Quasilinear Riccati type equations and quasiminimizers

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Abstract

Solutions u of the equation $-\nabla \cdot A(x, \nabla u) = b(x)|\nabla u|^q$ are studied in the subcritical case $p-1 \leq q \leq p/n+p-1$, $q < p < n$. Here $A(x,h)$ satisfies $A(x,h)\cdot h \approx |h|^p$ and b a bounded function in an open set $\Omega \subset \mathbf{R}^n$. It is shown that the solutions u are local quasiminimizers in Ω. From the quasiminimizing property several conclusions on the behavior of the solutions u can be derived In particular, the method provides an easy solution to the uniqueness problem in the class $W_0^{1,p}(\Omega)$.

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1 Introduction

We consider solutions u of the equation

$$-\nabla \cdot A(x, \nabla u) = b(x)|\nabla u|^q \tag{1.1}$$

where $A(x,h)\cdot h \approx |h|^p$. For the precise assumptions on p, A, b and q see (1.3)–(1.6) below. The solutions u are understood in the weak sense. Hence u is a solution of (1.1) in an open set $\Omega \subset \mathbf{R}^n$ if u belongs to the local Sobolev space $W_{loc}^{1,p}(\Omega)$ and

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, dx = \int_\Omega \varphi(x) b(x) |\nabla u|^q \, dx \tag{1.2}$$

for all $\varphi \in C_0^\infty(\Omega)$.

We use the following assumptions in the sequel unless otherwise stated:

$$0 < p - 1 \le q < p/n + p - 1,\ q < p < n, \tag{1.3}$$

$$b \text{ is a bounded measurable function in } \Omega \tag{1.4}$$

and $A : \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ is a Caratheodory function such that for all $h \in \mathbf{R}^n$ and a.e. $x \in \Omega$

$$A(x, h) \cdot h \ge \alpha |h|^p, \tag{1.5}$$

$$|A(x, h)| \le \beta |h|^{p-1} \tag{1.6}$$

where $0 < \alpha \le \beta < \infty$.

Note that there is no assumption on the sign of the function b. Since we are not studying existence problems, we do not assume monotoneity of the operator A.

The prototype of (1.1) is the equation

$$-\Delta_p u = -\nabla \cdot (|\nabla u|^{p-2} \nabla u) = |\nabla u|^q. \tag{1.7}$$

This equation has been extensively studied. In particular existence and comparison problems have got a lot of attention. For $p - 1 < q \le p$ see [FMe] and [DP1–2], for $q = p - 1$ [BMMP] and [Me], for $q = p$ [ADP1–2], [FMu], [Me] and [TK] and finally for $q > p$ [Ng] and references therein.

Next we recall the concept of a quasiminimizer. Let Ω be an open subset of $\mathbf{R^n}$, $n \ge 1$, $p > 1$ and $K \ge 1$. A function u in the local Sobolev space $W^{1,p}_{loc}(\Omega)$ is called a (p, K)*–quasiminimizer* in Ω if for all open sets $\Omega' \subset\subset \Omega$

$$\int_{\Omega'} |\nabla u|^p \, dx \le K \int_{\Omega'} |\nabla v|^p \, dx \tag{1.8}$$

for all functions v such that $v - u \in W^{1,p}_0(\Omega')$. Note that if a function u belongs to $W^{1,p}(\Omega)$, then u is a K–quasiminimizer if and only if (1.8) holds for all open sets $\Omega' \subset \Omega$, i.e. $\overline{\Omega}'$ need not be a compact subset of Ω. In general we keep the number p fixed and use the abbreviation a K–quasiminimizer. For $K = 1$ the function u is minimizer and hence a p–harmonic function, i.e. u satisfies $\Delta_p u = 0$. For the theory of quasiminimizers see [GG1–2] and [KiM]. Although little is known of the structure of quasiminimizers, several properties like Hölder continuity, maximum and minimum principles, boundary behavior and the Harnack inequality have been investigated, see [GG1–2], [BA1], [KiM], [KiMaM], [BJ1], [Zie] and [DT]. For the theory of quasiminimizers in metric measure spaces see [KiS].

