TOEPLITZ OPERATORS ON BLOCH-TYPE SPACES AND CLASSES OF WEIGHTED SOBOLEV DISTRIBUTIONS

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ABSTRACT. We study Toeplitz operators between the analytic Bloch-type spaces of the unit disk. We construct suitable classes of distributions that generate bounded Toeplitz operators between these spaces. The classes are naturally connected to the corresponding results in the reflexive Bergman space setting and previously known results on A^1 and the Bloch space. We also study distributional symbols satisfying logarithmic BMO-condition. In addition, sufficient compactness criteria are provided.

1. INTRODUCTION.

Toeplitz operators on the reflexive Bergman spaces have been extensively studied for many years. Their theory is indeed largely understood. However, complete characterization of the symbol classes that generate bounded, compact or finite rank Toeplitz operators has been a long-standing problem. In fact, only the finite rank question has been satisfactorily settled quite recently by D. Luecking in [4]. A generalization of this result is given in [2]. Even less is known in the setting of the Bloch spaces $\mathcal{B}_d$. There are some results in these directions, see for instance [9, 10, 12, 15].

In this paper we study questions related to these spaces by extending the definition of Toeplitz operators to distributional symbols. We will look at boundedness and compactness of Toeplitz operators by using the machinery developed in [5, 6, 7, 10, 11].

Let $d > 0$ and $\mathbb{D}$ be the open unit disk centered at the origin. The analytic Bloch space $\mathcal{B}_d$ of the unit disk consist of those analytic functions f for which the following semi-norm:

$$\|f\|_{*,d} = \sup_{z\in\mathbb{D}}(1 - |z|^2)^d |f'(z)|$$

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is finite. We then equip the space $\mathcal{B}_d$ with the norm

$$\|f\|_{\mathcal{B}_d} = \|f\|_{*,d} + |f(0)|,$$

which will make it a Banach space. The little Bloch space $\mathcal{B}_{0,d}$ is the closed subspace of $\mathcal{B}_d$, whose elements satisfy

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^d|f'(z)| \to 0, \text{ when } |z| \to 1^-.$$

For reference on the Bloch spaces we mention [12, 15, 16, 17]. When $d = 1$, we write $\mathcal{B}$ and $\mathcal{B}_0$ to mean the spaces $\mathcal{B}_1$ and $\mathcal{B}_{0,1}$, respectively. Because we always assume that $d > 0$, there is no ambiquity in the notation $\mathcal{B}_0$.

Note that if $d > d' > 0$ and $1 \le p < \infty$, then

$$\mathcal{B}_{d'} \subset \mathcal{B}_{0,d} \subset \mathcal{B}_d \subset A^p_{d-1},$$

where A^p_{d-1} is a certain weighted Bergman space. Also, if $0 < d < 1$, the spaces $\mathcal{B}_d$ and $\mathcal{B}_{0,d}$ are contained in the algebra of bounded analytic functions.

The Bergman projection is defined for $f \in L^1$ by

$$Pf(z) = \int_{\mathbb{D}} \frac{f(w)dA(w)}{(1-z\bar{w})^2},$$

where $dA(z) = \pi^{-1}dxdy$ is the normalized Lebesgue area measure. It is known that P is bounded from L^p onto A^p for each $p \in (1,\infty)$. It is not bounded in the end-point cases, $p \in \{1,\infty\}$. However, P is known to map L^∞ (and even BMO_∂) boundedly onto $\mathcal{B}$. For this reason it is often more convenient to study Toeplitz operators on $\mathcal{B}$ and use duality to obtain results in A^1. For a reference on multipliers of these BMO-type spaces and continuity of the relevant projections we mention [13, 14, 15]. The other direction is also possible, see [9, 11].

Suppose f is analytic and a is a measurable function such that $af \in L^1$. Then is makes sense to define the Toeplitz operator with symbol a acting on f by

$$T_af(z) = \int_{\mathbb{D}} \frac{f(w)a(w)dA(w)}{(1-z\bar{w})^2}.$$

Since $w \mapsto f(w)(1-z\bar{w})^{-2}$ is smooth, it makes sense to define T_a for compactly supported distribution a by

$$T_af(z) = \langle f(w)(1-z\bar{w})^{-2}, a\rangle_w \ , \ z \in \mathbb{D}.$$

However, it is not difficult to see that such operator is compact $A^p \to A^p$ for all $p \in [1,\infty]$ and $\mathcal{B}_d \to \mathcal{B}_{d'}$ for $d, d' > 0$. Therefore, we will need to look for larger classes of suitable distributions.

Together with J. Taskinen and J. Virtanen the author studied a weighted Sobolev spaces of distributions, generating bounded Toeplitz operators on the Bergman and Fock space settings. See [5, 6, 7] for reference. This work bears more similarity to the Bergman space approach, since all the functions are defined on the unit disk. When dealing with Toeplitz operators $\mathcal{B} \to \mathcal{B}_d$ for $d > 0$, we will need an additional logarithmic weight. We will actually make use of several distributional classes to cover Toeplitz operators acting between the spaces $\mathcal{B}_d$ and $\mathcal{B}_{d'}$ with the possibility of $d \neq d'$. This kind of setting has not been actively studied before, but it works nicely with the families of distributions presented. This paper is in a sense a generalization of the results in [11].

