On stochastic difference equations in insurance ruin theory

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Abstract

We study the insurance ruin problem in a model where in addition to the basic insurance business, the company operates in the general financial market. The development of the capital is described as the solution to a stochastic difference equation. Basic estimates for ruin probabilities are recalled from the literature and qualitative descriptions of the range and the limit of the capital are given.

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1 Introduction

A basic financial operation of an insurance company is to pay *claim amounts* on occurences of fires, storms, car accidents, etc. The payments are based on the contracts between the company and the insureds. As the price of the contract, each insured pays a *premium* to the company. Typically, the premium exceeds the mean of the associated claim amounts. If the company has a number of contracts with similar and independent insureds then by the law of large numbers, the total premium of the company suffices for the total claim amount with a high probability. This result is central even for the existence of the industry. In addition to the basic insurance business, the company usually operates in the general financial market. As a consequence, it receives (possibly negative) returns on the investments. The importance of this additional operation has been under an extensive study during the last few decades. A recent survey on the area is given by Paulsen (2008). For a wide description of insurance processes and their properties, we refer the reader to Daykin et al. (1994).

Suppose now that the company starts the business at the beginning of year one. To do this, it needs an initial capital $U_0 = u$. Denote by U_n the capital at the end of year n for $n = 1, 2, \ldots$. Let B_n be the *net loss* of the company in the basic insurance business, that is, B_n equals the total claim amount less the total premium in year n. As an approximation, we assume that this payment takes place at the beginning of year n. After the payment, the company invests the rest of its capital. Let r_n be the *rate of return* on the investments. The development of the capital can now be described as the solution to a stochastic difference equation, namely,

$$U_n = (1 + r_n)(U_{n-1} - B_n), \quad n = 1, 2, \ldots. \tag{1.1}$$

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In order to carry on, the company has to show a reasonable capital to meet its obligations. Define the time of *ruin* $T = T(u)$ by

$$T = \begin{cases} \inf\{n \in \mathbb{N}; U_n < 0\} \\ \infty \text{ if } U_n \geq 0 \text{ for } n = 1, 2, \dots. \end{cases} \tag{1.2}$$

A typical requirement is that the ruin probability within an appropriate time horizon must be below a predescribed low level.

The focus of the present paper is on the infinite time ruin probability $\mathbb{P}(T < \infty)$. We recall some basic estimates for the probability from the literature. As extensions of earlier results, we describe rather exceptional phenomena associated with the range and the limit of the solution to (1.1).

2 Descriptions of the development of the capital

Let the processes $\{r_n\}$, $\{B_n\}$ and $\{U_n\}$ be as described in Section 1. We begin by specifying the model in our interest in detail and by recalling some general observations concerning the time of ruin.

We will consider a basic model where

$$(r, B), (r_1, B_1), (r_2, B_2), (r_3, B_3), \dots$$

are independent and identically distributed random vectors. The first vector (r, B) is generic and is introduced for notational simplicity. We will assume that $\mathbb{P}(r > -1) = 1$. Without the assumption, ruin typically occurs with probability one. This can be expected from (1.1), and has been observed in a related model by Paulsen (1993), Remark 2.2. We also assume that $\mathbb{P}(r < 0) > 0$. From the applied point of view, this means that there is a risk associated with the investments.

Consider now representations for the time of ruin and for the probability $\mathbb{P}(T < \infty)$. By iterating (1.1), it is seen that

$$U_n = (1 + r_1) \cdots (1 + r_n)u - \sum_{i=1}^{n} (1 + r_i) \cdots (1 + r_n)B_i. \tag{2.1}$$

Let

$$A = \frac{1}{1 + r} \quad \text{and} \quad A_n = \frac{1}{1 + r_n},$$

and write

$$Y_n = B_1 + A_1 B_2 + A_1 A_2 B_3 + \cdots + A_1 \cdots A_{n-1} B_n. \tag{2.2}$$

Then

$$U_n = (1 + r_1) \cdots (1 + r_n)(u - Y_n). \tag{2.3}$$

Thus the time of ruin can be expressed as

$$T = \begin{cases} \inf\{n \in \mathbb{N}; Y_n > u\} \\ \infty \text{ if } Y_n \leq u \text{ for } n = 1, 2, \dots. \end{cases} \tag{2.4}$$

This representation is especially useful in models where Y_n converges to a finit limit with probability one. Observe that

$$(A, B), (A_1, B_1), (A_2, B_2), \dots$$

are independent and identically distributed random vectors. Our assumptions concerning the rate of return r mean that

$$\mathbb{P}(A > 0) = 1 \quad \text{and} \quad \mathbb{P}(A > 1) > 0. \tag{2.5}$$