We say that a function u in Ω is a *local K–quasiminimizer* if every $x \in \Omega$ has a neighborhood U such that $u|U$ is a K–quasiminimizer. For us there are three important concepts that take into account the behavior of u on the boundary. We say that u is a *local K–quasiminimizer in $\overline{\Omega}$* if every $x \in \overline{\Omega}$ has a radius $r > 0$ such that $u|B(x, r) \cap \Omega$ is a K–quasiminimizer and u is called a *local K–quasiminimizer uniformly in $\overline{\Omega}$* if there is $r > 0$ such that every $x \in \overline{\Omega}$, $u|B(x, r) \cap \Omega$ is a K–quasiminimizer. There is a somewhat stronger version of the last condition. The function u is called a *K–quasiminimizer in small sets in* Ω if there is $\delta > 0$ such that u is K–quasiminimizer in every open set $U \subset \Omega$ whenever $m(U) < \delta$.

Our main result in the next section says that all solutions $u \in W^{1,p}(\Omega)$ to (1.1) are quasiminimizers in small sets in Ω. We then use this property and its local counterpart in Section 3 to derive uniqueness and Hölder continuity results for solutions. The scope of our method is not restricted to the exponent range (1.3) but the quasiminimizing property appears most naturally in this range.

2 Main results

We first consider the case where a solution of (1.1) belongs to $W^{1,p}(\Omega)$.

Theorem 2.1 *Suppose that $u \in W^{1,p}(\Omega)$ is a solution of the equation (1.1) in an open set $\Omega \subset \mathbf{R}^n$ where p, q, A and b satisfy the assumptions (1.3) – (1.6). Then u is a K–quasiminimizer in small sets in Ω. More precisely, there is $\delta = \delta(p, q, n, \alpha, \beta, M, |\nabla u|) > 0$ such that u is a K–quasiminimizer in $\Omega' \subset \Omega$ with constant $K = K(p, q, n, \alpha, \beta, M, |\nabla u|)$ whenever $m(\Omega') < \delta$. Here $M = \text{ess sup}_\Omega |b|$. In the case $q = p - 1$ the numbers δ and K are independent of $|\nabla u|$.*

Proof. Let $\Omega' \subset \Omega$ be an open set and let $P = P(u, \Omega)$ be the function which minimizes the p–Dirichlet integral with boundary values u in Ω', i.e.

$$\int_{\Omega'} |\nabla P|^p \, dx = \inf_v \int_{\Omega'} |\nabla v|^p \, dx$$

over all functions $v - u \in W_0^{1,p}(\Omega')$. Such a unique function always exists, see e.g. [HKM, Chapter 5]. If we set $P = u$ in $\Omega \setminus \Omega'$, then in potential theoretic terms P is the p–Poisson modification of the function u in Ω', see [HKM, 7.13]. For the quasiminimizing property (1.8) of u we need to show that

$$\int_{\Omega'} |\nabla u|^p \, dx \le K \int_{\Omega'} |\nabla P|^p \, dx. \tag{2.1}$$

Fix an open set $\Omega' \subset \Omega$ and let $P = P(u, \Omega')$ be as above. Now $u - P$ belongs to $W_0^{1,p}(\Omega')$ and $u - P$ can be used as a test function for the equation (1.2). This gives

$$\int_{\Omega'} A(x, \nabla u) \cdot \nabla(u - P) \, dx = \int_{\Omega'} (u - P) b(x) |\nabla u|^q \, dx. \tag{2.2}$$

We estimate the left and the right hand side of (2.2) separately.

