A NOTE ON THE FREDHOLM PROPERTIES OF TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES WITH MATRIX-VALUED SYMBOLS

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Abstract. We characterize the essential spectra of Toeplitz operators T_a on weighted Bergman spaces with matrix-valued symbols; in particular we deal with two classes of symbols, the Douglas algebra $C + H^\infty$ and the Zhu class $Q := L^\infty \cap VMO_\partial$. In addition, for symbols in $C + H^\infty$, we derive a formula for the index of T_a in terms of its symbol a in the scalar-valued case, while in the matrix-valued case we indicate that the standard reduction to the scalar-valued case fails to work analogously to the Hardy space case.

1. Introduction

Fredholm theory of Toeplitz operators T_a on the Bergman space A^2 with continuous matrix-valued symbols and scalar-valued $C+H^\infty$ symbols was developed by Coburn [4], McDonald [9], and Venugopalkrishna [14] in the 1970s. Part of the theory has been generalized to the reflexive Bergman spaces A^p for symbols in $C + H^\infty$ of the unit ball—see [3] and [15], of which the former characterizes the essential spectrum of $T_a : A^p \to A^p$ when $a \in C + H^\infty$. However, a formula for the Fredholm index appears in the literature only in the Hilbert space case and only for continuous symbols.

In this note we consider similar questions in a more general setting when the underlying space is a weighted reflexive Bergman space A^p_α and a is a matrix-valued symbol in $C + H^\infty$. In particular, we characterize the essential spectrum of T_a, and in the scalar-valued case we also derive the usual index formula (analogous to the Hardy space case). Regarding the index of T_a on the Hardy space H^p with $a \in (C+\overline{H^\infty})_{N\times N}$, recall that T_f and T_g commute modulo finite rank operators when acting on Hardy spaces H^p provided that f, g are trigonometric polynomials, from which the matrix-valued case can be easily obtained (see,

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e.g., [2]); however, when considering Toeplitz operators on Bergman spaces, we cannot proceed in a similar fashion since there are no non-constant trace class Hankel operators. The other way of dealing with matrix-valued symbols in Hardy spaces is known as factorization (see, e.g., [8]), which seems unsuitable here as functions in Bergman spaces need not have boundary values in L^p. We finish the note by listing some open problems, which could stimulate further research into questions involving matrix symbols in Bergman spaces (indeed, there is a huge amount of literature on the Hardy space counterpart, but very few results dealing with Toeplitz operators on Bergman spaces with matrix symbols).

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2. Preliminaries

Let B_n denote the open unit ball in $\mathbb{C}^n$ with normalized volume measure $dA(z)$. For $1 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of all analytic functions in $L^p(B_n, dA_\alpha)$, where

$$dA_\alpha(w) = c_\alpha(1 - |w|^2)^\alpha dA(w)$$

with a positive normalizing constant c_α. The Bergman projection P_α of L^p onto A^p_α is the integral operator

$$P_\alpha f(z) = \int_{B_n} K_z^{(\alpha)}(w) f(w) dA_\alpha(w) = \int_{B_n} \frac{f(w)}{(1 - \langle w, z\rangle)^{n+1+\alpha}} dA_\alpha(w)\,.$$

Recall [19, Theorem 2.11] that, for $p \geq 1$ and $\alpha, t > -1$, the operator P_α is a bounded projection of $L^p(B_n, dA_t)$ onto A^p_t if and only if $p(\alpha+1) > t+1$. Let $a \in L^p(B_n, dA_t)$ and define the Toeplitz operator T_a and the Hankel operator H_a by setting

$$T_a = P_\alpha M_a \quad \text{and} \quad H_a = Q_\alpha M_a = (I - P_\alpha)M_a,$$

where M_a stands for the multiplication operator; the function a is referred to as the symbol of the given operator. It is clear that, for $1 < p < \infty$, $T_a : A^p_\alpha \to A^p_\alpha$ and $H_a : A^p_\alpha \to L^p(B_n, dA_\alpha)$ are both bounded whenever $a \in L^\infty(B_n, dA_\alpha)$.