A further expression for the ruin probability can be given by means of the variable

$$\bar{Y} = \sup\{Y_1, Y_2, \ldots\} \in (-\infty, \infty]. \tag{2.6}$$

Namely, we obviously have

$$\mathbb{P}(T < \infty) = \mathbb{P}(\bar{Y} > u). \tag{2.7}$$

An advantage of this representation is that $\bar{Y}$ satisfies a random equation. Namely,

$$\bar{Y} = \max\left(B_1, B_1 + A_1 \sup\{B_2 + A_2 B_3 + \cdots + A_2 \cdots A_{n-1} B_n; n = 2, 3, \ldots\}\right),$$

and hence,

$$\bar{Y} =_L B + A \max(0, \bar{Y}) \tag{2.8}$$

where $=_L$ means equality of probability laws. The variables A and B are independent of $\bar{Y}$ on the right hand side of (2.8).

2.1 Estimates for ruin probabilities

We recall in this section some basic results on ruin theory associated with model (2.2). Denote by Λ the cumulant generating function of $\log A$, that is,

$$\Lambda(\alpha) = \log \mathbb{E}(A^\alpha) \tag{2.9}$$

for $\alpha \in \mathbb{R}$. Let

$$\mathbf{r} = \sup\{\alpha;\ \Lambda(\alpha) \le 0\}. \tag{2.10}$$

Clearly, $\mathbb{P}(A > 1) > 0$ implies that $\Lambda(\alpha)$ tends to infinity as α tends to infinity. Thus $\mathbf{r} \in [0, \infty)$. Theorem 6.2 of Goldie (1991) shows that under suitable assumptions,

$$\mathbb{P}(T < \infty) = (1 + o(1)) C u^{-\mathbf{r}}, \quad u \to \infty, \tag{2.11}$$

where C is a constant. The main requirements for (2.11) are that $\mathbf{r} > 0$, $\Lambda(\mathbf{r}) = 0$ and $\mathbb{E}(|B|^{\mathbf{r}}) < \infty$. An interesting feature from the applied point of view is that for large u, the magnitude of the ruin probability is determined by the investment side, namely, by the parameter $\mathbf{r}$. The basic insurance processes only affect the constant C. The situation may be different if $\mathbb{E}(|B|^{\mathbf{r}}) = \infty$. To explain this, write

$$\mathbf{s} = \sup\{\alpha;\ \mathbb{E}((B\mathbf{1}(B > 0))^\alpha) < \infty\} \in [0, \infty]. \tag{2.12}$$

Consider the special case where $B \ge 0$ and

$$\mathbb{P}(B > t) = L(t) t^{-\beta}, \quad t > 0, \tag{2.13}$$

where $\beta \in (0, \mathbf{r})$ is a constant and L is a slowly varying function at infinity. Then $\mathbf{s} = \beta$. We refer to Feller (1971), Section VIII.8. By Grey (1994),

$$\mathbb{P}(T < \infty) = (1 + o(1)) D L(u) u^{-\mathbf{s}}, \quad u \to \infty, \tag{2.14}$$

where D is a positive constant. In this case, the magnitude of the ruin probability is determined by the insurance side, mainly by the parameter $\mathbf{s}$. We also refer the reader to Tang and Tsitsiashvili (2003) where investment and insurance risks are compared.

It is interesting to compare (2.11) and (2.14) with related results of de Haan et al. (1989). Assume in addition to (2.5) that $\mathbb{P}(B > 0) = 1$. Let $Z_0 = 0$ and

$$Z_n = A_n Z_{n-1} + B_n \tag{2.15}$$

for $n \geq 1$, and let $T_Z = \inf\{n \in \mathbb{N}; Z_n > u\}$ be the associated time of ruin. Obviously, Z_n has the same distribution as Y_n. However, the dependence structures of $\{Y_n\}$ and $\{Z_n\}$ are different. Also the ruin probabilities differ drastically since in contrast to (2.11) and (2.14),

$$\mathbb{P}(T_Z < \infty) = 1 \tag{2.16}$$

for every $u > 0$. This is clear if $\mathbb{P}(B > t) > 0$ for every $t > 0$ since then

$$\mathbb{P}(T_Z < \infty) \geq \mathbb{P}(B_n > u \text{ for some } n \in \mathbb{N}) = 1. \tag{2.17}$$

It is not difficult to extend this to concern with the general case. The result is also in the scope of de Haan et al. (1989) although the focus of the paper is on finite time ruin probabilities.