For the left hand side we first use (1.5), (1.6) and the Hölder inequality to obtain

$$\int_{\Omega'} A(x, \nabla u) \cdot \nabla(u - P) \, dx$$

$$\ge \alpha \int_{\Omega'} |\nabla u|^p \, dx - \beta \left(\int_{\Omega'} |\nabla u|^p \, dx \right)^{(p-1)/p} \left(\int_{\Omega'} |\nabla P|^p \, dx \right)^{1/p} \tag{2.3}$$

$$= \left(\int_{\Omega'} |\nabla u|^p \, dx\right)^{(p-1)/p} \left(\alpha \left(\int_{\Omega'} |\nabla u|^p \, dx\right)^{1/p} - \beta \left(\int_{\Omega'} |\nabla P|^p \, dx\right)^{1/p}\right).$$

Let $M = \operatorname{esssup}_{\Omega}|b|$. To estimate the right hand side of (2.2) let $\gamma = n(p-q)/(n-p)$. Now $\gamma > 1$ because $q < p$ and $q < p/n + p - 1$ by (1.3) and we can use the Hölder inequality twice to obtain

$$\int_{\Omega'} (u-P)b(x)|\nabla u|^q \, dx$$

$$\leq M \left(\int_{\Omega'} |u-P|^{p/(p-q)} \, dx\right)^{(p-q)/p} \left(\int_{\Omega'} |\nabla u|^p \, dx\right)^{q/p} \tag{2.4}$$

$$\leq M m(\Omega')^{(\gamma-1)(p-q)/(\gamma p)} \left(\int_{\Omega'} |u-P|^{np/(n-p)} \, dx\right)^{(p-q)/(\gamma p)} \left(\int_{\Omega'} |\nabla u|^p \, dx\right)^{q/p}.$$

The Sobolev imbedding theorem, see e.g. [GT, Theorem 7.10], yields

$$\int_{\Omega'} |u-P|^{np/(n-p)} \, dx \leq c \left(\int_{\Omega'} |\nabla(u-P)|^p \, dx\right)^{n/(n-p)}$$

$$\leq c \left(\int_{\Omega'} |\nabla u|^p \, dx\right)^{n/(n-p)} \tag{2.5}$$

where we have used the minimizing property of P for

$$\int_{\Omega'} |\nabla u|^p \, dx \geq \int_{\Omega'} |\nabla P|^p \, dx.$$

Here c is a generic constant depending only n and p. From (2.4) and (2.5) and taking the value of γ into account we obtain

$$\int_{\Omega'} (u-P)b(x)|\nabla u|^q \, dx \leq cM \, m(\Omega')^{\delta'} \left(\int_{\Omega'} |\nabla u|^p \, dx\right)^{\frac{1}{p}+\frac{q}{p}} \tag{2.6}$$

where $\delta' = \frac{n(p-q-1)+p}{np} > 0$.

If now

$$\int_{\Omega'} |\nabla u|^p \, dx \leq 1, \tag{2.7}$$

then writing

$$\frac{1}{p} + \frac{q}{p} = \frac{1}{p} + \frac{p-1}{p} + \frac{q-(p-1)}{p}$$

and noting that $q \geq p-1$ we obtain from (2.2), (2.3) and (2.6) that

$$(\alpha - cMm(\Omega')^{\delta})^p \int_{\Omega'} |\nabla u|^p \, dx \leq \beta^p \int_{\Omega'} |\nabla P|^p \, dx. \tag{2.8}$$

Thus inequality (2.1) is satisfied with

$$K = \left(\frac{\beta}{\alpha - cMm(\Omega')^{\delta'}} \right)^p \tag{2.9}$$

provided that

$$\alpha - cMm(\Omega')^{\delta'} > 0 \tag{2.10}$$

and (2.7) hold.

We now complete the proof as follows. Since $u \in W^{1,p}(\Omega)$ there exists $\delta > 0$ such that

$$\int_{\Omega'} |\nabla u|^p \, dx \leq 1$$

provided that $m(\Omega') < \delta$. Choosing δ small enough we can also assume that

$$m(\Omega')^{\delta'} < \frac{\alpha}{cM}$$

where $M = \text{ess sup}_{\Omega}|b|$. If now $\Omega' \subset \Omega$, then (2.1) holds for K as in (2.9). Hence u is a K–quasiminimizer in small sets in Ω as required.

