Limits of Teichmüller maps

Pekka Tukia

Abstract. A Teichmüller map is a quasiconformal homeomorphism of $\overline{\mathbb{C}}$ so that f is conformal outside the closed unit disk, denoted Δ^*, and in the unit disk Δ the complex dilatation of f is of the form $k\bar{\varphi}/|\varphi|$ where $0 \leq k < 1$ and φ is holomorphic. We consider sequences of such Teichmüller maps f_j whose complex dilatations are of the form $k_j\bar{\varphi}_j/|\varphi_j|$ where $k_j \to 1$ and φ_j are holomorphic mappings such that φ_j tend toward a holomorphic mappings φ uniformly on compact subsets. We assume that the L^1-norms of φ_j and φ are uniformly bounded. If f_j are suitably normalized, it is possible to pass to a subsequence so that f_j tend toward a conformal limit outside the unit disk. Since f_j are not uniformly quasiconformal, such a limit need not exist in $\bar{\Delta}$ but we show that a modified form of limit in $\bar{\Delta}$ exists for a subsequence. We call it the extended limit and it is constructed using a partition of $\bar{\Delta}$, denoted $\mathcal{D}$, whose elements are closed sets constructed from vertical trajectories of φ as well as some closed arcs and points of $\partial\Delta$. The extended limit, denoted also f, is defined on $\Delta^* \cup \mathcal{D}$ and satisfies a continuity condition called semicontinuity. The image $f\mathcal{D} = \{f(X) : X \in \mathcal{D}\}$ is a family of closed sets of $\overline{\mathbb{C}}$ which is partition of $\overline{\mathbb{C}} \setminus f\Delta^*$. The extended limit is a limit of f_j's in a sense which we call semiconvergence. If sets of $\mathcal{D}$ are collapsed to points, and similarly likewise $f(X)$, $X \in \mathcal{D}$, are collapsed to points, then the quotient spaces are homeomorphic to $\overline{\mathbb{C}}$ and f is a homeomorphism between them.

1. Introduction

One considers in Teichmüller space theory quasiconformal maps f of the $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which are conformal in the exterior $\Delta^* = \{\infty\} \cup \{|z| > 1\}$ of the unit disk $\Delta = \{|z| < 1\}$ and in the unit disk the complex dilatation μ_f of f is given by

$$\mu_f = k\frac{\bar{\varphi}}{|\varphi|} \quad (0 \leq k < 1) \tag{1}$$

where φ is holomorphic and not identically zero in Δ; we will call such maps *Teichmüller maps*. If there is compatibility with the action of a Kleinian group, then φ satisfies a quadratic type relation and hence the holomorphic map φ in (1) is often called a *quadratic differential*. However, we will not consider here this situation although quadratic differential will be a synonym for a holomorphic map. We will obtain results for integrable holomorphic mappings, that is maps φ such that the L^1-norm

$$||\varphi||_1 = \int_\Delta |\varphi| dm < \infty \tag{2}$$

where dm is the euclidean area element.

The universal Teichmüller space can be identified with the space of such quasi-confomal mappings which are suitably normalized. We are interested in what happens when $k \to 1$, that is, we approach the boundary of the Teichmüller space. If φ is fixed in (1), then these maps are on a Teichmüller geodesic though we will also consider the situation that we have sequences k_j and φ_j vary so that $k_j \to 1$ and $\varphi_j \to \varphi$; the maps φ and φ_j are assumed to be integrable and we will later state the exact conditions. Let f_j be such a sequence. If we are outside the unit disk, then a suitable normalization, for instance f fixes 0, 2 and ∞, guarantees that there is a subsequence tending toward a conformal map f outside the unit disk. On the other hand, the maximal dilatation $K_j = \frac{1+k_j}{1-k_j}$ tends to ∞. Hence the normal family arguments working in the conformal case cannot be used.

We can say something of the situation inside the unit disk in terms utilizing the trajectory structure of φ to be defined below. On pointwise level, the distortion becomes too strong for the pointwise structure to be preserved in the limit but we can construct a coarser structure based on vertical trajectories of φ and this structure is mapped onto a similar structure by a map which can be described as a limit of a subsequence of f_j's. We denote this limit by f and it is continuous if continuity is defined as in Theorems 1 or 3.

We now define this trajectory structure. It differs from the usual definition in that we allow zeroes of φ into trajectories and the trajectory branches at such a point. A *non-critical vertical trajectory arc* of φ is an open arc τ outside the zeroes of φ and is an arc such that $\arg \varphi\, dz^2 = \pm\pi$ on τ when dz is an infinitesimal to the direction of τ. If $\tau \subset U \subset \Delta$, then we say that τ is a trajectory of U. We often omit the word vertical when considering such arcs.

If z_0 is a zero of order n of τ, then sufficiently small neighborhoods U of z_0 contain $n+2$ disjoint vertical non-critical trajectory arcs of $U \setminus \{z_0\}$ such that the closure of the trajectory is a closed arc with endpoint z_0; in that case we say that the trajectory has endpoint z_0, cf. Fig. 1 for the flow of trajectories. We now extend the definition of a vertical trajectory so that it may include zeroes of φ and that τ is a *vertical trajectory* if any two points z and w of τ can be connected by a sequence $\tau_1, .., \tau_n$ where $a \in \tau_1$ and $b \in \tau_n$ and each τ_j is either a non-critical trajectory arc or a zero of φ with the property that either $\tau_j \cap \tau_{j+1} \neq \emptyset$ or one of them is a zero of φ and the other is a non-critical trajectory arc whose endpoint the other is. In addition, we require that τ is a maximal set of this kind; we include the maximality in the definition since we consider only maximal trajectories. Again, the word vertical may be omitted.

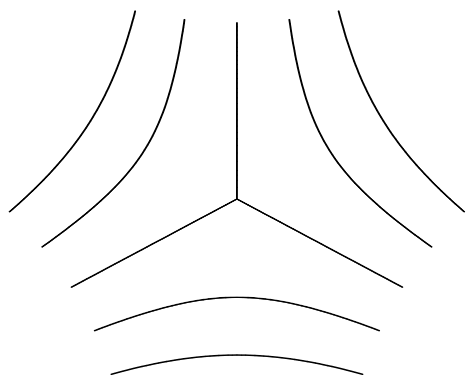

Fig.1

According to our definition, a vertical trajectory branches at zeroes of φ. Now, φ has only a countable number of zeroes and hence most trajectories do not contain zeroes of φ. We call such trajectories *simple* and one can show (Lemma 3.1) that the closure of a simple trajectory of an integrable φ is a cross cut of $\partial\Delta$, that is, it is obtained by adding two points on $\partial\Delta$ so that we obtain a closed arc with endpoints on $\partial\Delta$. A trajectory is always simply connected and is locally like the middle trajectory in Fig. 1 (if $\varphi(z_0) \neq 0$, then $n = 0$ in Fig.1), cf. [T1, Lemma 2.3].

We need still to extend the definition of a trajectory so that we can go to the boundary and beyond. The above definition are valid for any holomorphic φ but now we assume that φ is integrable. We will extend a trajectory τ to a set called *complete trajectory* and denoted τ^*. Its definition is somewhat more technical and involves the φ-metric d_φ defined by the element of length $\sqrt{|\varphi|}|dz|$ as well as the variant of φ-metric called the horizontal length whose element of length is $|\mathrm{Re}\sqrt{\varphi}dz|$ and which defines a pseudometric d^h_φ of Δ. The horizontal distance of two points on the same vertical trajectory is zero and $d^h_\varphi(z,w)$ is constant when z varies in one vertical trajectory and w in another.

Using the horizontal distance, we can define that a *complete trajectory* starting from a trajectory τ and let τ^*, the *completion of* τ, consists of points $z \in \bar{\Delta}$ such that τ and z are not separated in $\bar{\Delta}$ by the closure of any simple trajectory σ such that the horizontal distance $d^h_\varphi(\tau,\sigma) > 0$. *Complete trajectories* are sets of this form. Section 3 contains a more detailed definition and the following. A complete trajectory ν is a closed and connected subset of $\bar{\Delta}$ such that $\nu \cap \Delta$ is a union vertical trajectories which are locally finite in Δ and two distinct complete trajectories are disjoint (Theorem 3.6). Usually the completion of a trajectory τ is just the closure $\bar{\tau}$ of τ and is homeomorphic to a closed arc and in this case we say that $\bar{\tau}$ is a *simple complete trajectory.* There are only a countable number of exceptions (Lemma 3.8). Note that a simple trajectory is not the same as a simple complete trajectory.

The reason why vertical trajectories are important for us is that if z is not a zero of φ, then the differential of f at z maps infinitesimal circles centered at z to ellipsoids so that the direction of the vertical trajectory gives the direction which is mapped onto the minor axis of the image ellipsoid as can be seen of the formula for the derivative

$\partial_\alpha f$ of f to the direction $e^{i\alpha}$ which is

$$\partial_\alpha f = (1 + \mu_f e^{-2i\alpha}) f_z = (1 + k \frac{\overline{\varphi}}{|\varphi|} e^{-2i\alpha}) f_z. \tag{3}$$

This gives reasons to suspect that if f_j is a Teichmüller sequence satisfying (1) with $k = k_j$ and fixed φ, then, given a trajectory, there would be a subsequence which would tend toward a constant on the trajectory. This is indeed often the case. Perhaps surprisingly, if there is no subsequence with constant limit, then there is a subsequence tending toward an embedding of τ. Cf. [T1] for these results.

Horizontal trajectories are defined similarly using the condition $\arg \varphi \, dz^2 = 0$ but the vertical trajectories are more important for us. Therefore, in the sequel a *trajectory*, if not specified, means a *vertical* trajectory though we will often emphasize the verticality and include the word.

The paper [T1] considered pointwise convergence on a trajectory τ, obtaining the above mentioned dichotomy that a subsequence tends pointwise either to a constant or to an embedding or τ. It would be desirable to find the conditions under which f_j would have a continuous limit f which is conformal outside Δ and such that f maps vertical trajectories of φ to points. Our results in this paper are inspired by this aim and we will obtain results in this direction under the assumption that φ is integrable.

We will show that under the integrability assumption, assuming the existence of a conformal limit f on Δ^*, we can extend the limit f from the exterior Δ^* of $\bar{\Delta}$ to Δ so that the extension is a map not onto $\overline{\mathbb{C}}$ but to the family $\mathcal{C}$ of closed and non-empty subsets of $\overline{\mathbb{C}}$. The set $\mathcal{C}$ is topologized by means of the Hausdorff metric obtained using the spherical metric of $\overline{\mathbb{C}}$. It is also better to consider the extension to Δ not as a map of points but as a map defined on the set whose elements are complete trajectories as well some subsets of $\partial\Delta$. We denote this set by $\mathcal{D}$ and the precise definition is

$$\mathcal{D} = \mathcal{T} \cup \mathcal{V}$$

where $\mathcal{T}$ is the set of complete trajectories of $\mathcal{V}$ is the set of components of $\partial\Delta \setminus (\cup\mathcal{T})$. We will show that $\mathcal{D}$ is a partition of $\bar{\Delta}$ by closed sets (Lemma 3.11). We can regard $\overline{\mathbb{C}}$ in a natural way as a subset of $\mathcal{C}$ so that $x \in \overline{\mathbb{C}}$ is identified with $\{x\}$ and then the topologies given by the spherical metric of $\overline{\mathbb{C}}$ and the Hausdorff metric coincide. In the sequel, it will be convenient to interpret x as a one-point set $\{x\}$ as required. Thus we regard the limit of f_j in Δ^* as a map $\Delta^* \to \overline{\mathbb{C}} \subset \mathcal{C}$ and extend f to $\Delta^* \cup \mathcal{D}$.

We will show this extension of f to $\mathcal{D}$ is in a certain sense continuous. It is continuous if we pass to suitable quotients (Theorem 3) but considered as a map of $\mathcal{C}$ into $\mathcal{C}$ it need not be continuous in the Hausdorff metric of $\mathcal{C}$. However, it satisfies a weaker condition called semicontinuity. We say that $f : \Delta^* \cup \mathcal{D} \to \mathcal{C}$ is *semicontinuous* if, given $x \in \Delta^* \cup \mathcal{D}$ and a neighborhood U of $f(x)$ in $\overline{\mathbb{C}}$, there is a neighborhood V of x in $\overline{\mathbb{C}}$ so that $f(y) \subset U$ whenever $y \subset V$, $y \in \Delta^* \cup \mathcal{D}$. Similarly, if $x_n \in \mathcal{C}$ is a sequence and $x \in \mathcal{C}$, we say that x_n *semiconverges* to x if, given a neighborhood $U \subset \overline{\mathbb{C}}$ of x, then $x_n \subset U$ beginning from some n. Note that such a semiconvergent limit x is not well defined. The notion of semicontinuity is an extension of the notion of continuity in the sense that if f maps points onto points, then f is continuous if and only if it is semicontinuous.

We can now state our main theorem. Let φ_j and $\varphi \not\equiv 0$ be integrable holomorphic maps in Δ such that $||\varphi_j||_1$ are uniformly bounded and that $\varphi_j \to \varphi$ uniformly on compact subsets of Δ. If $\varphi_j \to \varphi$ in L^1, then these conditions are true as follows using the Cauchy formula. Let $\mathcal{D}$ be defined as above with respect to φ so that its elements are complete trajectories of φ as well as some subsets of $\partial\Delta$.

Theorem 1. *Suppose that f_j converge toward a conformal map f in the exterior Δ^* of Δ. Then there is a subsequence so that f admits a semicontinuous extension to a map $\Delta^* \cup \mathcal{D} \to \mathcal{C}$ as a map into the family $\mathcal{C}$ of closed non-empty subsets C of $\overline{\mathbb{C}}$. The extended map f is the limit of f_j in the sense that if $x \in \Delta \cup \mathcal{D}$, then $f_j(x)$ semiconverges to $f(x)$. The families $\mathcal{D}$ and $f\mathcal{D} = \{f(x) : x \in \mathcal{D}\}$ are partitions of $\bar{\Delta}$ and of $\overline{\mathbb{C}} \setminus f\Delta^*$, respectively, and f is a bijection of $\mathcal{D}$ onto $f\mathcal{D}$.*

We remark that if $f_j(\tau)$, $\tau \in \mathcal{D}$, have the Hausdorff limit ν, then $\nu \subset f(\tau)$ (Theorem 2.4).

The convergence of f_j to f on Δ^* is uniform on compact subsets. Uniform convergence on a compact set in the continuous case is equivalent to the fact that if x_j tend to x, then $f_j(x_j)$ tend to $f(x)$. This makes possible to formulate a uniform convergence result for this situation:

Theorem 2. *The map f of Theorem 1 has the following property. If $x_j \in \Delta^* \cup \mathcal{D}$ and x_j semiconverge to $x \in \Delta^* \cup \mathcal{D}$, then $f_j(x_j)$ semiconverge to $f(x)$.*

We have above thought the extension as a map of $\mathcal{C}$ into $\mathcal{C}$. This is useful since then we can express f as kind of limit of f_j's by means of semiconvergence. If we are interested only of the limit f, then passing to suitable quotients, f becomes a homeomorphism. We note that $\mathcal{D}$ is a partition of $\bar{\Delta}$ and hence we can form the quotient space of $\overline{\mathbb{C}}$, denoted by $\overline{\mathbb{C}}/\mathcal{D}$. A similar remark applies to $f\mathcal{D}$. Thus we can also regard the limit f as a map of $\overline{\mathbb{C}}/D$ into $\overline{\mathbb{C}}/f\mathcal{D}$ and have

Theorem 3. *The spaces $\overline{\mathbb{C}}/\mathcal{D}$ and $\overline{\mathbb{C}}/f\mathcal{D}$ are homeomorphic to $\overline{\mathbb{C}}$ and f is a homeomorphism between them.*

The part concerning homeomorphicity is based on a theorem of R.L. Moore [M] on upper semi-continuous decompositions of the plane. The proof is completed in the end of Section 4.

It is useful at this point to comment on the connection to [T1] where we considered the situation with fixed $\varphi_j = \varphi$ and with k_j tending to 1. We proved that given a trajectory τ, there is a subsequence so that f_j tend on τ toward a constant or an embedding of τ. The latter would seem to be exceptional (cf [T1, Theorem 2]). Suppose that f_j converge on τ toward a constant a. Then $a \in f(\tau^*)$, $\tau^* \in \mathcal{D}$ the completion of τ, but we cannot conclude that $f(\tau)$ is a one-point set. A similar remark applies if f_j converge toward an embedding of τ.

Quadratic differentials for Fuchsian groups. If φ is a non-trivial quadratic differential of an infinite Fuchsian group, then $||\varphi||_1 = \infty$. Thus our results do not apply in the Fuchsian case. However, it might be possible to obtain similar results if φ is a quadratic differential for a cocompact Fuchsian group G of Δ. We think that analogues of our theorems are true at least in the case that f_j define points in the Teichmüller space projecting into a compact set in the moduli space. Since the compactification of the moduli space is well understood, it might be that this restriction is not necessary.

Since $||\varphi||_1 = \infty$ in this case, we cannot use $\sqrt{|\varphi|}$ as a substitute for ϱ in (2.1) and (2.2) but using more complicated methods it might be possible to obtain the essential Lemma 2.1. The situation would seem to be simpler in the respect that the closure of a trajectory is already its completion. If τ and σ are distinct trajectories, they define a ring $R(\tau, \sigma)$ as in (2.7) and Theorem 2.3 should be valid for this ring. This is all that is needed for the construction of the extended limit.