2.2 On the range and the limit of the capital

There exist non-trivial cases where the solution to (1.1) stays positive or tends to infinity with probability one. These models are not natural in the insurance application but it is useful to identify them. We state the results in terms of the process $\{Y_n\}$. Denote

$$\bar{y} = \sup\{y \in \mathbb{R}; \mathbb{P}(\bar{Y} > y) > 0\}. \tag{2.18}$$

Obviously, $\bar{y} < \infty$ implies that $\mathbb{P}(T < \infty) = 0$ for $u > \bar{y}$.

The following result is a simplification and in part a generalization of Theorem 3 of Nyrhinen (2001).

Theorem 2.1 *Assume* (2.5)*. Then $\bar{y} = \infty$ if and only if there exists $k \in \mathbb{N}$ such that*

$$\mathbb{P}(Y_k > 0, A_1 \cdots A_k > 1) > 0. \tag{2.19}$$

A closely related problem is to give a similar description for the limit of the process $\{Y_n\}$. We will assume in the following theorem that

$$\mathbb{P}(B = 0) < 1 \quad \text{ and } \quad \mathbb{P}\left(\lim_{n\to\infty} A_1 \cdots A_{n-1} B_n = 0\right) = 1. \tag{2.20}$$

Under this assumption, we may write

$$Y_\infty = \sum_{n=1}^{\infty} A_1 \cdots A_{n-1} B_n \tag{2.21}$$

and

$$y_\infty = \sup\{y \in \mathbb{R}; \mathbb{P}(Y_\infty > y) > 0\}. \tag{2.22}$$

Namely, the series in (2.21) converges to a real number with probability one. We refer to Goldie and Maller (2000), Theorem 2.1.

Theorem 2.2 *Assume* (2.5) *and* (2.20). *Then* $y_\infty = \infty$ *if and only if* (2.19) *holds for some* $k \in \mathbb{N}$ *and*

$$\mathbb{P}(Y_m > 0, A_1 \cdots A_m \leq 1) > 0 \tag{2.23}$$

for some $m \in \mathbb{N}$.

Recall that u is the initial value of equation (1.1). Thus $\bar{y} < \infty$ implies that the solution to (1.1) satisfies

$$\mathbb{P}\left(\inf\{U_n; n = 1, 2, \ldots\} \geq 0\right) = 1 \tag{2.24}$$

for $u > \bar{y}$. If (2.20) holds and y_∞ is finite then

$$\mathbb{P}\left(\lim_{n\to\infty} U_n = \infty\right) = 1 \tag{2.25}$$

for $u > y_\infty$. This follows from (2.3) and Goldie and Maller (2000), Theorem 2.1.

Remark 2.1 There are simple but useful connections between $\bar{Y}$ and Y_∞. Namely, for a given $u > 0$, we have

$$\mathbb{P}(\bar{Y} > u) \geq \mathbb{P}(Y_\infty > u) \geq \mathbb{P}(\bar{Y} > u)\mathbb{P}(Y_\infty > 0). \tag{2.26}$$

The first inequality in (2.26) is trivial. The second one is given in a continuous time model by Paulsen (1993), Corollary 3.1. The proof in our case is similar but easier. Estimate (2.11) together with (2.26) can be used to sharpen an asymptotic estimate for the right tail of Y_∞. In fact, Theorem 4.1 of Goldie (1991) shows that under suitable conditions,

$$\mathbb{P}(Y_\infty > u) = (1 + o(1))C_\infty u^{-\mathbf{r}}, \quad u \to \infty,$$

where C_∞ is a constant. The representation of C_∞ is complicated and it is difficult to see directly whether or not it is strictly positive. Suppose now that (2.11) holds with $C > 0$ and that (2.5) and (2.20) hold. Then $\bar{y} = \infty$ so that (2.19) is satisfied for some $k \in \mathbb{N}$. It follows from (2.26) that the constant C_∞ is strictly positive if and only if (2.23) holds for some $m \in \mathbb{N}$. Sufficient conditions for $C > 0$ are known from Nyrhinen (2001). We also refer the reader to Klüppelberg and Kostadinova (2008) where the positivity of C_∞ is studied in a specific model.

We end the section by illustrating the results by means of examples. First observe that if A and B are independent then conditions (2.19) and (2.23) can be studied easily. Namely, it is sufficient to consider the positivity of the probabilities $\mathbb{P}(B > 0)$, $\mathbb{P}(A > 1)$ and $\mathbb{P}(A \leq 1)$. In general, the situation is more complicated. The first example below shows that y_∞ can be finite even if $\bar{y} = \infty$. Write

$$y_\infty^- = \sup\{y \in \mathbb{R}; \mathbb{P}(Y_\infty < -y) > 0\}. \tag{2.27}$$

Kesten (1973) gives general conditions under which either y_∞ or y_∞^- equals ∞. However, it is also of interest to understand the tails of Y_∞ separately. The second example provides a pair (A, B) such that both $\mathbb{P}(B > 0)$ and $\mathbb{P}(B < 0)$ are positive but $y_\infty < \infty$ and $y_\infty^- = \infty$.