For $q = p-1$ the exponent $\frac{1}{p} + \frac{q}{p}$ in (2.6) equals $\frac{1}{p} + \frac{(p-1)}{p}$ and the condition (2.7) is not needed. Hence the numbers δ and K are independent of $|\nabla u|$. The proof follows.

Remark 2.2 Note that for the p–harmonic operator $A(x, h) = |h|^{p-2}h$, where $\alpha = \beta = 1$, the number K in (2.9) can be chosen arbitrary close to 1 by choosing r small.

The proof of the above theorem immediately produces the following local version.

Theorem 2.3 *Suppose that $u \in W^{1,p}_{loc}(\Omega)$ is a solution of the equation (1.2) in an open set $\Omega \subset \mathbf{R}^n$ where p, q and A satisfy the assumptions (1.3), (1.5), (1.6) and b is a locally bounded measurable function in Ω. Then u is a local K–quasiminimizer in Ω. More precisely, for each $x \in \Omega$ there is $r = r(p, q, n, \alpha, \beta, M', |\nabla u|) \in (0, dist(x, \partial\Omega)/4)$ such that u is a K–quasiminimizer in $B(x, r)$ with constant $K = K(p, q, n, \alpha, \beta, M', |\nabla u|)$ with $M' = ess\ sup_{B(x,2r)}|b|$. For $q = p - 1$ the numbers r and K are independent of $|\nabla u|$.*

3 Consequences

Since K–quasiminimizers are locally Hölder continuous with exponent $\gamma = \gamma(n, p, K) > 0$, see [GG1–2] or [KiS], we obtain from Theorem 2.3 the following result on the Hölder continuity of solutions of (1.1). Most likely the result can be much improved. In particular all solutions to (1.1) are continuous. The Hölder continuity of the solutions up to the boundary is considered in Theorem 3.6.

Corollary 3.1 *Suppose that $u \in W^{1,p}_{loc}(\Omega)$ is a solution of the equation (1.2) in an open set $\Omega \subset \mathbf{R}^n$ where p, q and A satisfy the assumptions (1.3), (1.5), (1.6) and b is a locally bounded measurable function in Ω. Then for for each $x \in \Omega$ and each y sufficiently close to x*

$$|u(y) - u(x)| \leq c|y - x|^{\gamma}$$

where $c < \infty$, $\gamma = \gamma(p, q, n, \alpha, \beta, M', x, |\nabla u|) > 0$ and $M' = ess\ sup_U |b|$ for some neighborhood U of x. For $q = p - 1$ the exponent γ is independent of $|\nabla u|$.

We say that a function u in Ω satisfies the maximum principle, if $u \leq c$ in Ω and $u(x_o) < c$ imply that $u < c$ in the x_o–component of Ω. The minimum principle is defined similarly. Quasiminimizers satisfy the maximum and minimum principle, see [GG1–2] and [KiS]. Now Theorem 2.3 implies:

Corollary 3.2 *Let p, q, A and u be as in Theorem 2.3. Then u satisfies the maximum and minimum principle in Ω.*

Next we consider regularity up to the boundary. Sufficient conditions for boundary regularity of quasiminimizers are known, see [Zie] and [BJ1]. Typically these conditions employ a capacity thickness condition at a boundary point on a level p_1 where p_1 is less than p. We formulate a sufficient geometric condition which is adequate for our purposes and which holds for all $p > 1$. We give a similar condition for the Hölder continuity on the boundary.