It is expected that the limit set of a limit group of this kind is locally connected and if the limit group is totally degenerate, that is the discontinuity set has just one component, then the sets of $f\mathcal{D}$ are one-point sets which make up the limit set $L(H)$ of the limit group H. Thus we do not need collapse sets of $f\mathcal{D}$ to points but $\overline{\mathbb{C}}/f\mathcal{D}$ is actually $\overline{\mathbb{C}}$ and not only homeomorphic to $\overline{\mathbb{C}}$. Thus the the map of Theorem 3 becomes a homeomorphism $\overline{\mathbb{C}}/\mathcal{D} \to \overline{\mathbb{C}}$ so that $f\mathcal{D} = L(H)$. Note however, that the limit group is not necessarily totally degenerate. If there is a trajectory τ such that the stabilizer of τ in G is non-elementary, then τ will be mapped in the limit onto a component of $\Omega(H)$ distinct from $f\Delta^*$, see [T2].

We remark still that it H is a totally degenerate with locally connected limit set, not necessarily a limit group in the sense discussed above, then it is possible to define a geodesic lamination $\mathcal{L}$ invariant under a Fuchsian group G and that of Δ so that the natural action of H is topologically conjugate to the action of G on the space obtained by collapsing elements of the lamination to points so that the image of these collapsed elements is homeomorphic to $L(H)$ minus the endpoints of $L(H)$; endpoints of $L(H)$ are points of $L(H)$ not separating $L(H)$, cf. Abikoff [A, Theorem 3].

Organization of the paper. Section 2 contains the main ideas and outlines the proof. Section 3 is independent of Section 2 and contains the general properties of trajectories needed. Section 4 gives the exact construction of the extended limit and contains proofs of Theorems 1, 2 and 3, and it assumes the results of Sections 2 and 3. Finally, Section 5 contains the technicalities needed to complete proofs of Theorems 2.1 and 2.2 outlined in Section 2; this section is based on Section 3 and is independent of Section 4.

Definitions and notation. The *Hausdorff distance* is defined in the the family $\mathcal{C}$ of closed and non-empty subsets of $\overline{\mathbb{C}}$ and the Hausdorff distance of $X, Y \in \mathcal{C}$ is defined using the spherical metric of of $\overline{\mathbb{C}}$ and is the infimum of numbers δ such that $Y \subset U_\delta(X)$ and $X \subset U_\delta(Y)$ when $U_\delta(Z)$ is the δ-neighborhood of Z consisting of points whose spherical distance from Z is less than δ. The spherical metric is obtained by means of the stereographic projection and is denoted by q.

The following definitions assume that a fixed quadratic differential φ on Δ is given, that is a holomorphic map of Δ not vanishing identically. These definitions do not assume the integrability condition (2).

The *canonical coordinate* Φ for φ is a branch of $\int \sqrt{\varphi} dz$ and is defined in D

whenever D is a simply connected subdomain of Δ outside zeroes of φ. It maps arcs of U contained in a vertical/horizontal trajectory arcs onto vertical/horizontal line segments.

The φ-metric d_φ of Δ is given by the element of length $\sqrt{|\varphi|}|dz|$ and the horizontal distance is defined by the element of lengths $|\mathrm{Re}\sqrt{\varphi}dz|$ and is denoted d^h_φ; the element $|\mathrm{Re}\sqrt{\varphi}dz|$ is at first defined in smaller sets where φ has a branch but it is independent of the chosen branch and can be extended also to zeroes of φ. If $\Phi = u + iv$ is the canonical coordinate, then $|\mathrm{Re}\sqrt{\varphi}dz = |du|$. The corresponding lengths of a path γ are denoted $|\gamma|_\varphi$ and $|\gamma|^h_\varphi$. Thus $d_\varphi(z, w)$ is the infimum of $|\gamma|_\varphi$ when γ is a rectifiable path joining z and w in Δ and the definition of d^h_φ is analogous; $|\gamma|$ denotes the euclidean length.

The metric d_φ is Riemannian outside zeroes of φ and hence defines areal measure in Δ denoted m_φ; the euclidean areal measure is m.

A vertical/horizontal arc is an arc contained in a vertical/horizontal trajectory. It may contain zeros of φ; if it does not contain zeroes, we say that the arc is non-critical. When we say that two subsets X and Y of Δ of Δ are *joined by a horizontal arc* γ we assume that one endpoint of γ is in X and the other in Y but otherwise γ is disjoint from X and Y; usually, this could be obtained by passing to a subpath but we include this in the definition.

We will make use of *closed neighborhoods* of a closed set X of $\overline{\mathbb{C}}$ and U is a such a neighborhood if the interior $\mathrm{int}\, U$ of U is a neighborhood of X, that is, an open set containing X. A family $\mathcal{U}$ of closed subsets of $\overline{\mathbb{C}}$ is a *basis of closed neighborhoods* of X if each $U \in \mathcal{U}$ is a closed neighborhood of X and if every neighborhood of X (that is an open set containing X) contains some $U \in \mathcal{U}$. This is a modification of the definition of a *basis of neighborhoods* $\mathcal{U}$ of X which satisfies the same conditions except that each $U \in \mathcal{U}$ is a neighborhood of X that is an open set containing X.

Usually then closure cl and the boundary ∂ are taken in $\overline{\mathbb{C}}$ but if taken in some other set, this is indicated by an appropriate subscript like $\partial_\Delta A$.

2. Main ideas and the definition of the extended map

Our method is the estimation of moduli of rings. A *ring* is an open subset R of $\overline{\mathbb{C}}$ such that $\overline{\mathbb{C}} \setminus R$ has two components. The *modulus* of $M(R)$ of the ring R is defined as the infimum of the area integrals

$$M(R) = \inf_{\varrho} \int_R \varrho^2 dm \tag{2.1}$$

over all non-negative Borel maps $\varrho : R \to \mathbb{R} \cup \{\infty\}$ such that the path integral

$$\int_\gamma \varrho ds \geq 1 \tag{2.2}$$

for all paths γ joining the two components of $\overline{\mathbb{C}} \setminus R$; thus the endpoints of γ are outside R but we set $\varrho = 0$ outside R and so (2.2) is defined for all paths. We usually use the euclidean metric but if $\infty \in R$, we need to use the spherical metric. In situations where both metrics can be used they give the same result by the conformal invariance of the modulus. We will use also other Riemannian or similar metrics and denote

$$M_d(R) = \text{the modulus of } R \text{ with respect to } d$$

and $M(R)$ is the modulus with respect to the euclidean or the spherical metric.

The modulus is a conformal invariant can be expressed in two ways: if f is a conformal homeomorphism of R, then $M(fR) = M(R)$. Another way to express this is that if d is a Riemannian metric conformally equivalent to the euclidean (or spherical if it is used), then the modulus can also be expressed using the distance and area defined by d in (2.1) and (2.2); the conformal equivalence of two Riemannian metrics means that the angles defined by the two metrics are the same and is equivalent to the fact that the element of length of the one metric can be obtained from the element of length of the other metric by multiplying with a positive function. Our situation is more general than that and the metric in our case would be Riemannian outside zeroes of a quadratic differential and outside $\mathbb{R} \cap R$. The exact situation can be found in Section 5.

The conformal invariance of the modulus under conformal change of metric form the basis of the following observation which was the starting point of this paper. If f is a quasiconformal map of R, then it is possible to estimate $M(fR)$ knowing only the the complex dilatation μ_f of f. Assuming enough regularity, this can be explained as follows. Let f be a quasiconformal map of R. Then we can define a metric d_f on R so that the infinitesimal length element corresponding to the euclidean length element $|dz|$ is

$$|Df(z)dz|,$$

$Df(z)$ denoting the derivative of f at z. This gives a metric on R so that f is a local isometry of (R, d_f) onto (fR, d) where d is the euclidean metric. This metric is Riemannian outside the set where Df is non-singular. If this set is not too large, then

$$M_{d_f}(R) = M(fR). \tag{2.3}$$

If we multiply $|D_f(z)|$ by a positive, sufficiently regular, function we obtain a metric conformally equivalent to d_f. For instance, another natural metric is obtained if we multiply by the inverse of the Jacocian and obtain the length element

$$|J_f^{-1}(z)Df(z)dz|. \tag{2.4}$$

and the metric q_f obtained from this is conformally equivalent to d_f and hence (2.3) is valid also for this metric; at points where $J_f(z) = 0$, we set the length element to 0.

The expression (2.4) can be estimated from the shape of the dilatation ellipsoid of f at z. The dilatation ellipsoid is the image of an infinitesimal circle centered at z and the information we need to estimate (2.4) is the ratio of the major and minor axes, well as the directions which are mapped on the major and minor axes of the dilatation ellipsoid. This is exactly the information which we can obtain from the complex dilatation $\mu_f(z)$, cf. the formula (3) for the directional derivative. Thus μ_f determines q_f and $M(fR)$ can be determined from q_f.

This is applied as follows. Let φ be a quadratic differential on Δ and let τ and σ be two distinct simple trajectories. The closure of such a simple trajectory is a closed arc with endpoints on $\partial\Delta$ and otherwise contained in Δ (Lemma 3.1). Thus τ and σ bound a subset of Δ which is a Jordan domain. In addition, we assume that there is a horizontal arc κ joining them. Thus κ is a closed arc contained in a horizontal trajectory. If $z \in \kappa$, let τ_z be the vertical trajectory through z. Since φ has only countably many zeroes, τ_z is simple except for countably many z. Let $\kappa^\circ = \kappa \setminus \{\text{endpoints of } \kappa\}$.

Suppose that γ is a path joining a point of τ to a point of σ. Let $z \in \kappa^\circ$ be a point such that τ_z is simple so that $\bar{\tau}_z$ is a closed arc with endpoints on $\partial\Delta$. Thus γ must intersect τ_z. Let w be the point of intersection and let α be the subarc of τ_z with endpoints z and w. In this situation, there is a closed neighborhood W of α so the canonical coordinate $\Phi = \int \sqrt{\varphi} dz$ maps W isometrically onto a quadrilateral $Q = [a, b] \times [c, d]$ when W is provided with the φ-metric and Q with the euclidean metric. We can assume that $\Phi(W \cap \kappa) = Q \cap \mathbb{R}$ and that $\Phi(\alpha)$ is a subset of the imaginary axis. We can transform the situation to Q by means of Φ and can assume that γ contains a subpath γ_0 so that $\Phi\gamma_0$ joins a point on the left boundary $\{a\} \times [c, d]$ to a point on the right boundary $\{b\} \times [c, d]$. Now the euclidean length of $|\Phi\gamma_0|$ of $\Phi\gamma_0$ is at least $|a - b|$. Thus $|\Phi\gamma_0| = |\gamma_0|_\varphi$ is at least $|\kappa_0|_\varphi = |a - b|$ when $\kappa_0 = \Phi^{-1}[a, b]$. The path γ contains subpaths γ_j so that to each γ_j corresponds such an arc κ_j as above so that κ_j's cover κ minus the set $z \in \kappa$ such that τ_z contains z zero of φ. This latter set is countable and hence

$$|\gamma|_\varphi \geq |\kappa|_\varphi$$

for any such path.

Actually more is true. The horizontal length of φ is defined by the element of length $|\mathrm{Re}\, \sqrt{\varphi} dz|$. We can define the horizontal length of paths using using this element length. Now, Φ maps the horizontal element of length $|\mathrm{Re}\, \sqrt{\varphi} dz|$ to $|du|$ if $\Phi = u + iv$. Thus the above argument actually shows that

$$|\gamma|_\varphi \geq |\gamma|_\varphi^h \geq |\kappa|_\varphi^h = |\kappa|_\varphi > 0. \tag{2.5}$$

The proof of Lemma 3.3 gives the details. In particular, it follows that the horizontal distance $d_\varphi^h(\tau, \sigma) = |\kappa|_\varphi$.

A consequence of (2.5) is that if two simple trajectories τ and σ are joined by a horizontal arc, then the closures $\bar{\tau}$ and $\bar{\sigma}$ are disjoint: if there is $z_0 \in \bar{\tau} \cap \bar{\sigma}$, then, using (2.5), we could show that rings $0 < |z - z_0| < m$, m small, have positive modulus, contradicting the fact that the degenerate rings have zero modulus. Details are given in the proof of Lemma 3.4.

Let $r(z) = 1/\bar{z}$ be the reflection on $\partial\Delta$. Define the doubles of τ and σ by

$$\tilde{\tau} = \bar{\tau} \cup r(\tau) \text{ and } \tilde{\sigma} = \bar{\sigma} \cup r(\bar{\sigma});$$

these are Jordan curves since $\bar{\tau}$ and $\bar{\sigma}$ are arcs and since we now that they are disjoint, there is a ring $R = R(\tau, \sigma)$ so that $\partial R = \tilde{\tau} \cup \tilde{\sigma}$. Let $0 \leq k < 1$ and let f_k be a quasiconformal map of $\overline{\mathbb{C}}$ which is conformal in Δ^* and satisfies (1) with the parameter k in Δ. We will use (2.5) to estimate the moduli of $f_k R$ and will obtain an estimate independent of k. The quadratic differential will sometimes vary and to mark the dependence of f_k on the quadratic differential we may denote $f_{\varphi k}$.

Let γ be path joining τ and σ in Δ. We will start by considering the case $k = 0$ in which case we have a conformal map and hence it is enough to estimate $M(R)$. Now, (2.5) gives that

$$\int_\gamma \sqrt{|\varphi|}|dz| = |\gamma|_\varphi \geq |\kappa|_\varphi > 0. \tag{2.6}$$

So $\sqrt{|\varphi|}/|\kappa|_\varphi$ would seem to be a good candidate for ϱ in Δ. Going to Δ^*, we simply reflect φ to Δ^* by means of the reflection r to obtain a quadratic differential φ^* of Δ^*. details, see Section 5. We denote simply by φ the combined quadratic differential on $\Delta \cup \Delta^*$. The reflection r is an isometry of the φ-metric and it preserves horizontal/vertical trajectories. Thus, if γ joins $r(\tau)$ and $r(\sigma)$ in Δ^*, formula (2.6) is still true.

Of course, we would need to extend φ still to $\partial\Delta$ and consider paths joining the components of $\overline{\mathbb{C}} \setminus R$ which need not be contained Δ or in Δ^*. We can do this by some technical juggling, see Section 5. One gets perhaps a better picture of the situation if one notes that if $z \in \kappa$ and τ_z is a simple trajectory, then $\tilde\tau_z = \bar\tau \cup r(\bar\tau_z)$ is a topological circle and these form a family of disjoint concentric Jordan curves so that the value of the path integral $\int_\gamma \sqrt{|\varphi|}|dz|$ along γ increases at least by $d^h_\varphi f(z,w)$ when γ moves from $\tilde\tau_z$ to $\tilde\tau_w$.

Assuming that $\varrho = \sqrt{|\varphi|}/|\kappa|_\varphi$ will do the job, we would have the estimate

$$M(R) \leq m_\varphi(\Delta \cup \Delta^*)/|\kappa|^2_\varphi = 2||\varphi||_1/|\kappa|^2_\varphi$$

where m_φ is the area with respect to φ.

If $k > 0$, then we define a metric, denoted d_k, on $\Delta \cup \Delta^*$ conformally equivalent to d_{f_k} in (2.3) outside zeroes of φ which can be ignored as an isolated set of $\Delta \cup \Delta^*$. Since f_k is conformal on Δ^*, we can use the φ-metric (or more precisely, the φ^*-metric) on Δ^*. On Δ, we use the fact that f_k decreases infinitesimal distance to the vertical direction by the factor $K = (1+k)/(1-k)$ when compared to infinitesimal distance to horizontal direction. Thus we define a new metric on Δ so that the lengths of horizontal arcs are unchanged but lengths of vertical arcs are divided by K. This condition defines a unique metric on Δ, which is Riemannian outside zeroes of φ and conformally equivalent to d_{f_k}. Leaving again aside the problem of $\partial\Delta$, we have a metric d_k on $\Delta \cup \Delta^*$, conformally equivalent to d_{f_k} outside zeroes of φ.

We have changed the φ-metric on Δ to obtain d_k. Note that horizontal distances were not changed. Thus we can still define horizontal length by the element of length $|\mathrm{Re}\sqrt{\varphi}dz|$ and we have that $d^h_\varphi \leq d_k$. Thus if γ joins τ to σ in $R \cap \Delta$, and denoting the length of γ with respect to d_k by $|\gamma|_k$, we have $|\gamma|_k \geq |\gamma|^h_\varphi \geq |\kappa|_\varphi$. If m_k is the areal measure associated to d_k, the above argument is valid, modulo some details supplied in Section 5, and hence

$$|\kappa|^2_\varphi M(f_k R) \leq m_k(\Delta \cup \Delta^*) = m_\varphi(\Delta^*) + m_k(\Delta) \leq 2||\varphi||_1 < \infty$$

where we have used the facts that the area m_k with respect to d_κ satisfies $m_k \leq m_\varphi$ and that $m_\varphi(\Delta^*) = m_\varphi(\Delta) = ||\varphi||_1$.

We sum up what we have proved (modulo details to be supplied in Section 5). We have above explained the situation in the simplest case and assumed that τ and σ are simple trajectories but it is useful to drop this assumption. If τ and σ are any two distinct trajectories such that $d^h_\varphi(\tau, \sigma) > 0$, then there is a component D of $\Delta \setminus (\tau \cup \sigma)$ such that $\partial_\Delta D \cap \tau = \alpha$ and $\partial_\Delta D \cap \sigma = \beta$ where $\alpha \subset \tau$ and $\beta \subset \sigma$ are arcs such that $\bar{\alpha}$ and $\bar{\beta}$ are disjoint cross cuts of $\partial\Delta$ (Lemmas 3.1 and 3.2). Thus there is a ring $R = R(\tau, \sigma)$ of $\overline{\mathbb{C}}$ such that

$$\partial R(\tau, \sigma) = \bar{\alpha} \cup \bar{\beta} \cap r(\alpha) \cup r(\beta) \tag{2.7}$$

and we will call this ring the *ring defined by τ and σ*. We formulate the next lemma in this situation; if κ joins τ and σ, then $d^h_\varphi(\tau, \sigma) = |\kappa|_\varphi > 0$ (Lemma 3.3).