Spaces of bounded mean (and vanishing) oscillation play an important role in connection with the general theory of Toeplitz and Hankel operators on Bergman spaces. However, when symbols are restricted to be continuous, one can develop Fredholm theory without reference to these spaces. Despite this, we include a brief look at them in the following as this allows us to easily refer to results on compactness of Hankel operators. The Bergman ball $D(z, r)$ with center z and radius r is defined by $D(z, r) = \{w \in B_n : \beta(z, w) < r\}$, where $\beta(z, w)$ is the Bergman metric. For a locally integrable function $f : B_n \to \mathbb{C}$, the averaging function $\hat{f}_r$ is defined by

$$\hat{f}_r(z) = \frac{1}{|D(z,r)|}\int_{D(z,r)} f(w)dA(w) \quad (z \in B_n),$$

where $|D(z,r)|$ is the volume of $D(z,r)$. The space of bounded mean oscillation BMO_r^p in the Bergman metric consists of all locally L^p integrable functions for which

$$\|f\|_{r,p}^p := \sup_{z\in B_n} \frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w) - \hat{f}_r(z)|^p dA(w) < \infty$$

If, in addition,

$$\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w) - \hat{f}_r(z)|^p dA(w) \to 0$$

as $|z| \to 1$, we say that f is in VMO_r^p, which is a closed subspace of BMO_r^p. As pointed out by K. Zhu [17], the definition of BMO_r^p depends on p (unlike in the case of the classical BMO for the unit circle) and $BMO_r^p \subset BMO_r^q$ properly for $q < p$; note also that the definition above is independent of r and we write BMO_∂^p for BMO_r^p and VMO_∂^p for VMO_r^p.

Suppose that $p \geq 1$ and $p(\alpha+1) > \lambda + 1 > 0$. According to K. Zhu [17], $a \in BMO_\partial^p$ if and only if the Hankel operators $H_a = (I - P_\alpha)M_a$ and $H_{\overline{a}}$ are both bounded from A_α^p into $L^p(B_n, dA_\lambda)$; and in addition, $a \in VMO_\partial^p$ if and if the Hankel operators $H_a = (I - P_\alpha)M_a$ and $H_{\overline{a}}$ acting from A_α^p into $L^p(B_n, dA_\lambda)$ are both compact. Note, however, that when $\alpha = \lambda = 0$ (that is, we have the standard Bergman projection P and the standard Bergman space A^p), the two theorems above require that $p > 1$; the case $p = 1$ with bounded scalar-valued symbols in $BMO_\partial^2(\mathbb{D})$ was recently considered in [11].

3. Compact Toeplitz operators

Let $1 < p < \infty$ and $\alpha > -1$. Denote by $\tau(A_\alpha^p)$ the closed subalgebra of $\mathcal{L}(A_\alpha^p)$ generated by Toeplitz operators T_a with $a \in L^\infty(B_n, dA_\alpha)$. We define the Berezin transform $B(T)$ of $T \in \mathcal{L}(A_\alpha^p)$ by

$$B(T)(z) = (1-|z|^2)^{1+n+\alpha}\langle TK_z^{(\alpha)}, K_z^{(\alpha)}\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the integral pairing and $K_z^{(\alpha)}$ is the reproducing kernel of A_α^2 given by

$$K_\zeta^{(\alpha)}(w) = \frac{1}{(1-\langle w,\zeta\rangle)^{n+1+\alpha}}$$

for $\zeta, w \in B_n$.

Let $T \in \mathcal{L}(A^p)$. In [10], it was recently observed that T is compact on A^p if and only if $T \in \tau(A^p)$ and $B(T)(z) = 0$ for all $z \in \partial B_n$. There seems to be no reason why this result would fail in the more general case of weighted Bergman spaces A_α^p. However, we only need such a characterization for Toeplitz operators with continuous symbols, which can be easily obtained by following Coburn's original approach valid for $p = 2$ and $\alpha = 0$ (see [4]).

Theorem 1. *Let* $1 < p < \infty$*,* $\alpha > -1$*, and* $a \in C(\overline{B_n})$*. Then* T_a *is compact on* A^p_α *if and only if* $a(z) = 0$ *for all* $z \in \partial B_n$*.*

We finish this section with a remark on commutator ideals. Denote by τ the Toeplitz algebra generated by T_a with $a \in C(\overline{\mathbb{D}})$. As in the Hilbert space case, the commutator ideal $\mathcal{I}$ of τ coincides with the space of all compact operators on A^p_α. Indeed, we have $\mathcal{I} \subset \mathcal{K}$ according to the formula

$$T_a T_b = T_{ab} - P M_a H_b = I - T_{1-ab} - P M_a H_b \,, \tag{3.1}$$

which holds even for symbols in $L^\infty(B_n, dA_t)$; note that the Hankel operator H_b is compact—see Section 1. It remains to note that all rank one operators are contained in $\mathcal{I}$.