We will first review some theory of weighted Sobolev spaces. All the results are more or less well-known and obtainable by using [1]. Similar things were done in more detail in [5] and [6]. We will omit the proofs here. In section 3 we will consider boundedness of Toeplitz operators with distributional symbols between various Bloch spaces, and after that take a look at the compactness in the section 4. In the fifth section we study logarithmic BMO-type symbol classes in setting of $\mathcal{B}$ and, finally, in the section 6 we obtain the results in the Bergman space A^1.

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2. Preliminanies on Sobolev spaces.

In this paper we will refer to [8] for general theory of distributions, [1] for Sobolev spaces, and [17] for operator theory and analytic function spaces. In what follows we will encounter several spaces of functions and distributions, but they all are defined on $\mathbb{D}$. For the norm of an element f of a Banach function space X we use the notation $\|f; X\|$; for the operator norm of a bounded linear operator $T : X \to Y$ we write $\|T : X \to Y\|$. This notation is not the most standard one, but seems appropriate, as we deal with operators between several spaces that already have somewhat complex notation. The standard space of infinitely smooth compactly supported test functions of the unit disk is denoted by $C_0^\infty = C_0^\infty(\mathbb{D})$, and its dual, the space of distributions on $\mathbb{D}$, is $\mathcal{D}' = \mathcal{D}'(\mathbb{D})$. All our symbols will be assumed to be members of $\mathcal{D}'$. The order of a multi–index $\alpha \in \mathbb{N}^2$, where $\mathbb{N} := \{0, 1, 2, , \ldots\}$, is denoted by $|\alpha| := \alpha_1 + \alpha_2$. The notation $\alpha \geq \beta$ for the multi–indices α, β means that $\alpha_j \geq \beta_j$ for $j = 1, 2$. As for derivatives, the notation $D^\alpha f$ stands for

$$\frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial y^{\alpha_2}} f,$$

if f is a function of $z = x + iy$, where x, $y \in \mathbb{R}$, and α is a multi–index. The same notation is used for both classical and distributional derivatives. We also write $D_w^\alpha f$, if it is necessary to indicate the differentiation of a function f with respect to its variable w. For an analytic function f of the variable $z \in \mathbb{D}$, we denote by $f^{(l)}$ the l:th derivative with respect to z, for all $l \in \mathbb{N}$. By C, C', C_1, c etc. (respectively, C_n etc.) we mean positive constants independent of functions, variables or indices occurring in the given calculations (respectively, depending only on n). These may vary from place to place, but not in the same group of inequalities.

We define $\nu : \mathbb{D} \to \mathbb{R}^+$ to be the standard weight on the unit disk;

$$\nu(z) = 1 - |z|^2.$$

We will also need the logarithmic correction weight $\ell : \mathbb{D} \to \mathbb{R}^+$ on the unit disk. It is defined by

$$\ell(z) = 1 + |\log(\nu(z))|.$$

Given $m \in \mathbb{N}$ and $t > -1$ denote by $W^{m,1}_{t,\nu} = W^{m,1}_{t,\nu}(\mathbb{D})$ the weighted Sobolev space consisting of measurable functions f on $\mathbb{D}$ such that

$$\|f; W^{m,1}_{t,\nu}\| := \sum_{|\alpha|\le m} \int_{\mathbb{D}} |D^\alpha f(z)| \nu(z)^{-t+|\alpha|} dA(z) < \infty. \tag{2.1}$$

Similarly, $\mathcal{L}W^{m,1}_{t,\nu} = \mathcal{L}W^{m,1}_{t,\nu}(\mathbb{D})$ is the weighted logarithmic Sobolev space consisting of measurable functions f on $\mathbb{D}$ such that

$$\|f; \mathcal{L}W^{m,1}_{t,\nu}\| := \sum_{|\alpha|\le m} \int_{\mathbb{D}} |D^\alpha f(z)| \nu(z)^{-t+|\alpha|} \ell(z)^{-1} dA(z) < \infty. \tag{2.2}$$

We will use the following density result, which is similar to that proven in [5]. Note that this result is not true in the standard, unweighted Sobolev space $W^{m,1}$, unless $m = 0$.

Lemma 2.1. *Let $t > -1$. The subspace C_0^∞ of compactly supported infinitely smooth functions on $\mathbb{D}$ is dense in $W^{m,1}_{t,\nu}$ and $\mathcal{L}W^{m,1}_{t,\nu}$.*

Proof. The proof is essentially the same as the one given in [5]; The norm in $W^{m,1}_{t,\nu}$ (or $\mathcal{L}W^{m,1}_{t,\nu}$) just needs to add extra multiple of ν for higher order derivatives, the weight in case $\alpha = 0$ is irrelevant. □