Lemma 2.1. *If τ and σ are trajectories whose horizontal distance $d^h_\varphi(\tau, \sigma) > 0$, then $\bar{\tau}$ and $\bar{\sigma}$ are disjoint and if $f = f_{\varphi_k}$ satisfies (1) with fixed φ but with varying k, then the ring $R(\tau, \sigma)$ defined by τ and σ satisfies*

$$M(f_{\varphi k}(R(\tau, \sigma)) \leq 2||\varphi||_1 / d^h_\varphi(\tau, \sigma)^2.$$

In particular, this estimate is true for $M(R(\tau, \sigma))$.

This is our starting point. Using it, we can extend it to the situation that the quadratic differential varies, though not far from a fixed φ. We have the following lemma. Note that we still have the fixed quadratic differential φ and trajectories τ and σ are φ-trajectories.

Lemma 2.2. *Let φ, τ, and σ be as in Lemma 2.1. Let $m > 0$. Then there are $\varepsilon > 0$ and a compact set $K \subset \Delta$ as well as $c > 0$ with the following property. Let ψ be a a quadratic differential such that $||\psi||_< m$ and $|\varphi - \psi| < \varepsilon$ in K. If $f_{\psi k}$ satisfies (1) with $\varphi = \psi$, then $M(f_{\psi k}(R(\tau, \sigma)) \leq c$ independently of k.*

The proof will be given in Section 5 and is based on the fact that if ψ satisfies the conditions of the lemma, then $R(\tau, \sigma)$ contains a ring $R(\tau', \sigma')$ where τ' ad σ' are ψ-trajectories whose horizontal ψ-distance is bounded from below by a positive constant (Lemma 5.4).

We do not need so much this results than the following consequence of it. We assume that we have the situation of Theorem 1, that is $f_j = f_{k_j}$ are quasiconformal maps of $\overline{\mathbb{C}}$ conformal on Δ^* and on Δ they satisfy (1) with $k = k_j$ and $\varphi = \varphi_j$ such that $k_j \to 1$ and $\varphi_j \to \varphi$ uniformly on compact sets and the L^1-norms of φ and φ_j are bounded by a constant $m > 0$,

Corollary 2.3. *Let τ and σ be as in Lemma 2.2 and assume that f_j tend toward a conformal embedding on Δ^*. Then there is $c > 0$ independent of i such that whenever x and y are in different components of the complement of the ring $R(\tau, \sigma)$, their spherical distance satisfies*

$$q(f_j(x), f_j(y)) \geq c > 0 \tag{2.8}$$

for all $j > 0$.

Proof. Let $R = R(\tau, \sigma)$. Both components of $\overline{\mathbb{C}} \setminus R$ contain an open non-empty subset of Δ^* and since f_j tend toward a conformal embedding on Δ^*, it follows that the f_j-images of these components have spherical diameter bounded from below by a positive constant independently of i. Since the moduli of the rings are bounded from above by Lemma 2.2, known properties of the moduli imply (2.8), cf. [V, 12.7]. We have given a simple normal family proof in the Appendix.

Remarks. 1. We have used $||\varphi||_1$ in the estimate for $M(R)$ since this is all we need but it could be replaced by $m_\varphi(R \cap \Delta) = \int_{R\cap\Delta} |\varphi| dm$ where $R = R(t, \sigma)$ If τ and σ are joined by the horizontal trajectory κ, then $||\varphi||_1$ could be replaced by $m_\varphi(V)$ when V is the union of trajectories intersecting κ.

2. If τ and σ are joined by a horizontal arc κ, then an unessential modification of the proof in section 5 (involving Lemma 5.4) shows that, given $\varepsilon' > 0$, we can choose ε and K in Lemma 2.2 so that $M(f_{\psi k}(R(\tau, \sigma)) \leq 2m/d^h_\varphi(\tau, \sigma)^2 + \varepsilon'$. Here m is the upper bound for the L^1-norms of φ and ψ_j and could be replaced by an upper bound for $m_{\psi_j}(R \cap D)$ The assumption that τ and σ are joined by a horizontal arc is probably not necessary for this kind of estimate but this seems to require lengthy arguments.

Construction of the extended limit. The first idea is that we find a subsequence so that $f_j(\tau)$ have the Hausdorff limit for any complete trajectory τ and this Hausdorff limit will be $f(\tau)$. This does not work for two reasons. The first is that we cannot assume that the Hausdorff limit exists for uncountably many sets $\tau \in \mathcal{T} \cup \mathcal{V}$, $\mathcal{T}$ and $\mathcal{V}$ as in the Introduction. We must also take into account what happens outside Δ and consider instead τ the double $\tilde{\tau} = \tau \cup r(\bar{\tau})$ of $\bar{\tau}$. We construct the extended limit as follows.

Recall that if τ is a trajectory, the completion τ^* is a *simple complete trajectory* if τ^* is a closed arc which is the closure of τ. The countability of the zeroes of φ implies that τ is simple, i.e. $\varphi \neq 0$ on τ, except for countably many τ but to see that, apart from a countable set, $\bar{\tau}$ is already the completion of τ requires slightly more complicated argument, see Theorem 3.8. Next, we choose a countable dense set $\mathcal{S}$ of trajectories so that $\bar{\tau}$ is a simple complete trajectory. The density of $\mathcal{S}$ means that if κ is a horizontal arc, then $\{\kappa \cap \tau : \tau \in \mathcal{S}\}$ is a dense subset of κ. Using the fact that $\bar{\tau} = \tau^*$ except for countably many τ, we see that there is such a set $\mathcal{S}$; remember that $\Delta \setminus \{\text{zeroes of } \varphi\}$ can be covered by countably many closed sets which the canonical coordinate maps to quadrilaterals with sides parallel to the coordinate axes.

We need a few properties for $\tau \in \mathcal{S}$. First, we need that $\bar{\tau}$ is closed arc with endpoint on $\partial\Delta$ but otherwise contained in Δ, cf. Lemma 3.1. We need also the fact that the horizontal distance $d^h_\varphi(\tau, \sigma) > 0$ for distinct $\tau, \sigma \in \mathcal{S}$ and hence $\{\bar{\tau} : \tau \in \mathcal{S}\}$ is a family of disjoint closed arcs (Theorem 3.6).

The double $\tilde{\sigma} = \bar{\sigma} \cup r(\bar{\sigma})$, $\sigma \in \mathcal{S}$, is a Jordan curve and $\{\tilde{\sigma} : \sigma \in \mathcal{S}\}$ is a family of disjoint Jordan curves. Let $\tilde{\mathcal{U}}$ consist of the closures of the sets C such that C is a component of $\overline{\mathbb{C}} \setminus (\cup\mathcal{F})$ where $\mathcal{F} \subset \{\bar{\tau} : \tau \in \mathcal{S}\}$ is finite. Thus $\tilde{\mathcal{U}}$ is a countable family of compact sets. We will see (this is a consequence of Lemmas 3.5 and 3.9) that if ν is a complete trajectory, then

$$\tilde{\mathcal{U}}_\nu = \{U \in \tilde{\mathcal{U}} : \nu \subset \operatorname{int} U\} \tag{2.9}$$

is a basis of closed neighborhoods of the double $\tilde{\nu}$ of ν; that is $\operatorname{int} U \supset \nu$ and every open set containing ν contains some $U \in \tilde{\mathcal{U}}_\nu$. Here $\nu \in \mathcal{T}$ but this formula defines also a basis of closed neighborhood for ν if $\nu \in \mathcal{V}$ (in which case $\tilde{\nu} = \nu$).

Let f_j be as in Theorem 1 and suppose that and assume that f_j tend on Δ^* toward a conformal embedding of Δ^*. Since the family of closed subsets of $\bar{\Delta}$ is compact in the Hausdorff metric, given $U \in \mathcal{U}$, it is possible to pass to a subsequence so that $f_j(U)$ have a limit in the Hausdorff metric. Since $\mathcal{U}$ is countable we can assume that $f_j(U)$ have a Hausdorff limit, denoted U^∞, for every $U \in \mathcal{U}$. This is the subsequence for which the extended limit exists.

Let now ν be a complete trajectory. We define the extended limit of $\tilde{\nu} = \nu \cup r(\nu)$ as the intersection

$$f(\tilde{\nu}) = \bigcap \{U^\infty : U \in \tilde{\mathcal{U}}_\nu\} \tag{2.10}$$

and the extended limit of ν is

$$f(\nu) = f(\tilde{\nu}) \setminus f(\Delta^*). \tag{2.11}$$

Note that we have the natural relation $f(\tilde{\nu}) = f(\nu) \cup \{f(z) : z \in \tilde{\nu} \setminus \Delta^*\}$. If $\nu \in \mathcal{V}$, then the extended limit $f(\nu)$ is simply the intersection (2.10).

We have now defined the extended limit. The proof that the extended limit has the properties claimed is much based on the fact that $\{U^\infty : U \in \tilde{\mathcal{U}}_\nu\}$ is a basis of closed neighborhoods of $f(\tilde{\nu})$ and of a similar basis of closed neighborhoods for $f(\nu)$ (Lemmas 4.3 and 4.4). Details can be found in Section 4.

We have the following connection to the Hausdorff limit. If X_j is a sequence of closed subsets of $\overline{\mathbb{C}}$, denote by $\limsup X_j$ the set X such that $x \in X$ if and only if every neighborhood of x intersects infinitely many X_j; this is the Hausdorff limit of X_j if it exists. The next theorem assumes that we have passed to the subsequence constructed above.

Theorem 2.4. *If* $\nu \in \mathcal{T} \cup \mathcal{V}$, *then* $\limsup f_j(\nu) \subset f(\nu)$.

Proof. Suppose that $x \in \limsup f_j(\nu)$ where $\nu \in \Delta^* \cup \mathcal{D}$. The case that $\nu \in \Delta^*$ is obvious and the case that $\nu \in \mathcal{V}$ similar to the case that $\nu \in \mathcal{T}$ which we now treat. If $\nu \in \mathcal{T}$, then $x \in \limsup f_j U = U^\infty$ for every $U \in \tilde{U}_\nu$, implying that $x \in \bigcap_{U \in \tilde{\mathcal{U}}_\nu} U^\infty = f(\tilde{\nu})$. If $x \in f\Delta^*$, then $x \in f_j \Delta^*$ for large n and hence x has neighborhood W such that $W \cap f_j(\nu) = \emptyset$ for large n and hence $x \notin \limsup f_j(\nu)$, a contradiction. Thus $x \in f(\tilde{\nu}) \setminus f(\Delta^*) = f(\nu)$.

Remark. Once we have found the subsequence so that $f_j U$ have the Hausdorff limit for all $U \in \mathcal{U}$, it is easy to check (use Lemmas 4.1 and 4.4) that if we use replace $\mathcal{S}$ by another countable dense set of simple vertical trajectories, we arrive to the same extended limit.

3. Complete trajectories and partitioning $\bar{\Delta}$ by means of trajectories

We now start the detailed treatment. This section is independent of section 2.

We first study what happens when a trajectory approaches boundary. The basic situation is straightforward: A trajectory τ is simple if it does not contain zeroes of φ. If τ is such a trajectory, then $\bar{\tau}$ is homeomorphic to a closed interval so that $\bar{\tau} \setminus \tau$ consists of two points of $\partial\Delta$. Here we need only to refer to Strebel's book (which uses somewhat different terminology; Strebel does not allow branching at zeroes of φ and so our trajectories are trajectories in Strebel's sense if they do not contain zeroes of φ, that is, they are simple).

We only use here vertical or horizontal trajectories but it is useful to recall the notion of a geodesic. A *(closed) geodesic arc* is a closed arc which is locally isometric to a closed interval of $\mathbb{R}$, with respect to the φ-metric and the euclidean metric, respectively. A *geodesic ray* of φ is the image of an injective map $j : [0, a[\to \mathbb{R}$, $0 < a \leq \infty$, which is a local isometry (with respect to these metrics) and so that j cannot be extended from $[0, a[$ to any larger interval which would still be a local isometry. Similarly, a *geodesic* is a local isometry in these metrics $]b, a[\to \nu$ where $-\infty \leq b < a \leq \infty$ and which cannot be extended to any larger interval as a local isometry. We remark that in our situation these are actually global isometries [S, 14.2]. Thus, in particular, we cannot have closed geodesics.

If γ is a geodesic arc contained in an open set U where there is a branch of the canonical coordinate $\Phi = \int \sqrt{\varphi} dz$, then $\Phi\gamma$ is a line segment. All simple trajectories are geodesics and an arc contained in a trajectory is a geodesic arc. If τ is a vertical or horizontal trajectory and $z \in \tau$, it is obvious (see Fig. 1) that there is an arc $\gamma \subset \tau$ containing z which is a geodesic arc and, using local considerations appealing to Fig. 1, and remembering that τ is simply connected [T1, Lemma 2.3], we see that γ can be extended to a geodesic contained in τ; this fact will be needed later.

The next lemma is contained in Theorems 19.4 and 19.6 of Strebel [S].

Lemma 3.1. *If τ is a simple trajectory (i.e. does not contain zeroes of φ), then $\bar{\tau}$ is a cross cut of $\partial\Delta$, i.e. $\bar{\tau}$ is homeomorphic to a closed interval so that $\bar{\tau} \setminus \tau$ consists of two points of $\partial\Delta$, and this is also true if τ is a geodesic.*

This implies the following

Lemma 3.2. *If τ is a trajectory, then $\bar{\tau} \setminus \tau$ is a subset of $\partial\Delta$ and contains at least two points and if τ is simple, it contains exactly two points. If D is a component of $\bar{\Delta} \setminus \bar{\tau}$, then D is a Jordan domain whose boundary is the union of an arc of $\partial\Delta$ and of a geodesic contained in τ.*

If τ and σ are two distinct trajectories, then there is a unique component D of $\Delta \setminus (\tau \cup \sigma)$ such that D is a Jordan domain and $\partial_\Delta D = \alpha \cup \beta$ where α and β are geodesics so that $\alpha \subset \tau$ and $\beta \subset \sigma$.

Proof. As we have seen, τ contains a geodesic and hence by Lemma 3.1, $\bar{\tau} \cap \partial\Delta$ contains at least two points. If τ is simple, then τ is a geodesic and hence $\bar{\tau} \cap \partial\Delta$ contains two points.

To prove the remainder, we note that every $z \in \Delta \cap \tau$ has a neighborhood U such that $U \cap \tau$ consists of one vertical arc, or if z is a zero of φ, of the point z and a finite number of non-critical vertical arcs with endpoint z, that is we have the situation of Fig. 1. This follows since otherwise there would be a horizontal trajectory intersecting τ more than once and this is not possible [T1, Lemma 2.3]. It follows that $\partial_\Delta \tau = \tau$. If D is a component of $\Delta \setminus \tau$ and $z \in \partial_\Delta D$, then it follows from above by local considerations (see Fig. 1) that $\partial_\Delta D$ contains a geodesic $\alpha \subset \tau$ such that $z \in \alpha$. We know by Lemma 3.1 that $\bar{\alpha}$ is a closed arc such that $\bar{\alpha} \cap \partial\Delta$ consists of two points. Thus D is contained in one of the components of $\Delta \setminus \alpha$, call it D'. If there is $w \in \partial_\Delta D \setminus \alpha$, then $w \in \beta$ where $\beta \subset \tau$ is another geodesic. If $\alpha \cap \beta = \emptyset$, then τ could not be connected. Hence $\alpha \cap \beta \neq \emptyset$. Now, τ is simply connected [T1, Lemma 2.3 (b)] and hence $\alpha \cap \beta$ is either a point or an arc and hence there is a ray contained in β and which divides D' into two pieces and D is contained in one of them. It would follow that α cannot be contained in $\partial_\Delta D$. It follows that $\alpha = \partial_\Delta D$ and D is one of the components of $\Delta \setminus \alpha$.

The second paragraph is an obvious consequence of the first.

Next, we give in exact form the fact already outlined in Section 2 that if two trajectories are joined by a horizontal arc, then the φ-length of this arc gives the horizontal distance of the trajectories:

Lemma 3.3. *Let τ and σ be trajectories joined by a horizontal arc κ. If γ is a path of Δ joining τ and σ, then $|\gamma|_\varphi \geq |\gamma|_\varphi^h \geq |\kappa|_\varphi = |\kappa|_\varphi^h$ and thus $d_\varphi(\tau, \sigma) = d_\varphi^h(\tau, \varphi) = |\kappa|_\varphi$.*

Proof. Let κ° be κ minus the endpoints. If $z \in \kappa^\circ$, let τ_z the maximal non-critical vertical trajectory contained containing z. Thus we go from z to both vertical directions as far as we can without meeting a zero of φ; if z is a zero of φ, then set $\tau_z = \emptyset$. Set

$$V = \bigcup_{z \in \kappa^\circ} \tau_z$$

where κ° is the set of interior points of κ. Obviously, it is a subset of $\Delta \setminus (\tau \cup \sigma)$.

We claim that that V is open. Suppose that $w \in V$. Thus $w \in \tau_z$ for some $z \in \kappa^\circ$. Let α be the closed subarc of τ_z with endpoints z and w. There is a simply connected neighborhood $U \subset \Delta$ of α not containing zeroes of φ. Thus there is a branch of the canonical coordinate $\Phi = \int \sqrt{\varphi} dz$ on U and we can choose it so that $\Phi\alpha$ is contained in the imaginary axis and that ΦU contains a quadrilateral with sides parallel to the coordinate axes whose interior contains $\Phi\alpha$. If β is a vertical line segment of Q joining opposite sides of Q, then $\Phi^{-1}\beta$ is contained in some τ_w, $w \in \kappa$. Thus $\Phi^{-1}Q \subset V$ and since the interior of $\Phi^{-1}Q$ contains α, it follows that V is open.