4. Fredholm properties

A bounded linear operator A on a Banach space X is said to be Fredholm if both its kernel and cokernel are finite-dimensional; the index of a Fredholm operator is defined to be

$$\operatorname{Ind} A = \dim \ker A - \dim \operatorname{coker} A.$$

The winding number of a nonvanishing continuous function a is denoted by $\operatorname{ind} a$. The essential spectrum $\sigma_{\text{ess}}(A)$ of A consists of all $\lambda \in \mathbb{C}$ for which $T_a - \lambda I$ is not Fredholm, that is,

$$\sigma_{\text{ess}}(A) = \sigma_{\mathcal{L}(X)/\mathcal{K}(X)}(\pi(A)),$$

where π is the natural map.

It is well known that, for an $n \times n$ matrix-valued symbol a with entries in $C(\bar{B}_n)$, the Toeplitz operator $T_a : A^2_n \to A^2_n$ is Fredholm if and only if $\det a(z) \neq 0$ for any $z \in \partial B_n$ (see [4]). This can also be easily generalized to the weighted Hilbert space case. In what follows we deal with weighted Bergman spaces that are reflexive, that is, we consider the case $1 < p < \infty$. The next theorem follows from a more general result on symbols in $C(\overline{\mathbb{D}}) + H^\infty(\mathbb{D})$ (see Theorem 3 below); however, we still indicate how one can prove it.

Theorem 2. *Let* $1 < p < \infty$*,* $\alpha > -1$*, and* $a \in C(\overline{B_n})$*. Then* T_a *is Fredholm on* A^p_α *if and only if* $a(z) \neq 0$ *for any* $z \in \partial B_n$*; in which case* $\operatorname{Ind} T_a = -\operatorname{Ind} a_{\restriction \partial B_1}$ *when* $n = 1$ *and* $\operatorname{Ind} T_a = 0$ *when* $n > 1$*.*

Proof. Sufficiency can be proved by constructing a regularizer (as in the Hardy space case; see, e.g., [2, Theorem 2.42]) and using Theorem 1 and Zhu's characterization of compact Hankel operators.

For $n = 1$, the index formula can be proved similarly to the corresponding result in [11]. When $n > 1$, we can proceed as in the proof of [14, Theorem 1.4].

When $n = 1$, necessity can be proved as in the Hardy space case using the index formula. In the general case, we can apply the approach used in [4, 15]. □

Let $f \in H^\infty(\mathbb{D})$ be nonzero. Then there are a Blaschke product B and a function $g \in GH^\infty(\mathbb{D})$ such that $f = Bg$. Suppose that there are $\epsilon > 0$ and $\delta > 0$ such that $|f(z)| \geq \epsilon$ for $\delta < |z| \leq 1$. Proceeding as in the proof of [2, Theorem 2.64], we get

$$\operatorname{Ind} T_f = -\operatorname{ind} f_r, \tag{4.1}$$

where $f_r(t) = f(rt)$ $(t \in \mathbb{T})$ with $\delta < r < 1$. Note that $\operatorname{Ind} T_f$ is independent of the choice of r since the index is constant on connected components.

Theorem 3. *Let $a \in C(\overline{B}_n) + H^\infty(B_n)$ and $\alpha > -1$. Then T_a is Fredholm on $A^p_\alpha(B_n)$ if and only if a is bounded away from zero near the boundary of B_n, that is, if there are $\delta > 0$ and $\epsilon > 0$ such that*

$$|a(z)| > \epsilon \quad \text{for} \quad \delta < |z| < 1; \tag{4.2}$$

if in addition $n = 1$, we have the following index formula

$$\operatorname{Ind} T_a = -\operatorname{ind} a_r, \tag{4.3}$$

where $a_r(t) = a(rt)$ with $\delta < r < 1$.

Proof. For $\alpha = 0$, the proof of the Fredholm criterion can be found in [3]; the general case can be dealt with similarly.