Definition 2.2. Given $m \in \mathbb{N}$ and $t > -1$ we denote by

$$\mathcal{Y}^m_t := W^{-m,\infty}_{t,\nu} = W^{-m,\infty}_{t,\nu}(\mathbb{D})$$

$$(\mathcal{L}\mathcal{Y}^m_t := \mathcal{L}W^{-m,\infty}_{t,\nu} = \mathcal{L}W^{-m,\infty}_{t,\nu}(\mathbb{D}))$$

the weighted Sobolev spaces consisting of distributions a on $\mathbb{D}$ which can be written in the form

$$a = \sum_{0\le|\alpha|\le m} (-1)^{|\alpha|} D^\alpha b_\alpha, \tag{2.3}$$

where

$$\|b_\alpha; L^\infty_{t-|\alpha|}\| := \|b_\alpha \nu^{t-|\alpha|}\|_\infty < \infty$$

$$(\|b_\alpha;\mathcal{L}L^\infty_{t-|\alpha|}\| := \|b_\alpha \ell\nu^{t-|\alpha|}\|_\infty < \infty).$$

Here every b_α is considered as a distribution like a locally integrable function, and the identity (2.3) contains distributional derivatives. We refer to representations like (2.3) by $(b_\alpha) := (b_\alpha)_{|\alpha|\le m}$.

The representation (2.3) is not unique in general. Hence, we define the norm in $\mathcal{Y}^m_t$ ($\mathcal{L}\mathcal{Y}^m_t$) by

$$\|a\| := \|a;\mathcal{Y}^m_t\| := \inf \max_{0\le|\alpha|\le m} \|b_\alpha; L^\infty_{t-|\alpha|}\|,$$

$$(\|a\| := \|a;\mathcal{L}\mathcal{Y}^m_t\| := \inf \max_{0\le|\alpha|\le m} \|b_\alpha; \mathcal{L}L^\infty_{t-|\alpha|}\|),$$

where the infimum is taken over all representations (2.3).

The following lemma was also essentially proven in [5]:

Lemma 2.3. *Let $t > -1$. The dual of $W^{m,1}_{t,\nu}$ (dual of $\mathcal{L}W^{m,1}_{t,\nu}$) is isometrically isomorphic to $\mathcal{Y}^m_t$ ($\mathcal{L}\mathcal{Y}^m_t$) with respect to the dual paring*

$$\langle f,a\rangle := \sum_{0\le|\alpha|\le m} \int_{\mathbb{D}} (D^\alpha f) b_\alpha dA, \tag{2.4}$$

where the functions b_α are as in (2.3).

Remark 2.4. Let $t > -1$. If $a \in \mathcal{Y}^m_t$ ($a \in \mathcal{L}\mathcal{Y}^m_t$), the value of the expression on the right hand side of (2.4) is unique, although the representation (2.3) is not. Namely, for every $\varphi \in C^\infty_0$, the value of

$$\sum_{0\le|\alpha|\le m} \int_{\mathbb{D}} (D^\alpha \varphi) b_\alpha dA$$

coincides with $\langle \varphi, a\rangle$, by the standard definition of distributional derivative, and the uniqueness of (2.4) follows from Lemma 2.1.

The space $\mathcal{Y}^m_0$ is the space $W^{-m,\infty}_\nu$ of [5, 7]. Note also that for each $t > -1$ and a compactly supported distribution a, there exists $m \in \mathbb{N}$ such that $a \in \mathcal{L}\mathcal{Y}^m_t \subset \mathcal{Y}^m_t$. Also, when $t \le 0$, the space $\mathcal{Y}^m_t$ contains all bounded functions; same is true about $\mathcal{L}\mathcal{Y}^m_t$, when $t < 0$. Also, if $-1 < t < t' < \infty$, then

$$\mathcal{L}\mathcal{Y}^m_{t'} \subset \mathcal{Y}^m_{t'} \subset \mathcal{L}\mathcal{Y}^m_t$$

for each $m \in \mathbb{N}$ and with bounded inclusions.

3. Bounded Toeplitz operators with distributional symbols.

In order to work effectively in the setting of Bloch spaces for different values of $d > 0$, we need a result on operators generated by Bergman-type kernels. The following lemmas can be found in [15, 17], for instance.

Lemma 3.1. *Let $d > 0$, $s > -1$ and t be real. Then the operator $J_{s,t}$ defined by*

$$J_{s,t}f(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^s f(w)dA(w)}{(1-z\bar{w})^{1+s+t}}$$

is bounded $L^\infty \to \mathcal{B}_d$ whenever $t \leq d$.

Lemma 3.2. *Let $d > 0$, $n \geq 1$ and suppose that f is analytic. Then*

$$\|\nu^{d-1+n} f^{(n)}\|_\infty \leq C_n \|f\|_{\mathcal{B}_d},$$

for some $C_n > 0$. An analytic function f belongs to $\mathcal{B}_d$ if and only if $\nu^{d-1+k} f^{(k)}$ is bounded.