We can use Q and Φ also to obtain the following. Let π be the projection $V \to \kappa$ so that $\pi(\zeta) = z$ if $\zeta \in \tau_z$. We can move from $\Phi^{-1}Q$ by means of the canonical coordinate Φ to Q and then π is transformed to the projection $\pi_0 : x + iy \mapsto x$ of Q and thus $\Phi\pi = \pi_0\Phi = u$ if $\Phi = u + iv$. The horizontal length element $|\mathrm{Re}\sqrt{\varphi}dz|$ is sent to $|du|$ by $\Phi = u + iv$ and hence $|\mathrm{Re}\sqrt{\varphi}dz| = |d\pi_0\Phi(z)| = |du|$. On the other hand, $|du| = |d\pi_0\Phi(z)| = |d\Phi\pi(z)| = |\sqrt{\varphi(\pi(z))}d\pi(z)|$. Hence if β is a path in $\Phi^{-1}Q$,

$$|\beta|_\varphi^h = \int_\beta |\mathrm{Re}\sqrt{\varphi}dz| = \int_{\pi_0\Phi\beta} |du| = \int_{\Phi\pi\beta} |du| = \int_{\pi\beta} |\sqrt{\varphi}d\zeta| = \int_{\pi\beta} \sqrt{|\varphi|}|d\zeta| = |\pi\beta|_\varphi.$$

Every $z \in V$ is contained in such a set $\Phi^{-1}Q$ and hence $|\beta|_\varphi^h = |\pi\beta|_\varphi$ whenever β is a path of V. If β is such a path, let κ_β be the set of points covered by $\pi\beta$ which is a subarc of κ. We have

$$|\beta|_\varphi^h = |\pi\beta|_\varphi^h \geq |\kappa_\beta|_\varphi.$$

Let J be the parameter interval of γ and thus $\gamma^{-1}V$ is a union of disjoint subintervals of J. We denote these intervals by J_j and let $\gamma_j = \gamma|J_j$.

Let $z \in \kappa_0$ and assume that the vertical trajectory through z does not contain zeroes of φ. Then we know by Lemma 3.1 that $\bar{\tau}_z$ is a closed arc with endpoints on $\partial\Delta$ but otherwise contained in Δ. Let U be the component of $\Delta \setminus (\tau \cup \sigma)$ whose boundary contains a geodesic $\alpha \subset \tau$ and $\beta \subset \sigma$ as in Lemma 3.2. Thus $\tau_z \subset U$ and since $\bar{\tau}_z$ is a crosscut of $\partial\Delta$ it divides Δ into two pieces which we call V_1 and V_2. Since τ_z is a vertical arc and κ_0 horizontal arc intersecting at z, κ contains points both in V_1 and V_2. Since $\tau_z \cap \kappa$ can contain at most one point [T1, Lemma 2.3], it follows that τ_z divides κ into two pieces, one of which is contained in V_1 and the other in V_2. Thus V_1 contains one of the trajectories τ and σ and V_2 the other. Thus τ_z separates τ and σ and hence γ must intersect τ_z. It follows that z is in some κ_j. There are only a countable number of z such that the vertical trajectory through z contains zeroes of φ and hence the set of such points z is a nullset for the linear φ-measure of κ. Thus

$$|\gamma|_\varphi^h \geq \sum_j |\gamma_j|_\varphi^h \geq \sum_j |\kappa_j|_\varphi \geq |\kappa|_\varphi$$

and the lemma is proved.

If τ and σ are distinct trajectories, then $\tau \cap \sigma = \emptyset$ but their closures can intersect. However, if their horizontal distance is positive, the closures are disjoint.

Lemma 3.4. *If τ and σ are two trajectories such that $d_\varphi^h(\tau, \sigma) > 0$, then $\bar{\tau}$ and $\bar{\sigma}$ are disjoint. In particular, this is true if τ and σ are joined by a horizontal arc.*

Proof. Let D be the domain of Δ bounded by τ and σ. Thus (Lemma 3.2) $\partial_\Delta D = \alpha \cup \beta$ where $\alpha \subset \tau$ and $\beta \subset \sigma$ are geodesics and thus $\bar{\alpha}$ and $\bar{\beta}$ are closed arcs with endpoints on $\partial\Delta$. It follows that if $\bar{\tau} \cap \bar{\sigma} \neq \emptyset$, then there is $z_0 \in \bar{\alpha} \cap \bar{\beta} \cap \partial\Delta$

We can now proceed as follows. There is $R > 0$ such that the circle $|z - z_0| = r$ intersects both α and β if $0 < r < R$. We derive a contradiction as follows.

Let A be the ring $0 < |z - z_0| < R$. This ring is degenerate and hence the modulus $M(A) = 0$. We can show, using φ, a positive lower bound for $M(A)$ which is a contradiction.

We use the following alternate definition for the modulus of a ring. We have

$$\frac{1}{M(A)} = \inf \int_A \varrho^2 dm,$$

where dm is the area element and where the infimum is taken over all non-negative Borel functions ϱ on A such that

$$\int_\gamma \varrho |dz| \geq 1$$

for all closed paths of A whose winding number with respect to z_0 is 1. Each such closed path γ contains a subpath γ_0 joining a point of τ to a point of σ but otherwise contained in the domain D of Δ bounded by τ and σ and thus, by the preceding lemma,

$$\int_{\gamma_0} \sqrt{|\varphi|}|dz| \geq c$$

where $c = d_\varphi(\tau, \sigma) > 0$. If we extend φ by 0 to all of A, this is still true and we can replace γ_0 by γ. Thus

$$\frac{1}{M(A)} \leq \frac{1}{c^2}\int_A |\varphi| dm \leq \frac{1}{c^2}\int_\Delta |\varphi| dm < \infty$$

and the lemma is proved.

If τ and σ are two vertical trajectories are joined by a horizontal arc κ, then we know that $|\kappa|_\varphi$ gives the horizontal distance $d^h_\varphi(\tau, \sigma)$. However, not all distinct trajectories can be joined by a horizontal arc. The next lemma addresses this situation if $d^h_\varphi(\tau, \sigma) > 0$.

The horizontal distance of τ and σ is the infimum of $|\gamma|^h_\varphi$ where γ is a rectifiable path joining τ and σ. It is not difficult to see by passing to suitable coordinate neighborhoods that we can take in the above infimum defining the horizontal distance of τ and σ only such paths γ which are composed of horizontal and vertical arcs alternating; thus $|\gamma|^h_\varphi$ is simply the composition of the φ-lengths of the horizontal arcs in the composition. We call such paths *horizontal/vertical paths*. If we have such a composition to horizontal and vertical arcs, it is simple to see that we can remove parts from γ so that γ is an arc with the original endpoints; we call such arcs *horizontal/vertical arcs.* If τ and σ are distinct trajectories, Lemma 3.2 implies that τ and σ bound a domain V of Δ. A simple argument adds the property that $d^h_\varphi(\tau, \sigma)$, $\tau \neq \sigma$, is the infimum of $|\gamma|^h_\gamma$ of horizontal/vertical paths with one endpoint on τ and the other on σ and contained except for endpoints in V.

Lemma 3.5. *Let τ and σ be two trajectories such that $d^h_\varphi(\tau, \sigma) > 0$. Then there is a simple trajectory ν such that $\bar{\nu}$ separates $\bar{\tau}$ and $\bar{\sigma}$ in $\bar{\Delta}$ and that $d^h_\varphi(\tau, \nu) > 0$ and $d^h_\varphi(\sigma, \nu) > 0$. Furthermore, there is a horizontal arc κ so that this is true whenever ν is a simple trajectory intersecting κ and ν is distinct from τ or σ.*

If τ and σ are joined by a horizontal arc κ', we can take $\kappa = \kappa'$.

Proof. Let V the subdomain of Δ bounded by τ and σ so that $\partial_\Delta V \subset \tau \cup \sigma$. As we have seen, there is a horizontal/vertical arc γ joining τ and σ in V. Thus γ is the product $\gamma_1 ... \gamma_n$ of arcs so that so that horizontal and vertical arcs alternate in the sequence; the first and last are horizontal.

At this point we need to introduce the following terminology. The subdomain V of Δ bounded by τ and σ bound is a Jordan domain since both $\tau \cap \partial V$ and $\sigma \cap \partial V$ are geodesics (Lemma 3.2) and hence their closures are closed arcs with endpoints on $\partial\Delta$ but otherwise contained in Δ. Thus the arc γ divides V into two parts V_1 and V_2 which are also Jordan domains. The intersection $J_j = \bar{V}_j \cap \partial\Delta$ is either a closed arc or a point; by the preceding lemma, it must actually be an arc.

We will prove the lemma by induction on n. If $n = 1$, then $\gamma = \gamma_1$ and our theorem is true in view of Lemmas 3.3 and 3.4; we can take $\kappa = \gamma_1$. If $z \in \kappa$ is not an endpoint of κ (i.e. $\tau \neq \tau_z \neq \sigma$) and τ_z is a simple vertical trajectory through z, then τ_z contains points both in V_1 and V_2 and hence one endpoint of $\bar{\tau}_z$ is in $\bar{V}_1 \cap \partial\Delta$ and another in $\bar{V}_2 \cap \partial\Delta$. It follows that $\bar{\tau}_z$ separates $\bar{\tau}$ and $\bar{\sigma}$ in $\bar{\Delta}$.

In particular, we see that if the horizontal arc κ' joins τ and σ, we can take $\kappa = \kappa'$.

Suppose then that the lemma is true for $n \leq k-1$. We will show that it is true also if $n = k$.

We order γ so that the point of γ on τ is the first point of γ and denote this order by $<$. We say that $x \in \gamma$ is smaller (or larger) than $y \in \gamma$ if $x < y$ (or $y < x$) with respect to this order.

So suppose that there are exactly k horizontal arcs in the composition $\gamma_1 ... \gamma_k$. The first arc γ_1 is horizontal and we let a and b be the endpoints of γ_1 so that $a < b$. Let τ_z be the vertical trajectory through z and define, using the order we have defined,

$$c = \inf\{z \in \gamma_1 : \tau_z \cap \gamma_3 ... \gamma_k \neq \emptyset\};$$

note that $\tau_b \supset \gamma_2$ and hence $\tau_b \cap \gamma_3 \neq \emptyset$ and so the above infimum is well defined.

If $c > a$, let κ_0 be the subarc of γ_1 with endpoints a and c. The points $z \in \kappa_0 \setminus \{a, c\}$ such that τ_z is simple are dense in κ_0. Let z be such a point. Now $d_\varphi(z, a) = d_\varphi^h(\tau_z, \tau)$ by Lemma 3.3 and hence, if z is close to a, then both $d_\varphi^h(\tau_z, \tau) > 0$ and $d_\varphi^h(\tau_z, \sigma) > 0$. It follows by Lemma 3.4 that $\bar{\tau}_z$ is disjoint from $\bar{\tau} \cup \bar{\sigma}$ and it can be seen as in the case $n = 1$ that $\bar{\tau}_z$ separates τ and σ. We can take for the arc κ any horizontal subarc of κ_0 a but whose points are so close to a that both $d_\varphi^h(z, \tau) > 0$ and $d_\varphi^h(z, \sigma) > 0$ if $z \in \kappa$.

Thus we are done if $c \neq a$. If $c = a$, we use the inductive assumption as follows. We choose a sequence of points z_j tending to $c = a$ so that τ_{z_j} intersects $\gamma_3 ... \kappa_k$ at w_j; by [T1, Lemma 2.3] this intersection can contain at most one point. We pass to a subsequence so that that $w_j \to d \in \gamma_3 ... \gamma_k$. Let τ' be the the vertical trajectory through d. Then $d_\varphi^h(\tau, \tau') = 0$ and hence $d_\varphi^h(\tau', \sigma) > 0$. Thus γ contains a subarc γ' joining τ' and σ but distinct form $\tau' \cup \sigma$ except for endpoints. We easily obtain from the horizontal/vertical decomposition of γ a horizontal/vertical decomposition of γ' with fewer arcs. Thus the inductive assumption is true for the horizontal/vertical arc γ' joining τ' and σ and hence γ' contains a horizontal arc κ so that if ν is a simple vertical trajectory intersecting κ, and $\tau' \neq \nu \neq \sigma$, then $\bar{\nu}$ separates $\bar{\tau}'$ and $\bar{\sigma}$ and both $d_\varphi^h(\nu, \tau')$ and $d_\varphi^h(\nu, \sigma)$ are positive. Since $d_\varphi^h(\tau, \nu) = d_\varphi^h(\tau', \nu) > 0$, $\bar{\nu}$ and $\bar{\tau}$ are disjoint. Furthermore, $\bar{\tau}$ and $\bar{\tau}'$ must be in the same component of $\bar{\Delta} \setminus \bar{\nu}$ since their horizontal distance is zero but they have positive horizontal distance to ν. The lemma is proved.

Complete trajectories. Even if we have extended the notion of a trajectory so that it may include critical points, we still need to extend this notion further so that a trajectory may be extended across the boundary. The reason is that we need that the closures of two trajectories are disjoint but this need not be always the case. For instance, suppose that φ is defined and holomorphic in a neighborhood of $\bar{\Delta}$. Then φ may have a zero on $\partial\Delta$ so that they are two trajectories of Δ meeting at the zero which are disjoint in Δ. Then the closures of these trajectories intersect and it is natural, and also necessary, to join these trajectories. Also, it is possible that φ has two zeroes on $\partial\Delta$ so that an arc of $\partial\Delta$ joining these zeroes is is a trajectory arc. In this case, we would join the trajectories going to these zeroes by adding this trajectory arc on $\partial\Delta$.

These examples illustrate what may happen although our definition includes much more complex situations. The idea is that we can join trajectories by going through boundary and thus we include boundary points in the trajectories so that they will be closed subsets of $\bar{\Delta}$. We call these trajectories complete trajectories.

Another way to think about complete trajectories is to use the horizontal distance between two trajectories. The horizontal distance $d^h_\varphi(\tau, \sigma)$ is a pseudometric in the set of all vertical trajectories. Thus there may be many trajectories at zero distance from a given trajectory τ. We want to extend a trajectory τ to include all these trajectories at zero distance as well as some points on the boundary joining them. This trajectory the *complete trajectory* defined by τ, denoted τ^* and called the completion of τ. The following definition is convenient. If σ is a simple trajectory such that $\bar{\sigma}$ is disjoint from $\bar{\tau}$, let

$$D_\sigma = D_\sigma(\tau) \tag{3.0}$$

be the closure of the component of $\bar{\Delta} \setminus \bar{\sigma}$ containing $\bar{\tau}$. We can now define the *completion* of τ as

$$\tau^* = \bigcap_\sigma D_\sigma(\tau) \tag{3.1}$$

where the intersection is taken over all simple trajectories σ such that $d_h(\tau, \sigma) > 0$; by Lemma 3.4 this condition implies that $\bar{\tau}$ and $\bar{\sigma}$ are disjoint. A *complete trajectory* ν is a trajectory which is the completion τ^* of some trajectory τ. Note that although a trajectory is a subset of Δ, a complete trajectory is a subset of $\bar{\Delta}$.

The following properties of complete trajectories will be needed.

Theorem 3.6. *Let ν be a complete trajectory. Then ν is a closed and connected subset of $\bar{\Delta}$ not separating $\overline{\mathbb{C}}$ (i.e. $\overline{\mathbb{C}} \setminus \nu$ is connected). If ν' is another complete trajectory distinct from ν, then ν and ν' are disjoint. If $\nu = \tau^*$ is the completion of a trajectory τ, then*

$$\nu \cap \Delta = \cup\{\sigma : \sigma \text{ is a vertical trajectory such that } d_h(\tau, \sigma) = 0\}. \tag{3.2}$$

Let $z \in \nu \cap \Delta$. Then z has a neighborhood V such that $\varphi \neq 0$ in V except possibly at z and such that $V \cap \nu$ consists of one trajectory arc if $\varphi(z) \neq 0$. If $\varphi(z) = 0$, then $\nu \cap \Delta$ consists of z and of a finite number of trajectory arcs of $V \setminus \{z\}$ with endpoint z.

Thus locally $\nu \cap \Delta$ is like in Fig. 1; the number of trajectory arcs is $n + 2$ if n is the order of φ at z ($n = 0$ if $\varphi(z) \neq 0$).

Proof. Obviously, ν is closed and to show that it is connected it is enough to show that every finite intersection of the sets $D_\sigma(\tau)$ in (3.1) is connected. This can be seen easily since the boundary of $D_\sigma(\tau)$ in $\bar{\Delta}$ is $\bar{\sigma}$ which crosscut $\partial\Delta$ and these are disjoint.

Thus each finite intersection of the sets $D_\sigma(\tau)$ in (3.1) is homeomorphic to a closed disk and hence is connected. Similarly, the complement in $\overline{\mathbb{C}}$ of each finite intersection of the sets in (3.1) is connected and it follows that also the complement of the intersection (3.1) is connected.

We then prove (3.2). Let $\nu = \tau^*$. If σ is a trajectory such that $d^h_\nu(\tau, \tau') > 0$, then Lemma 3.5 implies that there is a simple trajectory σ so that $d^h_\varphi(\tau, \sigma) > 0$ and $\bar{\sigma}$ separates $\bar{\tau}$ and $\bar{\tau}'$ in $\bar{\Delta}$. Thus τ'^* is disjoint from τ^*. On the other hand, if τ' is a trajectory such that $d^h_\varphi(\tau, \tau') = 0$, there cannot be a simple trajectory σ such that $\bar{\sigma}$ separates $\bar{\tau}$ and $\bar{\tau}'$ and $d^h_\varphi(\tau, \sigma) > 0$ as this would imply that $d^h_\varphi(\tau, \tau') \geq d_\varphi(\tau, \sigma) > 0$.