Let us consider the index formula when $n = 1$. Since polynomials (in z and $\bar{z}$) are dense in $C(\overline{\mathbb{D}})$, it suffices to prove the formula when $a = p+g$ for some polynomial p and $g \in H^\infty(\mathbb{D})$. Write $p+g = \bar{z}^m f + h$ for some $f \in H^\infty(\mathbb{D})$ and $h \in C(\overline{\mathbb{D}})$ with $h = 0$ on $\mathbb{T}$ (see the proof of [9, Theorem 3.2]). Since f is bounded away from zero, we can apply (4.1) to conclude

$$\begin{aligned}
\operatorname{Ind} T_{p+g} &= \operatorname{Ind}(T_{\bar{z}^m f} + T_h) = \operatorname{Ind} T_{\bar{z}^m} + \operatorname{Ind} T_f \\
&= -\operatorname{ind} \bar{z}^m - \operatorname{ind} f_r = -\operatorname{ind}(\bar{z}^m f)_r \\
&= -\operatorname{ind}(\bar{z}^m f + h)_r = -\operatorname{Ind}(p+g)_r
\end{aligned}$$

provided that r is sufficiently close to 1. □

Next we consider the symbol class $Q := L^\infty \cap VMO_\partial$, introduced by K. Zhu, who studied the properties of Toeplitz operators with these symbols in the Hilbert space context—see [16]. In what follows we study the Fredholm properties of Toeplitz operators T_a acting on the weighted Bergman spaces A^p_α when $a \in Q$.

Remark 4. *We have*

$$L^\infty \cap BMO_\partial = L^\infty \cap BMO_\partial^p$$

and

$$Q = L^\infty \cap VMO_\partial = L^\infty \cap VMO_\partial^2 = L^\infty \cap VMO_\partial^p$$

for $1 \le p < \infty$.

Proof. For $f \in L^\infty$ we clearly have $\hat{f} \in L^\infty$ and hence it is easily seen that $L^\infty \subset BMO_\partial^p$ for each $1 \le p < \infty$. For the second claim assume that $f \in L^\infty \cap VMO_\partial$. Then for a fixed $r > 0$ and $1 \le p < \infty$ we have

$$\begin{aligned}
&\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w) - \hat{f}(z)|^p dA_\alpha(w)\\
=&\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w) - \hat{f}(z)|^{p-1}|f(w) - \hat{f}(z)| dA_\alpha(w)\\
\le&C\frac{1}{|D(z,r)|}\int_{D(z,r)} |f(w) - \hat{f}(z)| dA_\alpha(w),
\end{aligned}$$

by the above remark about $\hat{f}$. Conversely, if $f \in L^\infty \cap VMO_\partial^p$, then by similar reasoning as above and by using Hölder's inequality we see that $f \in L^\infty \cap VMO_\partial$. □

As we are only concerned with bounded symbols, we need not deal with the general BMO_∂^p and VMO_∂^p spaces according to the preceding remark; compare this with the situation in [12].

It is also worth noting that there are symbols both in $(L^\infty \cap VMO_\partial) \setminus (C(\overline{\mathbb{D}}) + H^\infty(\mathbb{D}))$ and in $(C(\overline{\mathbb{D}}) + H^\infty(\mathbb{D})) \setminus (L^\infty \cap VMO_\partial)$. Indeed, suppose $H^\infty \subset VMO_\partial \cap L^\infty$. Then, given $a \in H^\infty$, we have $\overline{a} \subset VMO_\partial \cap L^\infty$ (since this set is a C^*-algebra). But then the Hankel operator $H_{\overline{a}}$ is compact, which implies that $a \subset \mathcal{B}_0$. So $H^\infty \subset \mathcal{B}_0$, which is a contradiction.

Theorem 5. *Let* $a \in Q := L^\infty \cap VMO_\partial$, $1 < p < \infty$, *and* $\alpha > -1$. *Then* T_a *is Fredholm on* A^p_α *if and only if* $B(a)$ *is bounded away from zero near the boundary* $\partial\mathbb{D}$, *in which case* $\operatorname{Ind} T_a = -\operatorname{ind} B(a)_{\restriction r\mathbb{T}}$ *for* r *sufficiently close to* 1.