Lemma 3.3. *The following three statements are true.*

1. *(1) If $0 < d < 1$ and $f \in \mathcal{B}_d$, then there exists a positive constant $C > 0$ so that $|f(z)| \leq C\|f\|_{\mathcal{B}_d}$ for each $z \in \mathbb{D}$.*
2. *(2) For $f \in \mathcal{B}$ and $z \in \mathbb{D}$, there exists a positive constant $C > 0$ so that $|f(z)|/\ell(z) \leq C\|f\|_{\mathcal{B}}$ for each $z \in \mathbb{D}$.*
3. *(1) If $1 < d < \infty$ and $f \in \mathcal{B}_d$, then there exists a positive constant $C > 0$ so that $\nu(z)^{d-1}|f(z)| \leq C\|f\|_{\mathcal{B}_d}$ for each $z \in \mathbb{D}$.*

Toeplitz operator with symbol $a \in \mathcal{Y}_t^m$ ($a \in \mathcal{LY}_t^m$) is defined like in [5, 7]. Note that $\mathcal{B}_d$ is contained in the predual of $\mathcal{Y}_t^m$ if and only if $t > d - 2$. In particular, one has to be careful when defining Toeplitz operators on $\mathcal{B}_d$ for $d > 1$.

Definition 3.4. Let $0 < d < \infty$, $t > -1$ and $t > d - 2$. Assume that the distribution $a \in \mathcal{D}'$ belongs to $\mathcal{Y}_t^m$ ($\mathcal{LY}_t^m$) for some m. Then the Toeplitz operator T_a is defined by the formula

$$T_a f(z) = \sum_{0\leq|\alpha|\leq m} \int_{\mathbb{D}} \Big(D_w^\alpha \frac{f(w)}{(1-z\bar{w})^2}\Big) b_\alpha(w) dA(w) \quad , \; f \in \mathcal{B}_d. \tag{3.1}$$

Since the above definition makes sense for $f \in A_t^1$ (see [5]), the operator is well-defined in the Bloch spaces as well.

The following theorems are the main results of this paper. The boundedness of $T_a : \mathcal{B}_d \to \mathcal{B}_{d'}$ depends on the various choices for d and d'. We remind the reader that, in our considerations, there are essentially three types of Bloch spaces $\mathcal{B}_d$; when $0 < d < 1$, the elements of $\mathcal{B}_d$ are bounded functions; the elements of $\mathcal{B}$ can have at most logarithmic singularity near the boundary; when $d > 1$, the functions in $\mathcal{B}_d$ can have at most singularity of order $d - 1$ near the boundary. After one differentiation, further differentiations of functions in any of the spaces will increase the order of the singularity by at most one. We state our main theorem in three parts to emphasize the behavior mentioned above.

Theorem 3.5. *Let $0 < d < 1$, $0 < d' < \infty$ and $a \in \mathcal{D}'$. The following propositions are true:*

(1) Suppose $d' < 2$ and $a \in \mathcal{Y}^m_{1-d'}$ for some $m \in \mathbb{N}$. Then T_a is bounded $\mathcal{B}_d \to \mathcal{B}_{d'}$ and $\|T_a; \mathcal{B}_d \to \mathcal{B}_{d'}\| \le C\|a; \mathcal{Y}^m_{1-d'}\|$ for some $C := C(d, d', m) > 0$.

(2) Suppose $d' \ge 2$ and $a \in \mathcal{Y}^m_t$ for some $t > -1$ and $m \in \mathbb{N}$. Then T_a is bounded $\mathcal{B}_d \to \mathcal{B}_{d'}$ and $\|T_a; \mathcal{B}_d \to \mathcal{B}_{d'}\| \le C\|a; \mathcal{Y}^m_t\|$ for some $C := C(d, d', t) > 0$.

Theorem 3.6. *Let $a \in \mathcal{D}'$. Then the following propositions are true:*

(1) Suppose $d' < 2$ and $a \in \mathcal{LY}^m_{1-d'}$ for some $m \in \mathbb{N}$. Then T_a is bounded $\mathcal{B} \to \mathcal{B}_{d'}$ and $\|T_a; \mathcal{B} \to \mathcal{B}_{d'}\| \le C\|a; \mathcal{LY}^m_{d-d'}\|$ for some $C := C(d', m) > 0$.

(2) Suppose $d' \ge 2$ and $a \in \mathcal{LY}^m_t$ for some $t > -1$ and $m \in \mathbb{N}$. Then T_a is bounded $\mathcal{B} \to \mathcal{B}_{d'}$ and $\|T_a; \mathcal{B} \to \mathcal{B}_{d'}\| \le C\|a; \mathcal{LY}^m_t\|$ for some $C := C(d', t) > 0$.

When $d > 1$, the Toeplitz operator can no longer be defined on $\mathcal{B}_d$ for all symbol classes $\mathcal{Y}^m_t$ $(t > -1)$; we need to restrict our attention to case $t > d - 2$.

Theorem 3.7. *Let $d > 1$ and $a \in \mathcal{D}'$. Then the following propositions are true:*

(1) Suppose $d' < 2$ and $a \in \mathcal{Y}^m_{d-d'}$ for some $m \in \mathbb{N}$. Then T_a is bounded $\mathcal{B}_d \to \mathcal{B}_{d'}$ and $\|T_a; \mathcal{B}_d \to \mathcal{B}_{d'}\| \le C\|a; \mathcal{Y}^m_{d-d'}\|$ for some $C := C(d, d', m) > 0$.