Let then $z \in \Delta \cap \nu$, $\nu = \tau^*$. Let V be a canonical neighborhood of z as shown in Fig. 1. If z is a zero of φ, z is the point in Fig. 1 where the vertical trajectory through z has $n+2$ branches at z when n is the order of the zero. Let τ_z be the trajectory of V through z. Thus $\tau_z \subset \nu$. Each point $w \in V \setminus \tau_z$ can be joined to by a horizontal arc to τ_z and hence $d^h_\varphi(\tau_z, w) = d_\varphi(\tau, w) > 0$ (Lemma 3.3) and consequently $\nu \cap V = \tau_z$. This implies the last paragraph.

Finally, let $\nu = \tau^*$ and $\nu' = \tau'^*$. If $\nu \neq \nu'$, then we have seen that $d^h_\varphi(\tau, \tau') > 0$ and thus, by Lemma 3.5, there is a simple trajectory σ such that $\bar{\sigma}$ separates $\bar{\tau}$ and $\bar{\tau}'$ and hence $\bar{\sigma}$ separates ν and ν'. It follows that $\bar{\nu}$ and $\bar{\nu}'$ are disjoint.

Lemma 3.7. *Let ν be a complete trajectory. Then components of $\Delta \setminus \nu$ are Jordan domains and if D is such a domain then $(\partial\Delta \cap \bar{D}) \setminus \nu$ is an open arc.*

Proof. Let D be a component of $\Delta \setminus \nu$. The set $\nu \cap \Delta$ is a union of trajectories τ_j, $j \in I$. It follows from Lemma 3.6 that $\{\tau_j\}$ is locally finite in Δ. Let D_j be the component of $D \setminus \tau_j$ containing D. We know (Lemma 3.1 and Corollary 3.2) that $\partial_\Delta D_j$ is an open arc δ_j whose closure is a closed arc with endpoints on $\partial\Delta$. Remembering that $\{\tau_j\}$ is locally finite in Δ, it easily follows that either $\delta_j \subset \partial_\Delta D$ or $\delta_j \cap \partial_\Delta D = \emptyset$. Since $\partial_\Delta D \subset \nu$, it follows that there is $J \subset I$ so that $\partial_\Delta D = \bigcup_{j \in J} \delta_j$ where $\{\delta_j\}_{j \in J}$ is a locally finite family.

Pick some δ_j, $j \in J$. We can find a horizontal arc $\kappa \subset \Delta$ with one endpoint in ν but otherwise contained in D. There is a simple trajectory σ traversing κ, then we see using Lemma 3.3 that $d^h_\varphi(\nu, \sigma) > 0$. This fact implies that ν is contained in a component of $\bar{\Delta} \setminus \bar{\sigma}$. Now, $\bar{\sigma}$ is a crosscut of $\partial\Delta$ and hence $\partial\Delta \cap \bar{D}$ contains one of the components of $\partial\Delta \setminus \bar{\sigma}$. Thus $(\partial D \cap \bar{D}) \setminus \nu$ contains at least one arc. If it contains another, ν could not be connected. So it remains to show that D is a Jordan domain.

The set D is the subdomain of Δ bounded by the arcs δ_j, $j \in J$. Each $\bar{\delta}_j$ is a cross cut of Δ and there is a well-defined closed subarc α_j of $\partial\Delta$ with the same endpoints than $\bar{\delta}_j$ and such that the Jordan domain bounded by $\bar{\delta}_j \cup \alpha_j$ is contained in Δ and does not intersect D. Thus α_j, $j \in J$, are disjoint except possibly for the endpoints.

There is a homeomorphism $f_j : \alpha_j \to \delta_j$ which keeps the endpoints fixed. If we set $f = f_j$ on α_j and and $f(x) = x$ elsewhere we obtain a bijective mappings $\partial\Delta \to \partial D$. If the number of δ_j's is finite, we are done. If it is infinite, we take J to be the set of natural numbers and f is a homeomorphism if we can show that the euclidean diameters

$$d(\delta_j) \to 0 \tag{3.3}$$

as $j \to \infty$; since the arcs α_j are disjoint, we have $d(\alpha_j) \to \infty$ as $j \to \infty$.

We now prove (3.3). The proof is similar to the proof that a maximal geodesic ray converges to a well-defined boundary point [S, 19.6].

Let a_j and b_j be the endpoints of δ_j. If (3.3) is not true, we can find a subsequence j_k and points $d_{j_k} \in \delta_{j_k}$ so that $|a_{_k} - d_{j_k}| \geq r > 0$. We can pass to a subsequence so that $a_{j_k} \to a$, $b_{j_k} \to b$, and $d_{j_k} \to d$ as $k \to \infty$. Here $a, b \in \partial\Delta$ and since $d(\alpha_j) \to 0$, we have that $a = b$. We chose the subsequence so that $d \neq a$. Since $\{\delta_j\}$ is locally finite in Δ, it follows that $d \in \partial\Delta$.

Choose a point $z_0 \in D$. If $x \in \partial\Delta$, let I_x be the line segment with endpoints z_0 and x. Let $\pi : \bar{\Delta} \setminus \{z_0\} \to \partial\Delta$ be the projection so that $\pi(y) = x$ if $y \in I_x$. We can assume that z_0 is so chosen that there is w in the interior of the arc $\bar{D} \cap \partial\Delta$ such that $I_w \setminus \{w\} \subset D$. Thus I_w intersects no δ_j. Let κ be the subarc of $\partial\Delta$ with endpoints a and d not containing w. Then, given $x \in J \setminus \{a, d\}$, $\pi(\delta_{j_k})$ contains x for large k since $w \notin \pi(\delta_j)$. Hence $I_x \cap \delta_{j_k}$ contains at least two points for large k.

The set of points $x \in \kappa$ such that I_x has finite φ-length has full linear measure in κ (the proof of Lemma 19.6 of [S] contains this result; Strebel assumed that $z_0 = 0$ but this is not essential) and hence we can fix two points $x, y \in \kappa \setminus \{a, d\}$ so that I_x and I_y have finite φ-length. We choose the notation so that x is between a and y on κ. As noted above, I_x intersects infintely many δ_{j_k} so that the intersection contains at least two points. We can assume that this is true of all δ_{j_k}'s and that the same is true also of I_y.

Let D'_k be the component of $\Delta \setminus \delta_{j_k}$ not intersecting D. Then D'_j is a Jordan domain and $I_x \cap D'_j$ contains a subsegment whose closure is a cross-cut of D'_k. It may contain several such subsegments, but we can find at least one, denoted J_k, so that J_k intersects I_y. Thus, if the endpoints of J_k are u_k and t_k, then u_k and t_k are the endpoints of a subarc δ'_k of δ_{j_k} so that $\delta'_k \cap I_y$ contains at least one point. We choose one of them and call it v_k. We let δ''_k be the subarc of δ'_k with endpoints u_k and v_k. The family $\{\delta_j\}$ is locally finite in Δ and hence $u_k \to x$ and $v_k \to y$ as $j \to \infty$.

Now

$$|J_k|_\varphi \geq |\delta'_k|_\varphi \geq |\delta''_k|_\varphi$$

where the first inequality is due to the fact δ_j's subarcs of geodesics and the second since $\delta''_k \subset \delta'_k$. Since J_k's are disjoint and $|I_x|_\varphi < \infty$, it follows that $|J_k|_\varphi \to 0$ as $j \to \infty$. Hence also $|\delta''_k|_\varphi \to 0$.

It is possible to define φ-distance for the points $x, y \in \partial\Delta$ so that $d_\varphi(x, y) = \liminf d_\varphi(u, v)$ when $u \to x$ and $v \to y$. This distance is always positive [S, Lemma 19.6] if $x \neq y$. However, we have seen that $d_\varphi(u_k, v_k) = |\delta''_k|_\varphi \to 0$ and this would imply that $d_\varphi(x, y) = 0$. This contradiction shows that (3.3) is true.

Recall that a complete trajectory ν is *simple* if $\nu = \bar{\tau}$ for some simple trajectory τ. Thus ν is a closed arc with endpoints on $\partial\Delta$.

Lemma 3.8. *There are only a countable number of complete trajectories which are not simple complete trajectories.*

Proof. Since the number of zeroes of φ is countable, we need to consider only such complete trajectories ν which do not contain zeroes of φ. Thus each component of

$\nu \cap \Delta$ is a simple trajectory. If ν is not the closure of a simple trajectory, then $\nu \cap \Delta$ contains at least two trajectories τ and σ. Now, $\bar{\tau}$ and $\bar{\sigma}$ are closed arcs and they can have at most one common endpoint since otherwise they would bound a Jordan domain D such that $\bar{D} \cap \partial\Delta$ consists of two points. However, D contains a component of $\Delta \setminus \nu$ and we know by the preceding lemma that the closure of such a component contains an arc of $\partial\Delta$. This is a contradiction.

Thus $\nu \cap \partial\Delta$ contains at least three points. Hence the hyperbolic convex hull of H_ν of $\nu \cap \partial\Delta$ is well defined and contain interior points. If ν' is another complete trajectory, then ν' is connected and hence $\nu' \cap \partial\Delta$ is contained in a component $\partial\Delta \setminus \nu$. Thus H_ν and $H_{\nu'}$ are disjoint and so the family of H_ν, ν a complete non-simple trajectory, is a family of disjoint sets with non-empty interior. Hence this family is countable and our lemma is proved.

The following lemma shows how we can find a neighborhood basis for a complete trajectory using simple trajectories. A *basis of closed neighborhoods* of a set C is a family $\mathcal{F}$ of closed sets such that $C \subset \operatorname{int} V$ for all $V \in \mathcal{F}$ and that if W is an open set containing C, then W contains some $V \in \mathcal{F}$.

Lemma 3.9. *Let τ be a trajectory. Let $\mathcal{F}$ be the family of finite intersections*

$$\bigcap_{j=1}^{n} D_{\sigma_j}(\tau) \tag{3.3}$$

where D_{σ_j} is the set D_{σ_j} as in (3.0) and where σ_j is a simple trajectory such that $d^h_\varphi(\tau, \sigma_j) > 0$. Then $\mathcal{F}$ is a basis of closed neighborhoods of the completion τ^* of τ in $\bar{\Delta}$. If $\mathcal{F}'$ is the family of intersections of this form where $\bar{\sigma}_j$ are complete simple trajectories, then $\mathcal{F}'$ is another basis of closed neighborhoods of τ^* in Δ^*.

Proof. Suppose that σ is a simple trajectory such that $d^h_\varphi(\tau, \sigma) > 0$. Using Lemma 3.5, we find a simple trajectory σ_0 such that $\bar{\sigma}_0$ separates $\bar{\tau}$ and $\bar{\sigma}$ and both $d^h_\varphi(\nu, \tau) > 0$ and $d^h_\varphi(\nu, \sigma) > 0$. Thus $\tau^* \subset D_{\sigma_0}$ and $D_{\sigma_0} \subset \operatorname{int} D_\sigma$. Using this fact we see that τ^* is in the interior of every intersection (3.3).

So, to conclude that $\mathcal{F}$ is a family of closed neighborhoods of τ^*, it is enough to show that if W is a neighborhood of τ^*, then W contains some $V \in \mathcal{F}$. We note that the family $\mathcal{E}$ consisting of W and sets $\bar{\Delta} \setminus D_\sigma$ such that σ is a simple trajectory with $d^h_\varphi(\tau, \sigma) > 0$, is an open cover of $\bar{\Delta}$ since if $z \in \bar{\Delta}$ and z is in every D_σ, then $z \in \tau^*$ by the definition of τ^*. Thus there is a finite subcover, consisting of W and some sets $\bar{\Delta} \setminus D_{\sigma_j}$, $j \le n$. Thus $W \supset \bigcap_j^n D_{\sigma_j} \in \mathcal{F}$.

The claim concerning $\mathcal{F}'$ follows similarly since, by Lemma 3.5, we can choose the trajectory σ_0 in the first paragraph of the proof from complete simple trajectories intersecting a horizontal trajectory κ and since all but countable number of trajectories intersecting κ are such that the closure is a complete simple trajectory. The second paragraph of the proof is modified similarly.

As a corollary we have the following characterization of a trajectory. If ν is a complete trajectory, then $d^h_\varphi(\nu \cap \Delta, \sigma) > 0$ for any trajectory σ not contained in ν and then $\nu \cap \bar{\sigma} = \emptyset$ and we have

Corollary 3.10. *If ν is a complete trajectory, then*

$$\nu = \bigcap D_\sigma$$

where D_σ is as in (3.0) and the intersection is taken over simple trajectories σ such that $\nu \cap \sigma = \emptyset$ or equivalently that $\nu \cap \bar{\sigma} = \emptyset$.

Recall that $\mathcal{T}$ was the family of complete trajectories and $\mathcal{V}$ was the family of components of $\partial\Delta \setminus (\cup\mathcal{T})$.

Lemma 3.11. *The family $\mathcal{T} \cup \mathcal{V}$ is a partition of $\bar{\Delta}$ such that each $\tau \in \mathcal{V}$ is either a point or a closed interval of $\partial\Delta$. If $\chi \in \mathcal{V}$, then χ has a closed neighborhood basis in $\bar{\Delta}$ whose elements are closures of components of $\bar{\Delta} \setminus \bar{\sigma}$ where $\bar{\sigma}$ is a complete simple trajectory.*

Proof. It follows from Theorem 3.6 that the set $\mathcal{T}\cup\mathcal{V}$ is a disjoint family. Obviously, every $z \in \bar{\Delta}$ is on some $\tau \in \mathcal{T} \cup \mathcal{V}$. The following argument shows that if χ is a component of $\partial\Delta \setminus (\cup\mathcal{T})$, then σ is closed and hence a point or a closed interval.

If χ is not a closed interval or a point, then it is an open or a half-open interval. Thus there is an endpoint z of χ such that $z \notin \chi$. In this case $z \in \tau^*$ where τ^* is the completion of a trajectory τ. Let now σ be a simple trajectory such that $d^h_\varphi(\tau, \sigma) > 0$. We know that $\bar{\sigma}$ and τ^* are disjoint and so are and $\bar{\sigma}$ and χ. Since $\tau^* \cup \chi$ is connected, it follows that τ and χ are contained in the same component of $\bar{\Delta} \setminus \bar{\sigma}$ for any such σ. It follows that $\chi \subset \tau^*$, contrary to the assumption. So χ is closed.

If σ is a simple trajectory, then obviously there is a component of $\Delta \setminus \bar{\sigma}$ containing χ. We denote the closure of this component by D_σ. Obviously, $\chi \subset \operatorname{int} D_\sigma$. We will find a sequence of complete simple trajectories σ_j so that if $D_j = D_{\sigma_j}$, then $D_{j+1} \subset \operatorname{int} D_j$ and that setting

$$\bigcap_j D_j = D', \tag{3.4}$$

then $D' = \chi$ and it follows that $\{D_j\}$ is a closed basis of neighborhood of χ.

Let the endpoints of χ be a and b (possibly $a = b$). There is a sequence of complete trajectories ν_j containing a point $z_j \in \nu_j \cap \partial\Delta$ such that $z_j \to a$. We can assume that the points z_j converge monotonously to a. We can find, using Lemma 3.5, a complete simple trajectory σ_j separating ν_j and ν_{j+1} in $\bar{\Delta}$. Hence we can assume that ν_j are complete simple trajectories. Let D_j be the closure of the component $\bar{D} \setminus \nu_j$ containing χ. We now claim that the intersection D' of (3.4) is χ.

If $D' \neq \chi$, pick a point $z \in D' \setminus \chi$. We claim that we can choose z so that $z \in \Delta$. If this is not the case, then D' would be a subarc of $\partial\Delta$ properly containing χ and there would be a complete trajectory ν' such that $\nu' \cap D' \neq \emptyset$. Thus $\nu' \subset D_j$ for every j and hence $\nu' \subset D'$. This is a contradiction and so $\partial_\Delta D' \neq \emptyset$ and there is such a z as claimed and we can further assume that $z \in \partial D'$. Then $d_\varphi(z, \nu_j \cap \Delta) \to 0$ and hence $d^h_\varphi(z, \nu_i \cap \Delta) \to 0$.

Let τ be the trajectory through z. We claim that $\chi \subset \tau^*$. To prove this we must show that if σ is a simple trajectory such that $d^h_\varphi(\tau, \sigma) > 0$ and D_σ is the closure of the component of $\bar{\Delta} \setminus \bar{\sigma}$ containing τ, then $D_\sigma \supset \chi$, implying that $\chi \subset \tau^*$ contrary to the assumption. We note that $d^h_\varphi(\tau, \nu_i \cap \Delta) \to 0$, and hence $\nu_j \subset D_\sigma$ for large j. Since the endpoints of $\bar{\sigma}$ are not points of χ and since $z_j \in \nu_j$ tend toward the endpoint a of χ, the sets χ and ν_j are in the same component of $\bar{\Delta} \setminus \bar{\sigma}$ for large j. These facts imply that $\chi \subset D_\sigma$. The lemma is proved.

4. The extended limit and semiconvergence

We now give the detailed construction of the extended limit whose construction was outlined in Section 2 "Main ideas".

We recall the notation. Let $r(z) = 1/\bar{z}$ be the reflection on the unit circle. If τ is a (vertical) trajectory, the complete trajectory τ^* defined by τ is given by (3.1). If $\nu = \tau^*$ is a complete trajectory, its double is $\tilde{\nu} = \nu \cup r(\nu)$; more generally, the double $\tilde{A}$ of a set A is

$$\tilde{A} = A \cup r(\bar{A}) \tag{4.1}$$

and so, if σ is a simple trajectory, $\tilde{\sigma}$ is a Jordan curve.

