbb{P}(\mathrm{LD})}\left[(1+o(1))\left(\mathbb{P}(\mathrm{LD}) - \mathbb{P}(\mathrm{LD}_0^\varepsilon) - \mathbb{P}(\mathrm{LD}_s^\varepsilon)\right)\right] && (6.2)\\
&\to_{n\to\infty} 1,
\end{aligned}$$

where in (6.1) we have used Lemma 4.3, and in (6.2) we have used the fact that LD is the disjoint union of $\mathrm{LD}_0^\varepsilon$, $\mathrm{LD}_1^\varepsilon$ and $\mathrm{LD}_s^\varepsilon$. □

Theorem 6.1 can be used to give an estimate for the probability of the large deviation.

Theorem 6.2 *Assume that μ is finite,* (UL), (LL) *and* (CS). *For any $\xi > 0$*

$$(1+o(1))n\overline{F}(n(a-\mu+\xi)) \leq \mathbb{P}(\mathrm{LD}) \leq (1+o(1))n\overline{F}(n(a-\mu-\xi))$$

Proof It is enough to consider small ξ, because increasing it makes the estimate worse. Choose ε and assume ξ small enough for Theorem 6.1 to hold. Using that and Lemma 4.1 gives

$$\mathbb{P}(\mathrm{LD}) = (1+o(1))\mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi}) \leq (1+o(1))n\overline{F}(n(a-\mu-\xi)).$$

The first inequality is a special case of Lemma 4.1. □

If the large deviation is not caused by a single random variable how does it happen? From our results it is not clear that the significant part of LD stabilizes to a structure that can be easily described. However, we have the following partial converse of Theorem 6.1.

Theorem 6.3 *Assume that μ is finite,* (UL), *and that*

$$\limsup_{x\to\infty} \frac{x\overline{F}(\kappa x)^2}{\overline{F}(x)} > 0$$

for some $\kappa > \frac{1}{2}$. Then for sufficiently small $\xi > 0$ and $0 < \varepsilon < a - \mu - \xi$, we have

$$\liminf_{n\to\infty} \mathbb{P}(\overline{\mathrm{LD}}_1^{\varepsilon,\xi}|\mathrm{LD}) < 1.$$

Proof Let the constant $c > 0$ and the sequence (n_i) be as in Lemma 5.2. Then

$$\begin{aligned}\mathbb{P}({}_{n_i}\overline{\mathrm{LD}}_1^{\varepsilon,\xi} \mid {}_{n_i}\mathrm{LD}) &= \frac{\mathbb{P}({}_{n_i}\overline{\mathrm{LD}}_1^{\varepsilon,\xi})}{\mathbb{P}({}_{n_i}\mathrm{LD})} \\ &\le \frac{\mathbb{P}({}_{n_i}\overline{\mathrm{LD}}_1^{\varepsilon,\xi})}{\mathbb{P}({}_{n_i}\mathrm{LD}_1^{\varepsilon}) + \mathbb{P}({}_{n_i}\mathrm{LD}_s^{\varepsilon})} \\ &\le \frac{\mathbb{P}({}_{n_i}\overline{\mathrm{LD}}_1^{\varepsilon,\xi})}{\mathbb{P}({}_{n_i}\overline{\mathrm{LD}}_1^{\varepsilon,\xi}) + c\mathbb{P}({}_{n_i}\overline{\mathrm{LD}}_1^{\varepsilon,\xi})(1+o(1))} \to_{i\to\infty} \frac{1}{1+c} < 1, \end{aligned} \tag{6.3}$$

where in (6.3) we have applied Lemma 5.2. □

7 Special cases

We show in this section that our conditions are satisfied for wide classes of distributions. We also consider a refinement of the asymptotic estimate of Theorem 6.1 in a special case.

It is natural to ask if condition (CS) holds automatically for appropriate values of $\overline{\alpha}$ and $\underline{\alpha}$. The following result was actually the original motivation of our research.

Proposition 7.1 *Assume* (UL) *and* (LL). *Let* $\overline{F}$ *be such that*

$$1 - 2\overline{\alpha} + \underline{\alpha} < 0. \tag{7.1}$$

Then (CS) *holds.*

Proof Let $\gamma > 0$. Using bounds (2.3) we get for $\kappa > 0$ and for large x

$$\frac{x\overline{F}^2(\kappa x)}{\overline{F}(x)} \le \frac{x x^{-2\overline{\alpha}+2\gamma}\kappa^{-2\overline{\alpha}+2\gamma}}{x^{-\underline{\alpha}-\gamma}} = \kappa^{-2\overline{\alpha}+2\gamma}x^{1-2\overline{\alpha}+\underline{\alpha}+3\gamma}.$$

Take small γ to see that the right hand side goes to 0 under (7.1). □

The following example shows that condition (7.1) is in essence strict for the result of Proposition 7.1.

Example 7.2 Let $-\overline{\alpha}$ and $-\underline{\alpha}$ be such that $-\overline{\alpha} < -1$ and $-\underline{\alpha} > -\infty$. Instead of (7.1), assume that

$$1 - 2\overline{\alpha} + \underline{\alpha} > 0. \tag{7.2}$$

We construct a distribution with finite mean and limits (2.1) and (2.2), but condition (CS) and also the conclusion of Theorem 6.1 fail.