Example 2.1 Let A be such that (2.5) holds and $\mathbb{E}(\log A) \in (-\infty, 0)$. Take $B = -1 + A$. Then

$$Y_n = -1 + A_1 \cdots A_n. \tag{2.28}$$

It is seen that $\bar{y} = \infty$. By the strong law of large numbers, the product in (2.28) tends to zero with probability one. Thus $y_\infty = -1$.

Example 2.2 Let A be as in Example 2.1. Assume further that $\mathbb{P}(A \in (0, 1/2)) > 0$ and $\mathbb{P}(A \in (1/2, 1)) > 0$. Let $B = 1 - 2A$. Then

$$Y_n = 1 - A_1 - A_1A_2 - \cdots - A_1 \cdots A_{n-1} - 2A_1 \cdots A_n. \tag{2.29}$$

It is seen that $y_\infty < \infty$. Apply Theorem 2.2 to the sequence $\{-Y_n\}$ to see that $y_\infty^- = \infty$.

3 Proofs

Proof of Theorem 2.1 Consider first a related result from Nyrhinen (2001). Let the A- and B-sequences be as in Section 2, and let $L, L_1, L_2, \ldots$ be such that

$$(A, B, L), (A_1, B_1, L_1), (A_2, B_2, L_2), \ldots \tag{3.1}$$

are independent and identically distributed random vectors. Assume that

$$\mathbf{r} > 0 \quad \text{and that} \quad \Lambda(\alpha) \text{ and } \mathbb{E}(|B|^\alpha) \text{ are finite for some } \alpha > \mathbf{r}. \tag{3.2}$$

Assume further that $\mathbb{E}((AL\mathbf{1}(L > 0))^\alpha)$ is finite for some $\alpha > \mathbf{r}$. Consider the process $\{Y_n^L\}$ defined by

$$Y_n^L = B_1 + A_1B_2 + \cdots + A_1 \cdots A_{n-1}B_n + A_1 \cdots A_nL_n \tag{3.3}$$

for $n \in \mathbb{N}$. Corresponding to (2.6) and (2.18), write

$$\bar{Y}^L = \sup\{Y_n^L; n \in \mathbb{N}\} \quad \text{and} \quad \bar{y}^L = \sup\{y \in \mathbb{R}\,;\, \mathbb{P}(\bar{Y}^L > y) > 0\}.$$

By Theorem 3 of Nyrhinen (2001), $\bar{y}^L = \infty$ if and only if there exists $h \in \mathbb{N}$ such that

$$\mathbb{P}(B + AL + Y_h/(\Pi_h - 1) > 0,\, \Pi_h > 1) > 0 \tag{3.4}$$

where Y_h is as in (2.2) and

$$\Pi_h = A_1 \cdots A_h. \tag{3.5}$$

Consider now Theorem 2.1. We work for a while under additional condition (3.2). Assume that (2.19) is satisfied. Then $\mathbb{P}(B > 0) > 0$. Take $L \equiv 0$ in (3.1). It is seen that requirement (3.4) is satisfied with $h = k$. Thus $\bar{y} = \bar{y}^L = \infty$. Assume now that $\bar{y} = \infty$. Take $L = -B/A$ in (3.1). Then $Y_1^L = 0$ and $Y_n^L = Y_{n-1}$ for $n \geq 2$. Thus $\bar{Y}^L = \bar{Y}\mathbf{1}(\bar{Y} > 0)$. It follows that $\bar{y}^L = \infty$ so that (3.4) holds for some $h \in \mathbb{N}$. But then (2.19) holds with $k = h$ since $B + AL \equiv 0$.