Proof. The index formula and sufficiency both follow from [12, Theorem 2.8] when $\alpha = 0$. For the weighted case, results in [17] and [16, Theorem 7] imply that $a - \tilde{a}$ is in Q and has vanishing Berezin symbol. Now [10, Theorem 9.5] implies that $T_{a-\tilde{a}} = T_a - T_{\tilde{a}}$ is compact. But this means that T_a is Fredholm if and only if $T_{\tilde{a}}$ is Fredholm. Moreover, if they are Fredholm, they have the same index. Finally, the index of $T_{\tilde{a}}$ can be computed using [16, Remark on p. 640]. □

5. Matrix-valued symbols

In this section we generalize Theorem 2 and part of Theorems 3 and 5 to the case of matrix-valued symbols using standard Banach algebra techniques. Let X be a Banach space and set $X_N = \{(f_1, \ldots, f_N) : f_k \in X\}$, which is also a Banach space when equipped with the norm

$$\|(f_1, \ldots, f_N)\|_{X_N} := \|f_1\|_X + \ldots + \|f_N\|_X$$

(or with any equivalent norm). Note that each operator $A \in \mathcal{L}(X_N)$ can be expressed as an operator matrix $(A_{ij})_{i,j=1}^N$ in $\mathcal{L}(X_{N\times N})$.

Recall the following results from matrix analysis; see, e.g., [7].

Theorem 6. (a) *Let A be a bounded linear operator on a Banach space X. Suppose that the entries A_{ij} of A pairwise commute modulo compact operators. Then A is Fredholm on X_N if and only if $\det A$ is Fredholm on X.*

(b) *Suppose that $\mathcal{A}$ is a subalgebra of $\mathcal{L}(X)$ and $A_{ij} \in \mathcal{A}$. If $\mathcal{A}$ contains all compact operators on X, if the commutator $AB - BA$ is compact for all $A, B \in \mathcal{A}$, and if $\Phi(X) \cap \mathcal{A}$ is dense in $\mathcal{A}$, then $\mathrm{Ind}(A_{ij}) = \mathrm{Ind}\det(A_{ij})$ whenever (A_{ij}) is Fredholm.*

(c) *Let X be a Banach space and let A be Fredholm on X_N. If the entries of A commute pairwise modulo finite-rank operators, then $\mathrm{Ind}\, A = \mathrm{Ind}\det A$.*

We can now give a necessary and sufficient condition for Fredholmness of Toeplitz operators with matrix-valued symbols.

Theorem 7. *Let $1 < p < \infty$, let $N \geq 2$ be an integer, and suppose that $\alpha > -1$.*

(a) *For $a \in C(\overline{B}_n)_{N\times N}$, T_a is Fredholm on $(A^p_\alpha)_N$ if and only if $\det a(z) \neq 0$ for any $z \in \partial B_n$; if in addition $n = 1$, we have*

$$\mathrm{Ind}\, T_a = \mathrm{Ind}\, T_{\det a} = -\,\mathrm{ind}\det a_{\restriction \partial B_n};$$

(b) *For $a \in (C(\overline{B}_n) + H^\infty(B_n))_{N\times N}$, T_a is Fredholm on $(A^p_\alpha)_N$ if and only if $\det a(z) \neq 0$ is bounded away from zero near the boundary ∂B_n.*

(c) *For $a \in Q = L^\infty \cap VMO_\partial$, T_a is Fredholm if and only if $B(\det a)$ is bounded away from zero near the boundary.*

Proof. We reduce the proof to the scalar case via the preceding theorem using the representation $T_a = (T_{a_{ij}})_{i,j=1}^N$. To verify the criterion, note that the Hankel operator H_f is compact for f in any of the classes above, and so, according to (3.1), we have $T_a T_b = T_b T_a$ modulo compact operators. Using (3.1) again, we get

$$\det T_a = \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) T_{a_{1\sigma(1)}\cdots a_{N\sigma(N)}} + K'$$

for some compact operator K', where S_n is the group of N-permutations and $\mathrm{sgn}(\sigma)$ stands for the sign of σ. Consequently, T_a is Fredholm if and only if $\det T_a = T_{\det a} + K'$ is Fredholm, that is, $\det a \neq 0$ on the boundary of B_n by the corresponding result in the scalar-valued case and Atkinson's theorem.

The index formula for continous symbols follows from the usual perturbation argument used in the scalar case and Theorem 6.

The other two statements can also be reduced to the scalar-valued case via Theorem 6. □

We would also like to say something about the index formula for the cases (b) and (c) in the previous theorem.