(2) Suppose $d' \ge 2$ and $a \in \mathcal{Y}^m_t$ for some $t > d - 2$ and $m \in \mathbb{N}$. Then T_a is bounded $\mathcal{B}_d \to \mathcal{B}_{d'}$ and $\|T_a; \mathcal{B}_d \to \mathcal{B}_{d'}\| \le C\|a; \mathcal{Y}^m_t\|$ for some $C := C(d, d', t) > 0$.

Proofs of theorems 3.5-3.7. The proof of theorem 3.6 is probably the most difficult as it contains the logarithmic weight. We therefore prove it and remark that the proofs for the other theorems are similar. Note that, for instance the condition $d' < 2$ means that for $d = 1$ we have $d' - d < 1$, so this part of the theorem is not different from the other theorems.

To prove (1) suppose $a \in \mathcal{LY}^m_{1-d'}$ with $d' < 2$ and $m \in \mathbb{N}$. Fix a representation (b_α) for a such that

$$\|a; \mathcal{LY}^m_{1-d'}\| \ge (1/2) \max_{|\alpha| \le m} \|b_\alpha; LL^\infty_{1-d'-|\alpha|}\|.$$

The operator T_a can now be represented as

$$T_a f(z) = \sum_{0 \le |\alpha| \le m} \int_{\mathbb{D}} \Big(D^\alpha_w \frac{f(w)}{(1 - z\bar{w})^2} \Big) b_\alpha(w) dA(w) \quad , \; f \in \mathcal{B}.$$

By using the Leibnitz formula, we get

$$T_a f(z) \quad = \sum_{|\alpha|\le m}\sum_{\beta\le\alpha} C_{\alpha,\beta}\int_{\mathbb{D}} \left[D^{\alpha-\beta} f(w)\right]\left[D^{\beta}(1-z\bar{w})^{-2})\right] b_\alpha(w)dA(w)$$

for some positive constants $C_{\alpha,\beta}$. Differentiating we get

$$\left[D^{\alpha-\beta} f(w)\right]\left[D^{\beta}(1-z\bar{w})^{-2})\right] = C'_{\alpha,\beta} z^{|\beta|} f^{(|\alpha|-|\beta|)}(w)(1-z\bar{w})^{-2-|\beta|}$$

for some complex constants $C'_{\alpha,\beta}$. We have thus shown that

$$T_a f(z) = \sum_{|\alpha|\le m}\sum_{\beta\le\alpha} C_{\alpha,\beta}C'_{\alpha,\beta} z^{|\beta|}\int_{\mathbb{D}} \frac{f^{(|\alpha|-|\beta|)}(w)b_\alpha(w)dA(w)}{(1-z\bar{w})^{2+|\beta|}}.$$

Since multiplication by polynomials is bounded on each $\mathcal{B}_d$, it is sufficient to prove that, under the assumptions of the theorem, the operators

$$R^{\alpha,\beta}_{b_\alpha} f(z) = \int_{\mathbb{D}} \frac{f^{(|\alpha|-|\beta|)}(w)b_\alpha(w)dA(w)}{(1-z\bar{w})^{2+|\beta|}} \tag{3.2}$$

are bounded $\mathcal{B}\to\mathcal{B}_{d'}$ and satisfy

$$\|R^{\alpha,\beta}_{b_\alpha};\mathcal{B}\to\mathcal{B}_{d'}\| \le C\|b_\alpha; LL^\infty_{1-d'-|\alpha|}\|$$

when $|\alpha|\le m$ and $\beta\le\alpha$.

First assume that $\beta=\alpha$. Then

$$R^{\alpha,\beta}_{b_\alpha} f(z) = R^{\alpha,\alpha}_{b_\alpha} f(z) = \int_{\mathbb{D}} \frac{f(w)b_\alpha(w)dA(w)}{(1-z\bar{w})^{2+|\alpha|}}.$$

Now, by lemma 3.3 and the assumption on b_α, we get

$$|f(w)b_\alpha(w)| \le C\ell(w)\|f;\mathcal{B}\|\ell(w)^{-1}\nu(w)^{1-d'+|\alpha|}\|b_\alpha; LL^\infty_{1-d'-|\alpha|}\|$$

for almost every $w\in\mathbb{D}$ and some $C>0$ independent of f and b_α. An application of 3.1 now gives the desired estimate.

Now assume that $\beta<\alpha$. Then

$$R^{\alpha,\beta}_{b_\alpha} f(z) = \int_{\mathbb{D}} \frac{f^{(|\alpha|-|\beta|)}(w)b_\alpha(w)dA(w)}{(1-z\bar{w})^{2+|\beta|}}.$$

By lemma 3.2 and the assumption on b_α, we get

$$\begin{aligned}|f^{(|\alpha|-|\beta|)}(w)b_\alpha(w)| &\le C\nu(w)^{|\beta|-|\alpha|}\|f;\mathcal{B}\|\ell(w)^{-1}\nu(w)^{1-d'+|\alpha|}\|b_\alpha; LL^\infty_{1-d'-|\alpha|}\| \\ &\le C\nu(w)^{1+d'+|\beta|}\|f;\mathcal{B}\|\|b_\alpha; LL^\infty_{1-d'-|\alpha|}\|\end{aligned}$$

for almost every $w\in\mathbb{D}$ and some $C>0$ independent of f and b_α. Again, lemma 3.1 applies here and gives the desired estimate. The case (1) is now proven.