First, we need a suitable neighborhood system for complete trajectories and their doubles. We now go through the construction in the end Section 2 in a more precise fashion. We first found a countable set $\mathcal{S}$ of simple complete trajectories, that is trajectories ν such that $\nu = \bar{\tau}$ for some simple trajectory τ and thus ν is a closed arc, and such that $\mathcal{S}$ is dense in the set of trajectories which means that if γ is a horizontal arc, then points of the form $\sigma \cap \gamma$ where $\sigma \in \mathcal{S}$ form a dense subset of γ.

Next, we define $\mathcal{U}$ to be the countable family of sets U such that U is the closure of a components of $\bar{\Delta} \setminus (\cup\mathcal{F})$ where $\mathcal{F} \subset \mathcal{S}$ is finite. If $\nu \in \mathcal{T}$, we define $\mathcal{U}_\nu$ by

$$\mathcal{U}_\nu = \{U \in \mathcal{U} : \nu \subset \operatorname{int} U\}.$$

Lemma 3.9 implies now that $\mathcal{U}_\nu$ is a countable basis of closed neighborhoods of ν in $\bar{\Delta}$. This implies that

$$\tilde{\mathcal{U}}_\nu = \{\tilde{U} : U \in \mathcal{U}_\nu\}$$

is a countable basis of closed neighborhoods of $\tilde{\nu}$ in $\overline{\mathbb{C}}$. (Note that the definition of $\tilde{\mathcal{U}}_\nu$ was slightly different in Section 2 since we went directly to $\tilde{\mathcal{U}}_\nu$ without considering $\mathcal{U}_\nu$.) We obtain a countable basis of closed neighborhoods for ν in $\overline{\mathbb{C}}$ if we set

$$U_r = \tilde{U} \cap \{|z| \leq r\}$$

and

$$\mathcal{U}^*_\nu = \{U_r : U \in \mathcal{U}_\nu, r > 1 \text{ rational}\}.$$

Not all points of $\bar{\Delta}$ are on some complete trajectory but some points of $\partial\Delta$ are in some component of $\partial\Delta \setminus (\cup\mathcal{T})$, that is, they are in some $\nu \in \mathcal{V}$ (in which case $\tilde{\nu} = \nu$). We need similar bases for $\nu \in \mathcal{V}$. If $\sigma \in \mathcal{S}$, we let D_σ be the closure of the component of $\bar{\Delta} \setminus \bar{\sigma}$ containing ν. As above, it follows from Lemma 3.11 using Lemma 3.5, that if we set $\mathcal{U}_\nu = \{D_\sigma : \sigma \in \mathcal{S}\}$, then $\mathcal{U}_\nu$ is a closed basis of neighborhoods of ν in $\bar{\Delta}$. If we set $\tilde{\mathcal{U}_\nu} = \{\tilde{U} : U \in U_\nu\}$, we obtain a basis of neighborhoods of ν in $\overline{\mathbb{C}}$. It is useful to set $\mathcal{U}^*_\nu = \tilde{\mathcal{U}}_\nu$ if $\nu \in \mathcal{V}$, since we can later treat simultaneously the cases that $\nu \in \mathcal{T}$ and $\nu \in \mathcal{V}$.

We now assume that we have a sequence f_j of quasiconformal mappings as in Theorem 1. In particular, f_j tend toward a conformal map f in Δ^*. Since the set $\mathcal{U}$ us countable, we can pass to a subsequence so that all the Hausdorff limits f_jU, $f_j\tilde{U}$, and f_jU_r as $j \to \infty$ exist for all $U \in \mathcal{U}$ and rational $r > 1$. We denote the Hausdorff limit by V^∞ if V is one of these sets. Thus we can define, if $\nu \in \mathcal{V}$, $f(\tilde{\nu})$ and $f(\nu)$ by (2.10) and (2.11).

We have now defined the extended limit. In order to show that the extended limit has the properties we have claimed, we need some lemmas. The main one is the following which is based on Corollary 2.3.

Lemma 4.1. *Let ν_j and $\nu_2 \neq \nu_1$ be elements of $\mathcal{T} \cup \mathcal{V}$. Then $\tilde{\nu}_1$ and $\tilde{\nu}_2$ have closed neighborhoods $V_j \in \tilde{\mathcal{U}}_{\nu_j}$ such that the spherical distances $q(f_jV_1, f_jV_2)$ have positive lower bound independent of j.*

Proof. Since $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are disjoint closed sets they have disjoint closed neighborhoods. It follows that there are $V_j \in \tilde{\mathcal{U}}_{\nu_j}$ so that V_1 and V_2 are disjoint. The boundary of V_j consists of a finite number of Jordan curves of the form $\tilde{\sigma}$ where σ is a complete simple trajectory and hence their horizontal distance is positive (Theorem 3.6). Hence Lemma 2.3 implies that if $\tilde{\sigma}$ and $\tilde{\sigma}'$ are two such boundary curves, the spherical distances $q(f_j(\tilde{\sigma}), f_j(\tilde{\sigma}'))$ are bounded from below by a positive constant independent of i. This implies the lemma.

Theorem 4.2. *The sets $f(\tau)$, $\tau \in \mathcal{D} = \mathcal{T} \cup \mathcal{V}$, together with the one point sets $f(z)$, $z \in \Delta^*$ form a partition of $\overline{\mathbb{C}}$ by closed sets. In addition, if $U \in \mathcal{U}_\nu$, $\nu \in \mathcal{D}$, then $\tilde{U}^\infty = \cup\{f(\tilde{\tau}) : \tau \subset U, \tau \in \mathcal{D}\}$*

Proof. We first show that the sets $f(\tau)$; $\tau \in \mathcal{D} \cup \Delta^*$ are disjoint. Disjointedness is a consequence of the preceding lemma since if ν_1 and ν_2 are distinct elements of $\mathcal{T} \cup \mathcal{V}$, and V_1 and V_2 are as in the lemma, then $f(\nu_k)$ is a subset of the Hausdorff limit V_k^∞ of $f_j(V_k)$ as $j \to \infty$. By Lemma 4.2, V_1^∞ and V_2^∞ are disjoint. If $\nu_1 \in \mathcal{T} \cup \mathcal{V}$ and $\nu_2 \in \Delta^*$, then we note that $f(\nu_j) \subset \overline{\mathbb{C}} \setminus f\Delta^*$ and disjointedness is true also in this case. Obviously it is true also if both ν_j are points of Δ^*.

To show that the union of these sets is $\overline{\mathbb{C}}$, we pick $w \in \overline{\mathbb{C}}$ and note that each f_j is a homeomorphism and hence there is z_j such that $f_j(z_j) = w$. Pass to a subsequence so that z_j tend to $z \in \overline{\mathbb{C}}$. If $z \in \Delta^*$, we are done since $f_j \to f$ locally uniformly in Δ^* and hence $f(z) = w$. Otherwise, $z \in \nu$ for some $\nu \in \mathcal{T} \cup \mathcal{V}$. If $V \in \tilde{\mathcal{U}}_\nu$, then $z_j \in V$ for large i, and hence $w \in V^\infty$. This is true for all $V \in \tilde{\mathcal{U}}$ and the definition of $f(\nu)$ implies that $w \in f(\tilde{\nu})$. If $w \in f(\Delta^*)$, then this would imply that $z \in \Delta^*$, and we know that in this case $f(z) = w$. Otherwise, $w \in f(\nu)$. This implies that the union is $\overline{\mathbb{C}}$.

Stated in other words, we have shown that $\overline{\mathbb{C}} = f\Delta^* \cup (\cup\{f(\tau) : \tau \in D\})$. Hence, in order to get the second paragraph, it is enough to show that that if $\tau \in \mathcal{D}$ and $\tau \not\subset U$, then $f(\tilde{\tau})$ is disjoint from U. But this is a consequence of Lemma 4.1 since if

$\tau \not\subset U$, then $\tau \cap U = \emptyset$ and hence τ has a neighborhood $V \in \mathcal{U}_\tau$ which is disjoint from U. Thus $\tilde{U}^\infty$ and $\tilde{V}^\infty \supset f(\tilde{\tau})$ are disjoint.

Lemma 4.3. *Let $\nu \in \mathcal{T} \cup \mathcal{V}$. Then $\tilde{\mathcal{W}}_\nu = \{\tilde{U}^\infty : U \in \mathcal{U}_\nu\}$ is a basis of closed neighborhoods of $f(\tilde{\nu})$ and $\mathcal{W}_\nu^* = \{U_r^\infty : U \in \mathcal{U}_\nu, r \in \mathbb{Q}, r > 1\}$ is basis of closed neighborhoods of $f(\nu)$.*

Given $U \in \mathcal{U}_\nu$ and rational $r > 1$, there are $W \in \mathcal{U}_\nu$, rational $s > 1$ and n_0 such that

$$f_j(\tilde{W}) \subset \operatorname{int} \tilde{U}^\infty \text{ and } f_j(W_s) \subset \operatorname{int} U_r^\infty \tag{4.2}$$

for all $i \geq n_0$.

Proof. We first prove that $\tilde{\mathcal{W}}_\nu = \{\tilde{U}^\infty : U \in \mathcal{U}_\nu\}$ is a basis of closed neighborhoods of $f(\tilde{\nu})$. We first show that $f(\tilde{\nu}) \subset \operatorname{int} \tilde{U}^\infty$ for every $U \in \mathcal{U}_\nu$. We prove this in the case that $\nu \in \mathcal{T}$; the modifications are obvious if $\nu \in \mathcal{V}$. Fix $U \in \mathcal{U}_\nu$. Thus

$$U = \bigcap_{j=1}^{n} D_j$$

where $\sigma_j \in \mathcal{S}$ are such that $d_\varphi^h(\sigma_j, \nu \cap \Delta) > 0$ and $D_j = D_{\sigma_j}$ is the closure of the component of $\bar{\Delta} \setminus \bar{\sigma}_j$ containing ν. Using Lemma 3.5 (apply it to a suitable component of $\nu \cap \Delta$), we can find another $\sigma_j' \in \mathcal{S}$ so that $d_\varphi^h(\sigma_j', \nu \cap \Delta) > 0$, $d_\varphi^h(\sigma_j', \sigma_j) > 0$ and that $\bar{\sigma}_j'$ separates ν and σ_j in $\bar{\Delta}$. Thus if we let D_j' be the component of $\bar{\Delta} \setminus \bar{\sigma}_j$ containing ν, then

$$W = \bigcap_{j=1}^{n} D_j' \subset \operatorname{int} U$$

is an element of $\mathcal{V}_\nu$. Since $\bar{\sigma}$, $\sigma \in \mathcal{S}$, are simple complete trajectories, their horizontal distance is positive and so Theorem 2.3 implies the existence of $c > 0$ such that $q(f_j(\tilde{\sigma}_p), f_j(\tilde{\sigma}_k')) \geq c > 0$ for all p and k independently of j. Now, $\partial \tilde{W} = \bigcup_k \tilde{\sigma}_k'$ and $\partial \tilde{U} = \bigcup_p \bar{\sigma}_p$ and hence (boundaries taken in $\bar{\Delta}$), and it follows that

$$q(f_j \tilde{W}, f_j \partial \tilde{U}) \geq c > 0 \tag{4.3}$$

for all j. These facts imply that the Hausdorff limit $\tilde{W}^\infty$ of $f_j W$ is in the interior of $\tilde{U}^\infty$ and (4.3) also implies the existence of n_0 such that the first inequality of (4.2) is true. Thus $f(\tilde{\nu}) \subset \tilde{W}^\infty \subset \operatorname{int} U^\infty$.

Thus, in order to show that $\tilde{\mathcal{W}}_\nu$ is a basis of closed neighborhoods of $f(\tilde{\nu})$, it is enough to show that if V_0 is a neighborhood of $f(\tilde{\nu})$, then it contains some $W \in \tilde{U}_\nu$. This follows if we can show that $\{V_0\} \cup \{\overline{\mathbb{C}} \setminus W : W \in \tilde{\mathcal{U}}_\nu\}$ is an open cover of $\overline{\mathbb{C}}$ and this is true if $\{\overline{\mathbb{C}} \setminus W : W \in \tilde{W}_\nu\}$ is an open cover of $\overline{\mathbb{C}} \setminus f(\tilde{\nu})$. This would imply that there are finitely many $W_k \in \tilde{U}_v$ so that $W = W_1 \cap ... \cap W_n \subset V_0$. However, the definition of $\tilde{U}_\nu$, and consequently of $\tilde{W}_\nu$, is such that finite intersections of members are still members.

This can be seen as follows if $\nu \in \mathcal{T}$; the case that $\nu \in \mathcal{V}$ is similar.

If $w \in \overline{\mathbb{C}} \setminus f(\tilde{\nu})$, then $w \in f(\tilde{\tau})$ for some $\tau \in \mathcal{T} \cup \mathcal{V}$ such that $\tau \neq \nu$ and hence $\tilde{\tau}$ and $\tilde{\nu}$ are disjoint. There are disjoint closed neighborhoods $U_\nu \in \tilde{\mathcal{U}}_\nu$ of ν and $U_\tau \in \tilde{\mathcal{U}}_\tau$ of τ. The boundaries ∂U_τ and ∂U_ν consist of a finite number of Jordan curves $\tilde{\sigma}$, $\sigma \in \mathcal{S}$. Thus there is $\sigma \in \mathcal{S}$ so that $\tilde{\sigma} \subset \partial U_\nu$ and such that $\tilde{\sigma}$ separates ν and τ. Let D_σ be the component of $\bar{\Delta} \setminus \bar{\sigma}$ containing $\bar{\nu}$. Thus $f(\tilde{\nu}) \subset \tilde{D}_\sigma^\infty$. On the other hand there is also $\sigma' \in \mathcal{S}$ such that $\bar{\sigma}$ separates ν and τ. Corollary 2.3 implies that the spherical distances $q(f_j(\tilde{\sigma}), f_j(\tilde{\tau}))$ have a positive lower bound independent of j. Hence $\tilde{D}_\sigma^\infty$ and $\tilde{D}_{\sigma'}^\infty$ are disjoint when $D_{\sigma'}^\infty$ is the closure of the component of $\bar{\Delta} \setminus \bar{\sigma}'$ containing τ. Thus $w \in f(\tilde{\tau}) \subset D_{\sigma'}^\infty \subset \overline{\mathbb{C}} \setminus D_\sigma^\infty$.

We need not add much to get the results for $\tilde{\mathcal{W}}_\nu^*$. We note that $f_j(\overline{\mathbb{C}} \setminus \{|z| \le r\})$ converge toward a set which is a neighborhood of $f(\bar{\Delta})$ (which we interpret as the union of $f(\tau)$, $\tau \in \mathcal{D} = \mathcal{T} \cup \mathcal{V}$, or equivalently, as $\overline{\mathbb{C}} \setminus f\Delta^*$) and any point not in $f(\Delta^*)$ is contained in some such set. Using these facts, and applying what we have proved above, one easily sees first that each element of $\tilde{\mathcal{W}}^*$ is a closed neighborhood of $f(\nu)$ and that the second inequality of (4.2) is true. Then, using a similar compactness argument as above, one shows that $\tilde{\mathcal{W}}_\nu^*$ is a basis of closed neighborhoods of ν.

Proofs of Theorems 1 and 2. Since elements of $\mathcal{T}$ are disjoint (Theorem 3.6), $\mathcal{D}$ is obviously a partition of $\bar{\Delta}$. That $f\mathcal{D}$ is a partition of $\overline{\mathbb{C}} \setminus f\Delta^*$ and f is a bijection $\mathcal{D} \to f\mathcal{D}$ follows by Lemma 4.2. What we need in addition is mainly contained in the preceding lemma. The definition of $f(\tilde{\nu})$ implies immediately, together with (4.2), that $f_j(\tilde{\nu})$ semiconverge to $f(\tilde{\nu})$ and this implies easily that $f_j(\nu)$ semiconverge to $f(\nu)$. If $\nu_j \in \Delta^* \cup \mathcal{T} \cup \mathcal{V}$ semiconverge to $\tilde{\nu}$, then the proof that $f_j(\nu_j)$ semiconverge to $f(\nu)$ is similar: if $\nu \in \mathcal{T}$, one shows first that $f_j(\tilde{\nu}_j)$ semiconverge to $f(\tilde{\nu})$ and that this implies the semiconvergence to $f(\nu)$.

We need to refine Lemma 4.3 for Theorem 3. We denote below the interior of a set X by X° for brevity; if confusion is possible, we may denote it also $\operatorname{int} X$. We also say that a set $U \subset \overline{\mathbb{C}}$ is $\mathcal{D}$-saturated if $\nu \cap U \neq \emptyset$ implies $\nu \subset U$ for $\nu \in \mathcal{D} = \mathcal{T} \cup \mathcal{V}$; the definition of a $f\mathcal{D}$-saturated set is similar; here $f\mathcal{D} = \{f(\tau) : \tau \in \mathcal{D}\}$.

Lemma 4.4. *If $\nu \in \mathcal{D}$, let $\mathcal{F}_\nu = \{U^\circ : U \in \mathcal{U}_\nu^*\}$. Then $\mathcal{F}_\nu$ is a basis of (open) neighborhoods of ν such that each $U \in \mathcal{F}_\nu$ is $\mathcal{D}$-saturated.*

Let $f\mathcal{F}_\nu = \{fU : U \in \mathcal{F}_\nu\}$ where fU is interpreted as $\cup\{f(\tau) : \tau \subset U, \tau \in \mathcal{D} \cup \Delta^*\}$. Then $f\mathcal{F}_\nu$ is a basis of (open) neighborhoods of $f(\nu)$, $\nu \in \mathcal{D}$, such that each $W \in fU$, $U \in \mathcal{F}_\nu$, is $f\mathcal{D}$-saturated.