Theorem 8. *Let $1 < p < \infty$, let $N \geq 2$ be an integer, and suppose that $\alpha > -1$ and $n = 1$. Let $a \in (C(\overline{B}_n) + H^\infty(B_n))_{N\times N}$ or $a \in Q = L^\infty \cap VMO_\partial$. Suppose that T_a is Fredholm and at least one of the following conditions hold*

(i) *The scalar Toeplitz operators $T_{a_{ij}}$ and $T_{a_{kl}}$ commute modulo trace class operators, where $a = (a_{ij})$;*

(ii) *T_{a_k} is Fredholm on $(A^p_\alpha)_k$ for each $k = 1,..,N$, where $a_k = (a_{ij})_{i,j\leq k}$;*

then

$$\operatorname{Ind} T_a = \operatorname{Ind} T_{\det a}.$$

Proof. This is a direct consequence of the above theorem and theorems 7.4 and 7.6 in [6]. □

Let us consider the index of T_a on the Hilbert space A^2 with $a \in (C(\overline{\mathbb{D}}) + H^\infty)_{N\times N}$ in some more detail. It is well known that if H is a Hilbert space, if T is Fredholm on H_N, and if the entries of T commute modulo trace class operators, then $\operatorname{Ind} T = \operatorname{Ind} \det T$ (note that [5] contains a slightly more general result, which, however, seems to offer no real advantages to the index computation here). Let $a, b \in C(\overline{\mathbb{D}}) + H^\infty$, which can be approximated by functions of the form $p+f$, where p is a polynomial in z and $\bar{z}$ and f is in H^∞. Note that it suffices to prove the index formula for a class of symbols that is dense in $(C(\overline{\mathbb{D}}) + H^\infty)_{N\times N}$. Now if $a = p_1 + f_1$ and $b = p_2 + f_2$ for some polynomials p_k and $f_k \in H^\infty$, then

$$T_a T_b = T_{ab} - PM_a H_{p_2}.$$

Here H_{p_2} is trace class only if p_2 is constant (see [1]), and so we cannot make use of the properties of Hankel operators the same way as in the Hardy space case where Hankel operators are finite rank (and hence trace class) for polynomial symbols. Also, if we could show that $PM_a \in S_q$ for any q, then we could conclude that T_a and T_b commute modulo trace class operators since $H_{p_2} \in S_p$ for $p > 1$.

On the other hand, note that we can write

$$T_{ab} - T_a T_b = H^*_{\bar{a}} H_b, \quad T_{ab} - T_b T_a = H^*_{\bar{b}} H_a.$$

Therefore,

$$T_aT_b - T_bT_a = H_{\bar{b}}^* H_a - H_{\bar{a}}^* H_b.$$

It is known that H_a and H_b are in the Schatten class S_p for any $p > 1$ and that neither of the operators $H_{\bar{b}}^*$ and $H_{\bar{b}}$ is compact. It seems that the Hankel products need not be in the trace class. While the difference could still be of trace class, we conjecture that there are Toeplitz operators with polynomial symbols that do not commute modulo trace class operators.

We can give an example of symbols in $a, b \in C(\overline{\mathbb{D}}) + H^\infty$ for which T_a and T_b do not commute modulo trace class operators. Recall that the disk algebra A is the set of all analytic functions continuous on $\overline{\mathbb{D}}$, and that the Dirichlet space D is the space of analytic functions with derivatives in L^2. Note first that neither of the spaces A and D is contained in the other. Indeed, the Riemann mapping theorem implies that there is an analytic function f that takes the unit disk to a simply connected, unbounded domain with finite area. Since the Dirichlet integral of an analytic function is the area of the image of the function (see, e.g., [20, Exercise 12 of Section 5.5]), it follows that f is in the Dirichlet space; however, since f is unbounded, it is not in the disk algebra. Consider now a function g defined by $g(z) = \sum_{n=0}^\infty a_n z^n$, where $a_n = n^{-1/2}$ when $n = k^4$ $(k = 1, 2, 3..)$ and $a_n = 0$ otherwise. Then the partial sums of g converge uniformly in $\mathbb{D}$ and hence $g \in A$. On the other hand,

$$(n+1)^2 |a_{n+1}|^2 = n + 1$$

whenever $n+1 = k^4$, which, by Parseval's identity, implies that g' does not belong to A^2. So $g \in A \setminus D$.

Let $a \in A \setminus D$. Then $T_a = M_a$ and $T_{\overline{a}} = T_a^* = M_a^*$. By the main result of [18], the trace of the commutator $[T_a, T_{\overline{a}}] = [M_a, M_a^*]$ is infinite.

While it seems to be widely assumed (or even considered well known!) that the index of T_a on A_N^p has a similar formula as in the H_N^p case, it is quite surprising that this has not been verified even in the Hilbert space case as far as we are aware. Because of Theorem 3, we still conjecture that the formulas for the index in Bergman and Hardy spaces are analogous.

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