We have proven that the claim of the theorem holds true under (3.2). Consider now the general case, but assume still that $\mathbb{P}(A < 1) > 0$. We make use of the method similar to the proof of Theorem 3 in Nyrhinen (2001). Let P be the distribution of (A, B). Define the distribution Q by

$$\frac{dQ}{dP}(y_1, y_2) = \kappa(a\mathbf{1}(y_1 \leq 1) + \mathbf{1}(y_1 > 1))e^{-y_1 - |y_2|}, \quad y_1, y_2, \in \mathbb{R},$$

where $a > 0$ is a constant and $\kappa = \kappa(a)$ has been chosen such that $Q(\mathbb{R}^2) = 1$. Consider a process which has the same structure as $\{Y_n\}$ in (2.2), but let Q be the distribution of (A, B). It is easy to see that for large a, requirement (3.2) is satisfied under Q. By the first part of the proof, the claim of the theorem is true under Q. But P an Q are mutually absolutely continuous so that the same is true under P. Consider finally the case where

$\mathbb{P}(A \geq 1) = 1$. The above method does not apply since we have $\mathbf{r} = 0$ both under P and Q. However, it is seen directly that $\bar{y} = \infty$ if and only if $\mathbb{P}(B > 0) > 0$, and the claim of the theorem easily follows. □

Proof of Theorem 2.2 Assume first that (2.19) and (2.23) are satisfied. By Theorem 2.1, $\bar{y} = \infty$. By (2.26), it is sufficient to prove that $\mathbb{P}(Y_\infty > 0) > 0$. Let Π_k be as in (3.5). By (2.23), we may find $\delta > 0$ such that

$$\mathbb{P}(Y_m \geq \delta, \, \Pi_m \leq 1) > 0. \tag{3.6}$$

Let $M \geq 0$ be such that $\mathbb{P}(Y_\infty > -M) > 0$. Obviously, Y_∞ satisfies the random equation

$$Y_\infty =_L Y_m + \Pi_m Y_\infty \tag{3.7}$$

where on the right hand side, Y_∞ is independent of Y_m and Π_m. By this observation,

$$\begin{aligned} \mathbb{P}(Y_\infty > -M + \delta) &\geq \mathbb{P}(Y_m \geq \delta, \, \Pi_m Y_\infty > -M) \\ &\geq \mathbb{P}(Y_m \geq \delta, \, \Pi_m \leq 1, Y_\infty > -M) \\ &= \mathbb{P}(Y_m \geq \delta, \, \Pi_m \leq 1)\mathbb{P}(Y_\infty > -M) > 0. \end{aligned} \tag{3.8}$$

Hence, $\mathbb{P}(Y_\infty > -M) > 0$ for a given $M \geq 0$ implies that $\mathbb{P}(Y_\infty > -M + \delta) > 0$. It follows that $\mathbb{P}(Y_\infty > 0) > 0$.

Assume now that $y_\infty = \infty$. Clearly, $\bar{y} = \infty$ so that by the first part of the proof, (2.19) holds for some $k \geq 1$. By Theorem 2.1 of Goldie and Maller (2000), $A_1 \cdots A_n$ tends to zero almost surely. Thus

$$\lim_{n \to \infty} \mathbb{P}(Y_n > 0, A_1 \cdots A_n \leq 1) = \mathbb{P}(Y_\infty > 0) > 0. \tag{3.9}$$

This implies (2.23) for m large enough. □

References

Daykin, C. D., T. Pentikäinen, and M. Pesonen (1994). *Practical Risk Theory for Actuaries.* London: Chapman & Hall.

de Haan, L., S. Resnick, H. Rootzen, and C. de Vries (1989). Extremal behaviour of solutions to a stochastic difference equation with applications to ARCH processes. *Stochastic Process. Appl.* **32**, 213–224.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications* (2nd ed.), Volume II. New York: John Wiley and Sons.

Goldie, C. M. (1991). Implicit renewal theory and tails of solutions of random equations. *Ann. Appl. Probab.* **1**, 126–166.

Goldie, C. M. and R. Maller (2000). Stability of perpetuities. *Ann. Probab.* **28**, 1195–1218.

Grey, D. R. (1994). Regular variation in the tail behaviour of solutions of random difference equations. *Ann. Appl. Probab.* **4**, 169–183.

Kesten, H. (1973). Random difference equations and renewal theory for products of random matrices. *Acta Math.* **131**, 207–248.

Klüppelberg, C. and R. Kostadinova (2008). Integrated insurance risk models with exponential Lévy investments. *Insurance Math. Econom.* **42**, 560–577.

Nyrhinen, H. (2001). Finite and infinite time ruin probabilities in a stochastic economic environment. *Stoch. Proc. Appl.* **92**, 265–285.

Paulsen, J. (1993). Risk theory in a stochastic economic environment. *Stoch. Proc. Appl.* **46**, 327–361.

Paulsen, J. (2008). Ruin models with investment income. *Probab. Surv.* **5**, 416–434.

Tang, Q. and G. Tsitsiashvili (2003). Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks. *Stoch. Proc. Appl.* **108**, 299–325.