Let $a_1 = 1$, $a_2 = 2$, and

$$a_{i+1} = a_i^{\underline{\alpha}/\overline{\alpha}},$$

for $i = 2, 3, \ldots$. Then $a_i = 2^{\rho^{i-2}}$ for $i \ge 2$ where $\rho = \underline{\alpha}/\overline{\alpha} > 1$. Take $\mathbb{P}\{X = a_1\} = 1 - 2^{-\overline{\alpha}}$. For $i \ge 2$, take

$$\mathbb{P}\{X = a_i\} = a_i^{-\overline{\alpha}} - a_{i+1}^{-\overline{\alpha}}$$

so that

$$\overline{F}(a_i) = a_{i+1}^{-\overline{\alpha}} = a_i^{-\underline{\alpha}}. \tag{7.3}$$

The distribution is concentrated on $\{a_1, a_2, \ldots\}$, limits (2.1) and (2.2) follow from construction, and the mean is finite because $X \geq 1$ and $-\overline{\alpha} < -1$.

We pick $0 < \kappa < 1$ and consider the quotient $\frac{x\overline{F}(\kappa x)^2}{\overline{F}(x)}$ along the sequence (a_i). We conclude that

$$\frac{a_i\overline{F}(\kappa a_i)^2}{\overline{F}(a_i)} \geq \frac{a_i\overline{F}(a_{i-1})^2}{\overline{F}(a_i)} = \frac{a_i a_i^{-2\overline{\alpha}}}{a_i^{-\underline{\alpha}}} = a_i^{-2\overline{\alpha}+\underline{\alpha}+1},$$

where we used $\overline{F}(\kappa a_i) \geq \overline{F}(a_i-) = \overline{F}(a_{i-1})$. It is seen that under (7.2)

$$\limsup_{x\to\infty} \frac{x\overline{F}(\kappa x)^2}{\overline{F}(x)} > 0$$

and (CS) fails. $\overline{F}$ satisfies the conditions of Theorem 6.3, which shows that the conclusion of Theorem 6.1 does not hold.

Proposition 7.3 *Assume* (UL) *and that* $\overline{F}$ *is of dominated variation i.e.*

$$\liminf_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} > 0 \tag{7.4}$$

for some $x > 1$*. Then* (CS) *holds.*

Proof First observe that (7.4) actually holds for any $x > 1$. Namely,

$$\liminf_{t\to\infty} \frac{\overline{F}(tx^2)}{\overline{F}(t)} \geq \liminf_{t\to\infty} \frac{\overline{F}((tx)x)}{\overline{F}(tx)} \liminf_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} > 0.$$

By repeating the process we see that (7.4) holds when x is replaced by x^n for any $n \in \mathbb{N}$. Because $\overline{F}$ is decreasing it holds for every $x > 1$. Let $0 < \kappa < 1$ and write

$$\frac{n\overline{F}(\kappa n)^2}{\overline{F}(n)} = n\overline{F}(\kappa n)\left(\frac{\overline{F}(n)}{\overline{F}(\kappa n)}\right)^{-1}.$$

The quotient on the right hand side is bounded from below by a positive constant and under (UL), $n\overline{F}(\kappa n)$ goes to 0 as $n \to \infty$. We see that condition (CS) is satisfied. □ In case of dominated variation, we can improve Theorem 6.2.

Theorem 7.4 *Assume that expectation* μ *exists,* (UL)*,* (LL) *and that* $\overline{F}$ *is of dominated variation. Write*

$$c = \lim_{x\to 1+} \liminf_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} \in (0, 1]. \tag{7.5}$$

We have

$$cn\overline{F}(n(a-\mu))(1+o(1)) \leq \mathbb{P}(\mathrm{LD}) \leq \frac{1}{c}n\overline{F}(n(a-\mu))(1+o(1)),$$

and in particular if $c = 1$

$$\mathbb{P}(\mathrm{LD}) = n\overline{F}(n(a-\mu))(1+o(1)).$$

Remark 7.5 A similar result was obtained by Tang and Yan (2002). They only assume finite expectation and dominated variation, but do not consider the size of the constants in the expression (c and $\frac{1}{c}$ in our formulation). If $c = 1$ then (7.5) means that $\overline{F}$ varies consistently. The result of the theorem in this case is known from Ng et al. (2004).

Proof of Theorem 7.4 Choose $\delta > 0$. Take ε small enough for

$$\frac{\overline{F}(n(a-\mu+\varepsilon))}{\overline{F}(n(a-\mu))} \geq (1-\delta)c$$

to hold for large n. Theorem 6.2 gives the estimate

$$\mathbb{P}(\mathrm{LD}) \geq (1-\delta)n\overline{F}(n(a-\mu+\varepsilon))$$

for large n. Combining these we get

$$\begin{aligned}\mathbb{P}(\mathrm{LD}) &\geq (1-\delta)n\overline{F}(n(a-\mu))\frac{\overline{F}(n(a-\mu+\varepsilon))}{\overline{F}(n(a-\mu))} \\ &\geq (1-\delta)^2 cn\overline{F}(n(a-\mu))\end{aligned}$$

for large n. Similarly, it is seen that

$$\mathbb{P}(\mathrm{LD}) \leq (1+\delta)^2\frac{1}{c}n\overline{F}(n(a-\mu))$$

for large n. The obtained estimates prove the theorem. □

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