Proof. The definition of U_r, $U \in \mathcal{U}_\nu$, is such that $\nu \subset U_r^\circ$ and since $\mathcal{U}_\nu^*$ is a basis of closed neighborhoods of ν (Lemma 3.9), $\mathcal{F}_v$ is a basis of neighborhoods of ν. Since $U_r \cap \Delta = U$ is a closed set of Δ bounded by a finite number of $\bar{\sigma}$, $\sigma \in \mathcal{S}$, it follows that U_r is $\mathcal{D}$-saturated.

To prove the second paragraph, we pick first $U \in \mathcal{U}_\nu$ and consider the set $f(\tilde{U}^\circ)$ defined as $\cup\{f(\tilde{\tau}), \tau \in \mathcal{D}, \tau \subset U^\circ\}$. We claim that $f(\tilde{U}^\circ)$ is open. If $z \in f(\tilde{U}^\circ)$, then $z \in f(\tilde{\tau})$ for some $\tau \in \mathcal{D}$, $\tau \subset U^\circ$. Thus $\tilde{U}^\infty \in \tilde{\mathcal{W}}_\tau$ which is a closed neighborhood of τ by Lemma 4.3. Hence there is $V \in \tilde{\mathcal{U}}_\tau$ so that $\tilde{V}^\infty \subset \operatorname{int} \tilde{U}^\infty$. We can assume that

$V \subset U^\circ$. However, Lemma 4.2 implies that $\tilde{V}^\infty = \cup\{f(\tilde{\sigma}) : \sigma \in \mathcal{D}, \sigma \subset V\} \subset f(\tilde{U}^\circ)$ since $V \subset U^\circ$. Now, $\tilde{V}^\infty$ is a closed neighborhood of τ and hence $f(\tilde{U}^\circ)$ is open.

Since $f_j \to f$ uniformly on compact subsets of Δ^*, we have that $U_r^\infty = \tilde{U}^\infty \backslash f\{|z| > r\}$ (note that we regard ∞ as a point of $\{|z| > \infty\}$). If $\tau \in \mathcal{D}$, set $\tau_r = \tau \cup \{z \in \tilde{\tau} \cap \Delta^*, |z| < r\} = \tilde{\tau} \setminus \{|z| \geq r\}$ and set $f(\tau_r) = f(\tau) \cup \{f(z) : z \in \tau_r \cap \Delta^*\} = f(\tilde{\tau}) \setminus f\{|z| \geq r\}$. Thus

$$\begin{aligned} fU_r^\circ &= \cup\{f(\tau_r) : \tau \in \mathcal{D}, \tau \subset U\} = \cup\{f(\tilde{\tau}) : \tau \in \mathcal{D}, \tau \subset U\} \setminus f\{|z| \geq r\} \\ &\subset \tilde{U}^\infty \setminus f\{|z| > r\} = U_r^\infty. \end{aligned}$$

where the inclusion is true because of Lemma 4.2. The equalities in this chain imply that fU_r° is open since we have shown that $f(\tilde{U}^\circ) = \cup\{f(\tilde{\tau}) : \tau \in \mathcal{D}, \tau \subset U\}$ is open. The second is implies that fU_r°'s form a basis of neighborhoods of ν since U_r^∞'s form a basis of closed neighborhoods of ν.

Proof of Theorem 3. Theorem 1 implies that f is a bijection $\overline{\mathbb{C}}/\mathcal{D} \to \overline{\mathbb{C}}/f\mathcal{D}$. Lemma 4.4 implies that $\mathcal{F}_v$ and $f\mathcal{F}_\nu$ project to neighborhoods of ν and $f(\nu)$ in the quotients $\overline{\mathbb{C}}/\mathcal{D}$ and $\overline{\mathbb{C}}/f\mathcal{D}$, respectively. Hence f is a homeomorphism in the quotient topologies.

The fact that $\overline{\mathbb{C}}/\mathcal{D}$ and $\overline{\mathbb{C}}/f\mathcal{D}$ are homeomorphic to $\overline{\mathbb{C}}$ is Theorem 22 of Moore [M]. One need only to translate this theorem into modern language and to add a few simple arguments. We note that $(\Delta^* \setminus \{\infty\}) \cup \mathcal{D}$ (here points of Δ^* or of $f\Delta^*$ are to be understood as one-point sets) is an upper semi-continuous collections of continua in the terminology of Moore. The partition $\mathcal{E}$ of $\mathbb{C}$ is an upper semi-continuous collection of the plane $\mathbb{C}$ if each $G \in \mathcal{E}$ is a compact connected set not separating $\mathbb{C}$, i.e. its complement is connected. In addition, if U is a neighborhood of G, then G has a smaller neighborhood V such that whenever $F \in \mathcal{E}$ and $F \cap V \neq \emptyset$, then $F \subset U$. We call this last condition the *semicontinuity condition.*

That $(\Delta^* \setminus \{\infty\}) \cup \mathcal{D}$ satisfies these conditions is seen as follows. If $\nu \in \mathcal{T}$, then $\nu \subset \mathbb{C}$ and is closed in $\overline{\mathbb{C}}$ by the definition (3.1). Hence ν is compact. By Theorem 3.6, it is connected and does not separate $\overline{\mathbb{C}}$ and hence does not separate $\mathbb{C}$. If $\nu \in \mathcal{V}$, then ν is either a point or a close arc (Lemma 3.11) and hence compact and connected and does not separate $\mathbb{C}$. The case that x (identified with $\{x\}$) is in $\Delta^* \backslash \{\infty\}$ is trivial. Now, $\mathcal{F}_\nu$ is a neighborhood basis for ν which is $\mathcal{D}$-saturated. This implies the semicontinuity condition. Hence Moore's theorem implies that $\mathbb{C}/\mathcal{D}$ is homeomorphic to the plane and so $\overline{\mathbb{C}}/\mathcal{D}$ is homeomorphic to $\overline{\mathbb{C}}$. Since we now know that f is a homeomorphism of $\overline{\mathbb{C}}/\mathcal{D}$ onto $\overline{\mathbb{C}}/f\mathcal{D}$, the same is true of $\overline{\mathbb{C}}/fD$.

5. Missing details in the section "Main ideas"

We now complete the outline given in Section 2. We complete the proofs of Lemmas 2.1 and 2.2 making use of the results of Section 3 which was independent of Section 2. Section 4 is not needed.

Completion of the proof of Lemma 2.1 First, we transform the situation by a Möbius transformation g so that Δ is mapped onto the upper halfplane U and Δ^* to the lower halfplane L. The quadratic differential φ is replaced by $\varphi(g(z))g'(z)^2$. The φ-length and area are unchanged and neither is the L^1-norm $||\varphi||_1$. The advantage is that now some steps are slightly easier to describe. For instance, we can extend φ to a quadratic differential of $U \cup L$ by the simple rule $\varphi(\bar{z}) = \overline{\varphi(z)}$. Defined in this manner, the reflection $r(z) = \bar{z}$ on $\mathbb{R}$ is an isometry of the φ-metric of U onto that of L which preserves horizontal and vertical trajectories.

We will consider rings R which are symmetric with respect to the reflection on $\mathbb{R}$. Both components of the complement of R will be Jordan domains and it is possible to choose the Möbius transformation conjugating Δ onto U so that ∞ is in the interior of one component of the complement. We assume in the following that we have this slightly simpler situation.

Let now the situation be as in Lemma 2.1. Thus we have two trajectories τ and σ of U such that $d^h_\varphi(\tau, \sigma) > 0$. Thus $\bar{\tau}$ and $\bar{\sigma}$ are disjoint (Lemma 3.4) and they contain geodesics τ' and σ' $\tau \cup \sigma'$ bound a component of $\Delta \setminus (\tau \cup \sigma)$. Thus $\bar{\tau}'$ and $\bar{\sigma}'$ are disjoint closed arcs (Lemma 3.1) and hence their doubles $\tilde{\tau}' = \bar{\tau}' \cup r(\bar{\tau}')$ and $\tilde{\sigma}'$ are disjoint Jordan curves and so they bound the ring $R = R(\tau, \sigma)$ of (2.7).

We will now derive the estimate of Lemma 2.1 for $M(R)$. The proof was outlined in Section 2 and we now complete it. If we extend φ to $U \cup L$ as indicated above, then the the extension may be very irregular near $\partial\Delta$. The basic idea is still valid and what we do here can be regarded as technical juggling.

If $t > 0$, let $U_t = \{\operatorname{Im} z > t\}$, $L_t = \{\operatorname{Im} z < t\}$ and $\mathbb{R}_t = \{\operatorname{Im} z = t\}$. We let f_{kt} be the normalized quasiconformal map which satisfies (1) in U_t but is conformal in L_t. A normalized map is a map fixing three given points; it does not matter what these points are but we fix three points which all normalized maps fix. We let f_k be the normalized solution which satisfies (1) in U and is conformal in L. Since $\mu_{f_{kt}} \to \mu_{f_k}$ a.e. as $t \to 0$, the good approximation theorem of quasiconformal mappings will make possible to estimate f_k and $M(f_k R)$ by means of f_{kt} and $M(f_{kt} R)$.

If t is small, then τ' and σ' have closed subarcs τ_t and σ_t whose endpoints are on $\mathbb{R}_t$ but otherwise lie on U_t. Although they might be several choices for fixed t, we can choose them so that τ_t and σ_t contain a fixed point of $U_t \cap \tau_t$ or of $U_t \cap \sigma_t$, respectively. So we can assume that they increase monotonously as $t \to 0$ and that their unions are τ' and σ', respectively.

Let r_t be the reflection on $\mathbb{R}_t$, that is $r_t(z) = \overline{z - it} + it$. Then $\tilde{\tau}_t = \tau_t \cup r_t(\tau_t)$ and $\tilde{\sigma}_t = \sigma_t \cup r_t(\sigma_t)$ are Jordan curves bounding a ring R_t. As $t \to 0$, R_t tend toward the ring R bounded by $\tilde{\tau}$ and $\tilde{\sigma}$ in the sense that the complement of R_t tends toward the complement of R in the Hausdorff metric, taken with respect to the spherical metric. We will see that the same is true of the rings $f_{kt} R_t$ and this fact makes possible to estimate $M(f_k R)$ by means of $M(f_{kt} k R_t)$.

Starting from the quadratic differential φ, defined on U, we form a new function φ_t which is holomorphic in $U_t \cup L_t$ so that $\varphi_t = \varphi$ in $\bar{U}_t$ and satisfies $\varphi_t(z) = \overline{\varphi_t(r_t(z))}$ in L_t. Although φ_t cannot be extended continuously to $\mathbb{R}_t$, the absolute value $|\varphi|$ and the real part can be extended continuously to $\mathbb{R}_t$. The element of horizontal length satisfies $|\operatorname{Re} \sqrt{\varphi(z)} dz| = |\operatorname{Re} \sqrt{\varphi_t(r_t(z)} \; \overline{dz}|$ and thus the φ_t-length and the horizontal φ_t-length well-defined and they are invariant under the reflection r_t. We denote the

φ_t-length and its horizontal variant of a path γ as usual by $|\gamma|_{\varphi_t}$ and $|\varphi|^h_{\varphi_t}$. Thus

$$|\gamma|^h_{\varphi_t} = \int_\gamma |\mathrm{Re}\sqrt{\varphi_t}dz|$$

and we have $|\gamma|^h_{\varphi_t} = |r_t\gamma|^h_{\varphi_t}$.

Let now γ be a path in R_t joining the components of the complement. Define a new path γ^* so that $\gamma^*(s) = \gamma(s)$ if $\gamma(s) \in \overline{U}_t$ but otherwise let $\gamma^*(a) = r_t(\gamma(s))$. Thus γ^* is path in $\overline{U}_t$ where $\varphi = \varphi_t$ and hence $|\gamma^*|^h_{\varphi_t} = |\gamma^*|^h_\varphi \geq d^h_\varphi(\tau,\sigma)$. Since the element of horizontal length is unchanged by the reflection, we obtain

$$|\gamma|^h_{\varphi_t} = |\gamma^*|^h_{\varphi_t} \geq d^h_\varphi(\tau,\sigma).$$

Fix now k and let $K = (1+k)/(1-k)$. We define a new metric d_{tk} from φ_t. The image of this metric under f_{tk} in $f_{tk}R$ will be conformally equivalent to the euclidean metric (at least outside some irregular points) and can be given by a metric density but d_{tk} itself is not conformally equivalent to the euclidean metric but it will be Riemannian outside zeroes of φ and the line $\mathbb{R}_t$. We let φ_{tk} be unchanged in L_t. However, in $\overline{U}_t$, the metric is not conformal (with respect to the euclidean metric) but is obtained from φ_t so that horizontal distances are unchanged but vertical distances are divided by K.

We can explain this more precisely as follows. We denote by $|dz|_{kt}$ the element of length of this metric which is defined as follows. If $z \in L_t$, set $|dz|_{tk} = \sqrt{|\varphi_t(z)|}|dz|$. If $z \in \overline{U}_t$, then we can express dz as the sum $dz = dx_h + dy_v$ where dx_h is the infinitesimal change in the horizontal direction and dy_h the change in vertical direction. We now set

$$|dz|_{tk} = \sqrt{|\varphi_t|(|dx_h|^2 + |dy|^2_h/K^2)}.$$

The d_{tk}-length of a path γ is

$$|\gamma|_{tk} = \int_\gamma |dz|_{tk}.$$

The important thing is that there is still associated to d_{tk} the element of horizontal length which is $\sqrt{|\varphi_t|}|dx_h|$ and it equals the horizontal length element $|\mathrm{Re}\sqrt{|\varphi_t|}\sqrt{\varphi_t}dz|$ associated to φ_t. Thus, if γ joins in R_t the components of the complement, denoting the horizontal length of γ with respect to d_{tk} by $|\gamma|^h_{tk}$, we have

$$|\gamma|_{tk} \geq |\gamma|^h_{tk} = \int_\gamma \sqrt{|\varphi|}|dx_h| = |\gamma|^h_{\varphi_t} \geq d^h_\varphi(\tau,\sigma). \tag{5.1}$$

Denote $w = f_{tk}(z)$. We will calculate $|dw|$ as a function of $|dz|_{tk}$. Note first that f_t is differentiable with non-vanishing Jacobian outside zeroes of φ_t in $\mathbb{C}\setminus\mathbb{R}_t$. If $z \in L_t$ is not a zero of φ_t, then $|dz|_{tk} = \sqrt{|\varphi_t(z)|}\,|dz|$ and hence $|dz|_{tk} = \varrho(w)|dw|$ where $\varrho(w) = \sqrt{|\varphi_t(z)|}/|f'_t(z)|$ since f_t is conformal in L_t. If $z \in U_t$ is not a zero of φ_t, let f_α be the directional derivative to the direction α. Thus $|f_\alpha|$ attains its maximal value when $\alpha = \alpha_h =$ the direction of horizontal trajectories and minimal when $\alpha = \alpha_v =$

the direction of vertical trajectories. If $dz = e^{i\alpha_h}dt$, then $|dz|_{tk} = \sqrt{|\varphi_t(z)|}|dz|$ and thus

$$|dz|_{tk} = \varrho(w)|dw| \tag{5.2}$$

where $\varrho(w) = \sqrt{|\varphi_t(z)|}/|f_{\alpha_h}(z)|$. If $dz = e^{i\alpha_v}dt$, then $|dz|_{tk} = \sqrt{|\varphi_t(z)|}/K$ and $|f_{\alpha_v}(z)| = |f_{\alpha_h}(z)|/K$. It follows that (5.2) is still true with the same ϱ. These facts imply that (5.2) is true in $U_t \cup L_t$ whenever z is not a zero of φ_t. It does not matter how ϱ is defined on zeros on φ and on $\mathbb{R}_t \cap R_t$, we can, for instance, set $\varrho = 0$ at these points.

Let Γ_0 be the path family whose elements are rectifiable paths of R_t joining the components of the complement of R_t such that also $f_{kt}\gamma$ is rectifiable. In addition, we assume that if $\gamma \in \Gamma_0$ is parametrized by the arc length, then the linear measure of parameter values t such $\gamma(t) \in \mathbb{R}_t$ vanishes. These assumptions imply that

$$\int_{f_{kt}\gamma} \varrho(w)|dw| = \int_\gamma |dz|_{tk} \geq d^h_\varphi(\tau, \sigma).$$

for all $\gamma \in \Gamma_0$. The family of paths of $f_{kt}R_t$ joining the components of the complement which are not of the form $f_{kt}\gamma$ for some $\gamma \in \Gamma_0$ has vanishing modulus (use Fuglede's theorem [V, 28.2] and remember that the areal measure of $f_t\mathbb{R}_t$ vanishes). This implies that

$$d^h_\varphi(\tau, \sigma)^2 M(f_tR_t) \leq \int_{f_{kt}R_t} \varrho^2 dm.$$

However, (5.2) implies that $\int_{f_{kt}R_t} \varrho^2 dm = m_{tk}(R_t)$ where m_{tk} is the area with respect to d_{tk}. We defined d_{tk} from φ_t so that the area was decreased (horizontal distances were unchanged but vertical distances divided by K) and hence $m_{tk} \leq m_{\varphi_t}$ when m_{φ_t} is the area with respect to the φ_t-metric. Thus

$$d^h_\varphi(\tau, \sigma)^2 M(f_{kt}R_t) \leq m_{tk}(R_t) = 2m_{tk}(U_t) \leq 2||\varphi||_1 < \infty.$$

Our maps are normalized and hence we can find a sequence t_j decreasing to zero so that f_{t_jk} tend toward a quasiconformal map g. The complex dilatations of f_{t_jk} tend a.e. to the complex dilatation of f_k. The good approximation theorem of quasiconformal mappings [LV, IV.5.6] implies that g and f_k have the same complex dilatation. Since they are normalized in the same manner, we conclude that $f_k = g$. Thus $f_{t_jk} \to f_k$ as $i \to \infty$ and, moreover, the convergence is uniform on compact subsets. Now, R_{t_j} tend toward R in the sense that complements converge with respect to the Hausdorff metric. In view of the uniform convergence, also $f_{t_j}R_{t_j}$ tend toward f_kR. It follows by a theorem of Gehring [G, Lemma 6] that $M(f_{t_jk}R_{t_j})$ tend toward $M(f_kR)$. We conclude that $2||\varphi||_1/d^h_\varphi(\tau, \sigma)^2$ is an upper bound for $M(f_kR)$ independently of k.