Let Ω be an open set in $\mathbf{R}^n$ and $x_o \in \partial\Omega$. We say that $x_o \in \partial\Omega$ is a *thick boundary point* if there are a number $\sigma > 0$, a sequence of points x_i and a sequence of radii r_i, $i = 1, 2, ...$, such that $\overline{B}(x_i, r_i) \subset \mathbf{R}^n \setminus \Omega$, $r_i \geq \sigma|x_i - x_o|$ and $r_i \to 0$ as $i \to \infty$. Note that there is no condition how fast the radii r_i approach zero and we say that x_o is a *uniformly thick boundary point* if, in addition, there is $\tau > 0$ such that $r_{i+1} > \tau r_i$ for all i.

We say that a function u in Ω is γ–Hölder continuous, $\gamma > 0$, at $x_o \in \overline{\Omega}$ if there is a neighborhood U of x_o and a constant $C < \infty$ such that

$$|u(x) - u(x_o)| \leq C|x - x_o|^{\gamma} \tag{3.1}$$

for all $x \in U \cap \Omega$. Note that if $x_o \in \partial\Omega$, then this means that u has an extension $u(x_o)$ to x_o so that (3.1) holds.

Lemma 3.3 *Let Ω be an open set in $\mathbf{R}^n$ and $x_o \in \partial\Omega$ a thick boundary point. Suppose that $w \in W^{1.p}_{loc}(\mathbf{R}^n)$ is continuous at x_o and u is a K–quasiminimizer in Ω with $u - w \in W^{1.p}_0(\Omega)$. Then $\lim_{x \to x_o, x \in \Omega} u(x) = w(x_o)$. If x_o is a uniformly thick boundary point and w is Hölder continuous at x_o, then u is Hölder continuous at x_o and $u(x_o) = w(x_o)$.*

Remark 3.4 The Hölder exponent $\gamma > 0$ for u in Lemma 3.3 depends on n, p, K σ, τ and the Hölder exponent of f at x_o, see the proof of Theorem 2.12 in [BJ1].

Proof for Lemma 3.3. Let x_o be a thick boundary point. and let $1 < p_1 < p$. By a result of W. P. Ziemer [Zie] the result follows if

$$\int_0^1 \left(\frac{cap_{p_1}(\overline{B}(x_o,r) \setminus \Omega, B(x_o,2r))}{cap_{p_1}(\overline{B}(x_o,r), B(x_o,2r))} \right)^\delta \frac{dr}{r} = \infty \tag{3.2}$$

for all $\delta > 0$. Here $cap_{p_1}(C, B(x_o,2r))$ refers to the ordinary p_1– capacity of a compact set C in the ball $B(x_o,2r)$, see [HKM, Chapter 2].

To show (3.2) fix $p_1 < p$ and let x_i, r_i and $\sigma > 0$ be as in the definition of the thick boundary point x_o. Set $R_i = |x_i - x_o|$. We may assume that $13\,R_i \le 1$ for all i. Let $r \in [2R_i, 6R_i]$. Now

$$\begin{aligned} cap_{p_1}(\overline{B}(x_o,r) \setminus \Omega, B(x_o,2r)) &\ge cap_{p_1}(\overline{B}(x_i,r_i), B(x_i,13\,R_i)) \\ &\ge cap_{p_1}(\overline{B}(x_i,\sigma R_i), B(x_i,13\,R_i)) = c\,R_i^{n-p_1} \end{aligned}$$

where c is a generic positive constant which depends only on n, p_1 and σ. See [HKM, 2.11] for the properties of the variational p_1–capacity. Hence for $r \in [2R_i, 6R_i]$

$$\frac{cap_{p_1}(\overline{B}(x_o,r) \setminus \Omega, B(x_o,2r))}{cap_{p_1}(\overline{B}(x_o,r), B(x_o,2r))} \ge c \frac{R_i^{n-p_1}}{r^{n-p_1}} \ge c \frac{R_i^{n-p_1}}{6\,R_i^{n-p_1}} \ge c > 0. \tag{3.3}$$