For the case (2), note that everything is similar, but $\mathcal{B}$ is not a subset of the predual of $\mathcal{LY}^m_t$ if $t\le -1$ (indeed, even constats fail to be in this space), we therefore cannot have $t \leq -1$.

We conclude that the Toeplitz operator is a finite sum of products of bounded operators and hence bounded. □

The main theorems can be applied in the form of the following corollary, which concerns the case when $d = d'$.

Corollary 3.8. *Suppose $0 < d < \infty$ and $a \in \mathcal{D}'$. The following propositions are true:*

(1) *Suppose $d < 1$ and $a \in \mathcal{Y}^m_{1-d}$ for some $m \in \mathbb{N}$. Then T_a is bounded on $\mathcal{B}_d$ and $\|T_a; \mathcal{B}_d \to \mathcal{B}_d\| \leq C\|a; \mathcal{Y}^m_{1-d}\|$ for some $C := C(d, m) > 0$.*
(2) *Suppose $a \in \mathcal{L}\mathcal{Y}^m_0$ for some $m \in \mathbb{N}$. Then T_a is bounded on $\mathcal{B}$ and $\|T_a; \mathcal{B} \to \mathcal{B}\| \leq C\|a; \mathcal{L}\mathcal{Y}^m_0\|$ for some $C := C(m) > 0$.*
(3) *Suppose $1 < d < 2$ and $a \in \mathcal{Y}^m_0$ for some $m \in \mathbb{N}$. Then T_a is bounded on $\mathcal{B}_d$ and $\|T_a; \mathcal{B}_d \to \mathcal{B}_d\| \leq C\|a; \mathcal{Y}^m_0\|$ for some $C := C(d, m) > 0$.*
(4) *Suppose $d \geq 2$ and $a \in \mathcal{Y}^m_t$ for some $t > d - 2$ and $m \in \mathbb{N}$. Then T_a is bounded $\mathcal{B}_d \to \mathcal{B}_d$ and $\|T_a; \mathcal{B}_d \to \mathcal{B}_d\| \leq C\|a; \mathcal{Y}^m_t\|$ for some $C := C(d, t) > 0$.*

4. Compact Toeplitz operators with distributional symbols.

In this section we prove compactness versions of the main theorems 3.5-3.7. As it often is the case, the results follow from the boundedness results by using approximation. It is known that compactly supported distributions generate compact Toeplitz operators.

Lemma 4.1. *Let a be a compactly supported distribution of $\mathbb{D}$. Then $T_a : \mathcal{B}_d \to \mathcal{B}_{d'}$ is compact for each $d, d' \in (0, \infty)$.*

Proof. It was noted before that all the presented distributional classes contain compactly supported distributions. Hence $T_a : \mathcal{B}_d \to \mathcal{B}_{d'}$ makes sense for all $d, d' \in (0, \infty)$. Compactness follows by usual arguments by using uniform boundedness on compact subsets of $\mathbb{D}$ and normal family techniques, for instance. □

We now achieve the compactness versions of theorems 3.5-3.7 with little effort. We will use the following auxiliary definition.

Definition 4.2. Let $d > -1$ and $a \in \mathcal{Y}^m_d$ ($a \in \mathcal{L}\mathcal{Y}^m_d$) for some $m \in \mathbb{N}$. Suppose also that a has a representation (b_α) such that

$$\operatorname{ess} \lim_{r \to 1^-} \sup_{r < |z| < 1} \nu(z)^{-|\alpha|} |b_\alpha(z)| = 0$$

$$(\operatorname{ess} \lim_{r \to 1^-} \sup_{r < |z| < 1} \ell(z) \nu(z)^{-|\alpha|} |b_\alpha(z)| = 0)$$

for each multi-index α with $|\alpha| \leq m$. Then a is said to belong to $\mathcal{V}^m_d$ ($\mathcal{L}\mathcal{V}^m_d$).

The symbol $\mathcal{V}$ indicates vanishing near the boundary. Note that not all representations of a need to satisfy the above definition in order to have $a \in \mathcal{V}_d^m$ or $a \in \mathcal{L}\mathcal{V}_d^m$. In particular, there might be representations for a not satisfying the definition but having norm smaller than a representation satisfying it. Also, it possible that $a \in \mathcal{V}_d^m$ and $a \in \mathcal{Y}_d^{m'}$ for $m > m'$ but $a \notin \mathcal{V}_d^{m'}$. Same goes for the logarithmic spaces, as well.