By deleting overlapping intervals we can assume that the intervals $[2R_i, 6R_i]$, $i = 1$, 2, ..., are disjoint. Then for $\delta > 0$ we obtain from (3.3)

$$\begin{aligned} &\int_0^1 \left(\frac{cap_{p_1}(\overline{B}(x_o,r) \setminus \Omega, B(x_o,2r))}{cap_{p_1}(\overline{B}(x_o,r), B(x_o,2r))} \right)^\delta \frac{dr}{r} \\ \ge \sum_i \int_{2R_i}^{6R_i} &\left(\frac{cap_{p_1}(\overline{B}(x_o,r) \setminus \Omega, B(x_o,2r))}{cap_{p_1}(\overline{B}(x_o,r), B(x_o,2r))} \right)^\delta \frac{dr}{r} \ge c^\delta \sum_i \log \frac{6R_i}{2R_i} = \infty \end{aligned}$$

as required.

Next we consider a uniformly thick boundary point x_o. By [BJ1, Theorem 2.12] it suffices to show that

$$\liminf_{\rho \to 0} \frac{1}{|\log \rho|} \int_\rho^1 \exp(-C_o \gamma(p_1,r)^{p/(p-p_1)}) \frac{dr}{r} > 0 \tag{3.4}$$

for all $C_o > 0$ and for some $1 < p_1 < p$. Here

$$\gamma(p_1,r) = \frac{r^{n-p_1}}{cap_{p_1}(\overline{B}(x_o,r) \setminus \Omega, B(x_o,2r))}.$$

Fix $C_o > 0$. Using the same notation as in the first part of the proof we obtain $\gamma(p_1,r) \ge c > 0$ for $r \in [2R_i, 6R_i]$. This gives

$$\int_{2R_i}^{6R_i} \exp(-C_o \gamma(p_1,r)^{p/(p-p_1)}) \frac{dr}{r} \ge c > 0. \tag{3.5}$$

Deleting some points x_i and adjusting the constant τ we can assume that $6R_{i+1} \leq 2R_i \leq 1/3$ for all $i = 1, 2, \dots$. Now for $\rho \in [R_{i+1}, R_i]$, (3.5) yields

$$\frac{1}{|\log \rho|} \int_{\rho}^{1} \exp(-C_o \gamma(p_1, r)^{p/(p-p_1)}) \frac{dr}{r}$$

$$\geq \frac{1}{|\log(R_{i+1}|} \sum_{j=1}^{i} \int_{2R_j}^{6R_j} \exp(-C_o \gamma(p_1, r)^{p/(p-p_1)}) \frac{dr}{r} \geq \frac{ic}{|\log(R_{i+1}|}. \tag{3.6}$$

On the other hand

$$R_{i+1} \geq \sigma r_{i+1} \geq \sigma \tau r_i \geq \dots \geq \sigma \tau^i r_1$$

and hence

$$i \geq \frac{|\log R_{i+1}| - |\log(\sigma r_1)|}{|\log \tau|}.$$

This together with (3.6) yields (3.4) as required.

Remark 3.5 The conditions in Lemma 3.3 allow more regular boundary points than the well–known cone and the curved cone conditions for the complement of Ω, see [HKM, pp. 123–124]. However, a sufficient and necessary condition for boundary regularity of quasiminimizers is not known.

If Ω is an open set in $\mathbf{R}^n$ such that each point $x_o \in \partial\Omega$ is a thick or uniformly thick boundary point, then for every $r > 0$ and every x the open set $B(x, r) \cap \Omega$ enjoys the same property. Hence from Theorem 2.1 and Lemma 3.3 we obtain the following result.

Theorem 3.6 *Suppose that $\Omega \subset \mathbf{R}^n$ is an open set such that each point $x_o \in \partial\Omega$ is a thick boundary point and let $w \in W^{1,p}(\mathbf{R}^n)$ be continuous. Then, under the assumptions (1.3) – (1.6), every solution u of (1.1) with $u - w \in W_0^{1,p}(\Omega)$ has a continuous extension as w to $\partial\Omega$. If w is Hölder continuous at each point of $\partial\Omega$ and each boundary point of Ω is uniformly thick, then u is Hölder continuous at each point of $\overline{\Omega}$ with $u = w$ in $\partial\Omega$.*

Remark 3.7 From Remark 3.4 it follows that the Hölder exponent γ for u in Theorem 3.6 depends on n, p, q, α, β, $|\nabla u|$, the Hölder exponent of w and the constants σ and τ associated with the condition for uniform thickness. For $q = p-1$ the exponent γ does not depend on $|\nabla u|$.

The maximum principle and Theorem 3.6 now imply that the only solution of (1.1) in the class $u \in W_0^{1,p}(\Omega)$ is the zero solution provided that Ω is a bounded open set whose every boundary point is a thick boundary point. We next show that this holds in every open set of $\mathbf{R}^n$.

Quasiminimizers or local quasiminimizers seldom provide a unique solution to the Dirichlet problem except in the case of the next lemma.

Lemma 3.8 *Suppose that Ω is an open set in $\mathbf{R}^n$ and u is a K–quasiminimizer in small sets in Ω. If $u \in W_0^{1,p}(\Omega)$, then $u = 0$.*

Proof. Suppose that u is not identically zero. We can assume that the open set $U_{t_o} = \{x \in \Omega : u(x) > t_o\}$ is not empty for some $t_o > 0$; the proof is similar in the case $t_o < 0$. Write $M_o = \sup_\Omega u$.

There are two possibilities; either

$$\lim_{t \nearrow M_o} m(U_t) = 0 \tag{3.7}$$

or

$$\lim_{t \nearrow M_o} m(U_t) = m_o > 0. \tag{3.8}$$

In the first case it follows from Theorem 2.1 that there is $t < M_o$ such that $u|U_t$ is a K–quasiminimizer for some $K < \infty$. Now $u - t$ is a K–quasiminimizer as well and since $u - t \in W_0^{1,p}(U_t)$, we can use the function $v = 0$ in the quasiminimizing property (1.8) to obtain $\nabla u = 0$ in U_t. Thus $u = t$ in U_t, a contradiction. In the case (3.8)

$$m(\cap_{t\in[t_o,M_o)} U_t) = \lim_{t \nearrow M_o} m(U_t) = m_o > 0$$

because $u \in W_0^{1,p}(\Omega)$ yields $m(U_{t_o}) < \infty$, see e.g. [Fe, Theorem 2.1.3]. By the continuity of u every point $x \in \cap_{t\in[t_o,M_o)} U_t$ satisfies $u(x) = M_o$. By the maximum principle for quasiminimizers this is a contradiction. Hence $u = 0$ in Ω as required.

The above lemma together with Theorem 2.1 immediately implies

Corollary 3.9 *Let Ω be an open set in $\mathbf{R}^n$ and let p, q, A and b satisfy the assumptions (1.3) – (1.6). Then the only solution $u \in W_0^{1,p}(\Omega)$ of (1.1) is the zero solution.*

Remark 3.10 It is well known that uniqueness does not hold for $q = p = 2$ in the class $W_0^{1,2}(\Omega)$, see for example [ADP1–2] and [Tu]. However, the zero function is the only bounded solution. The same effect happens for $q = p < n$ where the function

$$u(x) = (p-1)\log\left(\frac{|x|^{-(n-p)/(p-1)}}{R^{-(n-p)/(p-1)}}\right)$$

is a $W_0^{1,p}(B(0,R))$–solution of the equation (1.7), see [FMu], and in the case $0 < \frac{p}{n} + p - 1 < q < n$ the function

$$u(x) = c(|x|^s - R^s)$$

with $s = (q-p)/(1-p+q)$ and suitable $c > 0$ is a solution of equation (1.7) in the class $W_0^{1,p}(B(0,R))$, see [LL].

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