Theorem 4.3. *Let $0 < d < 1$, $0 < d' < \infty$ and $a \in \mathcal{D}'$. The following propositions are true:*

1. *Suppose $d' < 2$ and $a \in \mathcal{V}_{1-d'}^m$ for some $m \in \mathbb{N}$. Then T_a is compact $\mathcal{B}_d \to \mathcal{B}_{d'}$.*
2. *Suppose $d' \geq 2$ and $a \in \mathcal{V}_t^m$ for some $t > -1$ and $m \in \mathbb{N}$. Then T_a is compact $\mathcal{B}_d \to \mathcal{B}_{d'}$.*

Theorem 4.4. *Let $a \in \mathcal{D}'$. Then the following propositions are true:*

1. *Suppose $d' < 2$ and $a \in \mathcal{L}\mathcal{V}_{1-d'}^m$ for some $m \in \mathbb{N}$. Then T_a is compact $\mathcal{B} \to \mathcal{B}_{d'}$.*
2. *Suppose $d' \geq 2$ and $a \in \mathcal{L}\mathcal{V}_t^m$ for some $t > -1$ and $m \in \mathbb{N}$. Then T_a is compact $\mathcal{B} \to \mathcal{B}_{d'}$.*

Theorem 4.5. *Let $d > 1$ and $a \in \mathcal{D}'$. Then the following propositions are true:*

1. *Suppose $d' < 2$ and $a \in \mathcal{V}_{d-d'}^m$ for some $m \in \mathbb{N}$. Then T_a is compact $\mathcal{B}_d \to \mathcal{B}_{d'}$.*
2. *Suppose $d' \geq 2$ and $a \in \mathcal{V}_t^m$ for some $t > d-2$ and $m \in \mathbb{N}$. Then T_a is compact $\mathcal{B}_d \to \mathcal{B}_{d'}$.*

Proof of theorems 4.3-4.5. Suppose (b_α) is a representation for a satisfying definition 4.2. Then a can be approximated in norm by distributions a_r, $0 < r < 1$, with $r \to 1^-$. Indeed, define $b_{\alpha,r}(z) = \chi_{B(0,r)}(z) b_\alpha(z)$. The distributions a_r (having representations $(b_{\alpha,r})$) are compactly supported and so, by the respective theorem of boundedness and lemma 4.1, the operator T_a must be compact, since it can be approximated in norm by compact operators. □

Corollary 4.6. *Suppose $0 < d < \infty$ and $a \in \mathcal{D}'$. The following propositions are true:*

1. *Suppose $d < 1$ and $a \in \mathcal{V}_{1-d}^m$ for some $m \in \mathbb{N}$. Then T_a is compact on $\mathcal{B}_d$.*
2. *Suppose $a \in \mathcal{L}\mathcal{V}_0^m$ for some $m \in \mathbb{N}$. Then T_a is compact on $\mathcal{B}$.*
3. *Suppose $1 < d < 2$ and $a \in \mathcal{V}_0^m$ for some $m \in \mathbb{N}$. Then T_a is compact on $\mathcal{B}_d$.*
4. *Suppose $d \geq 2$ and $a \in \mathcal{V}_t^m$ for some $t > d-2$ and $m \in \mathbb{N}$. Then T_a is compact on $\mathcal{B}_d$.*

5. Weighted BMO and Toeplitz operators.

In this section we concentrate on Toeplitz operators acting on the most common Bloch space $\mathcal{B}$. So far we have shown that if $a \in \mathcal{LY}_0^m$, then $T_a : \mathcal{B} \to \mathcal{B}$ will be bounded. An obvious drawback for this approach is that $\mathcal{LY}_0^m$ does not contain constant functions, even though they obviously generate bounded Toeplitz operators on $\mathcal{B}$. However, it has been known for over two decades (see [13, 14]) that membership of the (function) symbol a in the space $BMO_{\log}$ will imply boundedness of T_a on $\mathcal{B}$. This was also recently studied in [11]. This section is devoted to the study of the related boundedness and compactness results in the case of distributional symbols.

Denote by $D(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$ (β is the Bergman metric, see [17]) the Bergman disk.

Let $f \in L^\infty$. Denote by $\hat{f}$ the averaging function:

$$\hat{f}(z) = |D(z,r)|^{-1} \int_{D(z,r)} f(w)dA(w).$$

In what follows, the choice of $r \in (0,1)$ is not important; Let us agree that $r = 1/2$, for instance.

Definition 5.1. Let $f \in L^\infty$. We say that f belongs to the logarithmic BMO_∂ ($f \in BMO_{\log}$) if

$$\|f; BMO_{\log}\| := \sup_{z\in\mathbb{D}} \ell(z)|D(z,r)|^{-1} \int_{D(z,r)} |f(w) - \hat{f}(z)|dA(w) < \infty.$$

The above definition is known to be equivalent to saying that

$$\ell(z)[B(|f|^2)(z) - |B(f)(z)|^2]^{1/2}$$

is bounded on $\mathbb{D}$. Here

$$B(f)(z) = \int_{\mathbb{D}} \frac{(1-|z|^2)^2 f(w)dA(w)}{|1-z\bar{w}|^4}$$

is the Berezin transform of f.

Lemma 5.2. *Let f be a measurable function on the unit disk. Then $fg \in BMO_\partial$ for each $g \in BMO_\partial$ is and only iff $\in BMO_{\log}$. That is, $BMO_{\log}$ is the pointwise multiplier of BMO_∂.*

Since the standard Bergman projection maps BMO_∂ boundedly to $\mathcal{B}$, the following lemma is intuitively clear. However, we present here a simple proof for the convenience of the reader.

Lemma 5.3. *Let $t \geq 0$, then the general projection P_t:*

$$P_t f(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t f(w)}{(1-z\bar{w})^{2+t}}$$

is bounded $BMO \to \mathcal{B}$.

Proof. For the proof of the case $t = 0$, see [13]. The general case follows by similar arguments: If $f \in BMO$, then the Hankel operators H_f and $H_{\bar{f}}$ are bounded on A^2_t. Hence the little Hankel operator (See [13, 17] for more details about the little Hankel operators. The notation here is that of [17].) $h_{\bar{f}}$ is bounded $A^2_t \to L^2_t$, but $h_{\bar{f}} = h_{\overline{P_t f}}$. Since the little Hankel operator with conjugate-analytic symbol g is bounded if and only if $g \in \mathcal{B}$, we conclude that $P_t f \in \mathcal{B}$. □

Theorem 5.4. *Let $a \in \mathcal{D}'$ be a member of $\mathcal{Y}^m_0$ for some $m \in \mathbb{N}$. Assume moreover, that there exists $m' \in \mathbb{N}$ such that a has a representation*

$$a = \sum_{|\alpha| \leq m'} D^\alpha b_\alpha,$$

where $b_\alpha / \nu^{|\alpha|} \in BMO_{\log}$ for each α. Then T_a is bounded on $\mathcal{B}$.

Proof. Let (b_α) be a representation for a given in the statement of the theorem. We will look at the operators like 3.2 (see the proof of theorems 3.5-3.7). Recall that

$$R^{\alpha,\beta}_{b_\alpha} f(z) = \int_{\mathbb{D}} \frac{f^{(|\alpha|-|\beta|)}(w) b_\alpha(w) dA(w)}{(1 - z\bar{w})^{2+|\beta|}}.$$

If $\alpha = \beta$, then we have $f^{(|\alpha|-|\beta|)} = f \in \mathcal{B}$ and $\nu^{-|\beta|} b_\alpha = \nu^{-|\alpha|} b_\alpha \in BMO_{\log}$ and so $R^{\alpha,\beta}_{b_\alpha}$ is just $P_{|\beta|}$ applied to the product $f \nu^{-|\beta|} b_\alpha$, which maps f boundedly to $\mathcal{B}$.

If $\alpha > \beta$, then $f^{(|\alpha|-|\beta|)} \nu^{|\alpha|-|\beta|} \in L^\infty \subset BMO_\partial$ and $\nu^{-|\alpha|} b_\alpha \in BMO_{\log}$, so $R^{\alpha,\beta}_{b_\alpha}$ is $P_{|\beta|}$ applied to the product $f^{(|\alpha|-|\beta|)} \nu^{|\alpha|-|\beta|} \nu^{-|\alpha|} b_\alpha$, which is again bounded. □

Remark 5.5. Note that the above result gives an estimate

$$\|T_a : \mathcal{B} \to \mathcal{B}\| \leq C \max_{|\alpha| \leq m'} \|\nu^{-|\alpha|} b_\alpha; BMO_{\log}\|$$

for each choice of (b_α) satisfying the requirements of the previous theorem; there is no easy way to compare a and the norm of T_a directly.

6. The Bergman space A^1 and examples.

We will finally briefly discuss the case of the Bergman space A^1. As it often happens, the results can be fairly easily achieved by using duality and studying operators of the type 3.2.

Corollary 6.1. *Suppose $a \in \mathcal{D}'$ belongs to $\mathcal{L}\mathcal{Y}^m_0$ for some $m \in \mathbb{N}$. Then T_a is bounded on A^1 and there exists a constant $C := C(m) > 0$ such that*

$$\|T_a : A^1 \to A^1\| \leq C \|a; \mathcal{L}\mathcal{Y}^m_0\|.$$

Corollary 6.2. *Suppose* $a \in \mathcal{D}'$ *belongs to* $\mathcal{LV}_0^m$ *for some* $m \in \mathbb{N}$. *Then* T_a *is compact on* A^1.

Corollary 6.3. *Let* $a \in \mathcal{D}'$ *be a member of* $\mathcal{Y}_0^m$ *for some* $m \in \mathbb{N}$. *Assume moreover, that there exists* $m' \in \mathbb{N}$ *such that* a *has a representation*

$$a = \sum_{|\alpha| \le m'} D^\alpha b_\alpha,$$

where $b_\alpha / \nu^{|\alpha|} \in BMO_{\log}$ *for each* α. *Then* T_a *is bounded on* A^1.

We finish the paper by looking at examples:

Example 6.4. Suppose

$$b_1(z) = \ell^{-1}(z)\nu(z)\sin(\exp^{100}(\nu(z)^{-1}))$$

and

$$a := D^{(1,0)} b_1.$$

Then $a \in \mathcal{LY}_0^1$ and so T_a is bounded on A^1 and $\mathcal{B}$. However, a is a function that is clearly not bounded (or even L^1). Modifying this example one can easily produce examples of bounded and compact Toeplitz operators $\mathcal{B}_d \to \mathcal{B}_{d'}$ for various choices of d and d', as well.

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