Proof of Lemma 2.2. We now have a fixed φ and consider another quadratic differential ψ such that ψ varies in a neighborhood of φ. We return to the original situation from the upper half-space, where we completed the proof of Lemma 2.1, and so φ and ψ are a integrable holomorphic maps of Δ. We assume Lemma 2.1, whose proof is now complete, and prove some lemmas.

Since we now have several quadratic differentials, we specify that a trajectory τ or an arc is a trajectory or an arc for a specific ψ, by saying that τ is a ψ-trajectory (or an arc, etc). If no quadratic differential is specified, φ is meant.

Lemma 5.1. *Let κ_0 be a closed horizontal φ-arc so that $\varphi \neq 0$ on κ_0. Let $\varepsilon > 0$. Then there are a compact subset K of Δ and $\delta > 0$ such that whenever ψ is integrable and holomorphic in Δ and $|\varphi - \psi| < \delta$ in K, then the following is true. If $c, d \in \kappa_0$ are distinct, then*

$$1 - \varepsilon < \frac{d^h_\psi(c,d)}{d^h_\varphi(c,d)} < 1 + \varepsilon. \tag{5.3}$$

There is a ψ-horizontal trajectory arc κ_ψ (not depending on d and c) such that if τ_c and τ_d are the ψ-vertical trajectories through c and d, respectively, then κ_ψ intersects τ_c and τ_d and hence a subarc of κ_ψ is ψ-horizontal arc joining τ_c and τ_d.

Let κ_1 be a φ-horizontal arc containing κ_0 without common endpoints with κ_0 such that $\varphi \neq 0$ on κ_1 and suppose that κ_1 joins φ-trajectories τ and σ. Then K, δ, and κ_ψ can be so chosen that κ_ψ can be extended to a horizontal ψ-trajectory arc κ'_ψ joining τ and σ.

Remark. Actually, we can take for K any compact set such that $\kappa_0 \subset \operatorname{int} K$, or, in the second paragraph, that $\kappa_1 \subset \operatorname{int} K$. Also, the holomorphic maps φ and ψ need not be integrable in this lemma.

Proof. This is mainly a question of continuity. We can always slightly extend κ_0 to such a φ-trajectory κ_1 as mentioned in the lemma if it is not already given. Thus we assume that κ_1 exists.

Since $\varphi > 0$ on κ_1, there are a closed neighborhood U of κ_1 and a branch of the canonical coordinate $\Phi = \int \sqrt{\varphi} dz$ on U so that $\operatorname{int} \Phi U$ contains a quadrilateral Q_1 with sides parallel to coordinate axes and that $\Phi \kappa_1 = [0, b] \subset \mathbb{R}$. We fix an endpoint z_0 of κ_1 and assume that $\Phi(z_0) = 0$. There is a smaller quadrilateral $Q_0 \subset \operatorname{int} Q_1$ with sides parallel to the coordinate axes such that $\Phi \kappa_0 = Q_0 \cap \mathbb{R}$.

We take $K = U$ and choose δ so small that $\psi \neq 0$ on U if $|\varphi - \psi| < \delta$ in K and choose a branch of the canonical coordinate $\Psi = \int \sqrt{\psi} dz$ so that $\operatorname{int} \Psi U$ contains a fixed quadrilateral Q (independent of ψ) with sides parallel to coordinate axes such that $\Psi(\kappa_0) \subset \operatorname{int} Q$.

If now $c, d \in \kappa$, then $\Psi(c), \Psi(d) \in Q$ and hence, if τ_c and τ_d are the trajectories mentioned in the theorem, $\Psi \tau_d \cap Q$ and $\Psi \tau_c \cap Q$ are vertical line segments. There is a horizontal line segment of Q which intersects these vertical line segments. Choosing one and transporting this segment back by Ψ^{-1}, we get the ψ-horizontal subarc of κ_ψ whose subarc joins τ_c and τ_d. If δ is small enough, another subarc can be taken for the arc κ_ψ in the second paragraph.

To prove (5.3), we note that if $c, d \in \kappa_0$, and if κ' is the subsegment of κ_0 with endpoints c and d, then $d^h_\varphi(c,d) = d_\varphi(c,d) = \int_{\kappa'} |\operatorname{Re} \sqrt{\varphi} dz|$ and $d^h_\psi(c,d) = \int_{\kappa'} |\operatorname{Re} \sqrt{\psi} dz|$; this latter equality assumes that $\operatorname{Re} \sqrt{\psi} dz$ does not change sign when moving monotonously from c to d on κ_0 which is true if δ is small enough and (5.3) follows.

We now fix such a horizontal arcs κ_0 and κ_1 not containing zeroes of φ as in Lemma 5.1. We consider a sequence φ_j of integrable holomorphic mappings of Δ so that $||\varphi_j||_1$ are uniformly bounded and that $\varphi_j \to \varphi$ uniformly on compact subsets.

We assume that $|\varphi - \varphi_j| < \varepsilon$ in K where ε and K are as in Lemma 5.1. If $z \in \kappa_0$, we let τ_z be the vertical φ-trajectory through z and τ_{jz} the ψ_j-trajectory through z. We now fix two distinct points z and w of κ_0.

Lemma 5.2. *Suppose that $\bar\tau_{zj}$ and $\bar\tau_{wj}$ have the Hausdorff limits χ_z and χ_w, respectively. Then χ_z and χ_w are disjoint and if τ_z and τ_w are simple, then $\bar\tau_z \subset \chi_z$ and $\bar\tau_w \subset \chi_w$.*

Proof. We first prove that $\bar\tau_z \subset \chi_z$ if τ_z is simple. We can argue as in Lemma 5.1. If τ is simple, given closed subarc α of τ_z, then α has a closed neighborhood U not containing zeroes of φ so that the canonical coordinate Φ is defined in a neighborhood of U such that $\operatorname{int} \Phi U$ contains a quadrilateral Q so that $\Phi\alpha$ is contained in a vertical line segment σ_z joining opposite sides of Q. For large j, there is a canonical coordinate Ψ_j for ψ_j is defined in U so that $\Psi_j \to \Phi$ uniformly on U and that $\Psi_j U$ contains Q. Thus $\sigma_{zj} = \Psi_j(\tau_{zj} \cap U) \cap Q$ is a vertical line segment passing through Q. Since $\Psi_j \to \Phi$ uniformly, the line segments σ_{zj} tend toward the line segment σ_z in the Hausdorff metric. Since also $\Psi_j^{-1} \to \Phi^{-1}$ uniformly, $\Psi_j^{-1}\sigma_{zj}$ tend toward $\Phi^{-1}\sigma_z$ in the Hausdorff limit. It follows that $\alpha \subset \Phi^{-1}\sigma_z$ is contained in the Hausdorff limit χ_z. Since α was an arbitrary subarc of τ_z, it follows tat $\bar\tau_z \subset \chi_z$. Similarly, $\bar\tau_w \subset \chi_w$ if τ_w is simple.

We then prove that χ_z and χ_w are disjoint. Let $R_j = R(\tau_{jz}, \tau_{jw})$ be the ring defined by τ and σ as in (2.7). Lemma 5.1 implies that τ_{jz} and τ_{jw} are joined by a horizontal arc κ_j for large j and that $|\kappa_j|_{\psi_j} = d^h_{\psi_j}(z,w) \to d^h_\varphi(z,w)$. Hence $m(R_j)$ are bounded from above (Lemma 2.1). Let C_j and D_j be the components of $\overline{\mathbb{C}} \setminus R_j$. The spherical diameters of C_j are bounded from below by the spherical distance of κ_0 to ∂D. Hence (cf. the Appendix) also the spherical distances $q(C_j, D_j) \geq c$ for some $c > 0$ indepedent of j. It follows that $q(\chi_z, \chi_w) \geq c > 0$. ⊃

Lemma 5.3. *Let τ and σ be the φ-trajectories through the endpoints of κ_0 and let $m > 0$. Pick $z \in \kappa_0$ which is not an endpoint of κ_0. Then there are a compact set $K \subset \Delta$ and $\varepsilon > 0$ with the following property. Let ψ be holomorphic such that $||\psi||_1 \leq m$ and such that $|\psi - \varphi| < \varepsilon$ in K. If ν be the ψ-trajectory through z, then $\bar\nu$, $\bar\tau$ and $\bar\sigma$ are disjoint and $\bar\nu$ separates $\bar\tau$ and $\bar\sigma$ in $\bar\Delta$,*

Proof. Arguing as in the proof of Lemma 5.2, we can assume that τ and σ are simple.

We first show that there are ε and K such that if ν, τ and σ are as in the theorem, then $\bar\nu$, $\bar\tau$ and $\bar\sigma$ are disjoint; note that in any case $\bar\tau$ and $\bar\sigma$ are disjoint since they are joined by a horizontal arc (Lemma 3.4). If there are not such K and ε as claimed, we can find ψ_j so that $||\psi_j||_1$ are uniformly bounded and such that $\psi_j \to \varphi$ uniformly on compact subsets with the property that if ν_j is the ψ_j-trajectory through z, then $\bar\nu_j$ intersects $\bar\tau$ or $\bar\sigma$. We can assume that $\bar\nu_j \cap \bar\tau \neq \emptyset$. Let w be the endpoint of κ_0 such that τ passes through w and let τ_j be the ψ_j-trajectory through w. Pass to a subsequence so that $\bar\nu_j$ has the Hausdorff limit χ_z and $\bar\tau_j$ have the Hausdorff limit χ_w. By Lemma 5.2, χ_z and χ_w are disjoint. By this same lemma, $\bar\tau \subset \chi_w$ and hence χ_z and $\bar\tau$ are disjoint. But we assumed that all the intersections $\bar\nu_j \cap \bar\tau$ are non-empty and this implies that $\emptyset \neq \chi_z \cap \bar\tau \subset \chi_z \cap \chi_w$. So our claim is proved.

We claim that K and ε can be so chosen that, in addition, $\bar{\nu}$ separates $\bar{\tau}$ and $\bar{\sigma}$. To see this we note that we can choose K and ε by Lemma 5.1 (apply it so that the present κ_0 is κ_1 of Lemma 5.1 and κ_0 of Lemma 5.1 is a subarc containing z) so that if $|\psi - \varphi| < \varepsilon$ in K, then there is a ψ-horizontal arc κ_ψ intersecting ν and which joins τ and σ. Since τ and σ are simple, they bound a Jordan domain D of Δ and κ_ψ is a crosscut of D with one endpoint on τ and the other on σ. Thus κ_ψ divides D into two subdomains D_1 and D_2.

If ν is simple, then $\bar{\nu}$ is a crosscut of $\partial\Delta$ and hence divides Δ into two components C_1 and C_2. Since ν is a vertical and κ_ψ a horizontal arc which intersect, κ_ψ contains points both in C_1 and C_1. Since $\kappa_\psi \cap \nu$ cannot contain more than one point ([T1, Lemma 2.3]), it follows that ν divides κ_ψ into two pieces of which one is contained in C_1 and the other in C_2. This implies that $\bar{\nu}$ separates τ and $\bar{\sigma}$.

If ν is not simple, the situation is basically the same. As above, $\nu \cap \kappa_\psi$ is a point $\{z\}$. Since κ_y is a horizontal arc intersecting the vertical trajectory ν at a point, we can find vertical arcs ν_1 and ν_2 contained in ν so that ν_j have endpoint z but otherwise are $\nu_j \subset D_j$. It is now obvious that ν_j can be completed to a ray ν_j so that still $\nu_j \subset \nu$ and $\nu_j \subset D_j$ (remember that ν_j is simply connected, cf. [T1, Lemma 2.3]). Thus $\nu' = \nu_1 \cup \nu_2$ is a geodesic and hence $\bar{\nu}'$ is a cross cut of $\partial\Delta$ (Lemma 3.1). Since κ divides ν into two pieces of which are in different components of $D \setminus \kappa$, we see that ν divides κ into two pieces which are in different components of $\bar{\Delta} \setminus \bar{\nu}'$ and hence $\bar{\nu}'$, and consequently $\bar{\nu}$, separate $\bar{\tau}$ and $\bar{\sigma}$.

If R_1 and R_2 are two rings, we say that R_1 *properly contains* R_2 if $R_1 \supset R_2$ and if, in addition, the components of $\overline{\mathbb{C}} \setminus R_j$ can be denoted as C_j and D_j so that $C_1 \subset C_2$ and $D_1 \subset D_2$. Thus $M(R_1) \le M(R_2)$ if R_1 properly contains R_2. Note that if $R_1 = R_2$, then R_1 properly contains R_2 according to our definition.

Recall the notation $R(\tau, \sigma)$ in (2.7).

Lemma 5.4. *Let τ and σ be two φ-trajectories such that $d^h_\varphi(\tau, \sigma) > 0$. Let $m > 0$. Then there are a compact set $K \subset \Delta$, $\varepsilon > 0$ and $c > 0$ and such that if ψ is holomorphic such that $||\psi||_1 < m$ and $|\varphi - \psi| < \varepsilon$ in K, then there are ψ-trajectories τ_ψ and σ_ψ such that $R(\tau, \sigma)$ properly contains $R(\tau_\psi, \sigma_\psi)$ and that τ_ψ and σ_ψ are joined by a ψ-horizontal arc κ_ψ such that $|\kappa_\psi|_\psi = d^h_\psi(\tau, \sigma) \ge c$.*

Proof. Let κ be the horizontal arc given by Lemma 3.5. Choose consecutive points z_1, z_2, z_3, z_4 on κ_0 so that z_1 and z_4 are endpoints and let χ_1 be the subarc of κ with having endpoints z_1 and z_2, χ_2 has endpoints z_2 and z_3 and finally χ_3 has endpoints z_3 and z_4.

Let τ_j be the φ-trajectory through z_j. Lemma 5.3 implies that there are a compact set $K \subset \Delta$ and $\varepsilon > 0$ such that if $|\varphi - \psi| < \varepsilon$ in K, then there is a ψ-trajectory τ_ψ intersecting χ_1 such that $\bar{\tau}_\psi$ separates $\bar{\tau}_1$ and $\bar{\tau}_2$ and that there is another ψ-trajectory σ_ψ intersecting χ_3 such that $\bar{\sigma}_\psi$ separates $\bar{\tau}_3$ and $\bar{\tau}_4$. It follows that $R(\tau, \sigma)$ properly contains $R(\tau_\psi, \sigma_\psi)$.

Lemma 5.1 implies that ε and K can be so chosen that there is a ψ-horizontal arc κ_ψ joining τ_ψ and σ_ψ. Since κ_ψ intersects both τ_2 and τ_3, $|\kappa_\psi|^h_\psi \ge d^h_\psi(\tau_2, \tau_3)$. Now $d^h_\psi(\tau_2, \tau_3) = d^h_\psi(z_2, z_3)$ and $|\chi_2|_\varphi = d^h_\varphi(z_2, z_3)$ and hence (5.3) implies that ε and K can be so chosen that

$$|\kappa_\psi|_\psi \geq |\chi_2|_\varphi/2.$$

Thus the lemma is true with this ε and $c = |\chi_2|_\varphi/2$.

Given Lemma 5.4, Theorem 2.2 is a consequence of Lemma 2.1.

6. Appendix

The following theorem is well known [V, 12.7] but we give a simple normal family argument.

Theorem 6.1. *Given $M > 0$ and $c > 0$, there is $d > 0$ with the following property. Let R be a ring such that $M(R) \leq M$ and that $\overline{\mathbb{C}} \setminus R$ has components C_1 and C_2 so the spherical diameter $q(C_i) \geq c$, $i = 1, 2$. Then $q(C_1, C_2) \geq d$.*

Proof. If the theorem is not true, there is a sequence of rings R_j so that $M(R_j) \leq M$ and that the components of $\overline{\mathbb{C}} \setminus R_j$ are C_{j1} and C_{j2} with $q(C_{jk}) \geq c$ but that $q(C_{j1}, C_{j2}) \to 0$ as $j \to \infty$. Let S_j be the ring $1 < |z| < r_j$ such that $M(S_j) = M(R_j)$ and thus $M(R_j) = 1/\log r_j$. We can assume that $r_j \to r$ where $1 < r \leq \infty$. Let $S = \{1 < |z| < r\}$. There is a conformal homeomorphism $f_j : S_j \to R_j$. Our assumptions imply that f_j avoids a set spherical diameter at least c and hence f_j is a normal family and it is possible to pass to a subsequence (denoted in the same manner) so that f_j tend in S uniformly on compact subsets toward a map $f : S \to \overline{\mathbb{C}}$ which is either a constant or a conformal embedding. Since both components of $\overline{\mathbb{C}} \setminus R_j$ have spherical diameter exceeding c, it follows that f cannot be a constant. It follows that $\limsup_{j\to\infty} q(C_{j1}, C_{j2}) \geq q(C_1, C_2) > 0$ where C_1 and C_2 are the components of $\overline{\mathbb{C}} \setminus fS$. This contradiction proves the lemma.

References

[A] Abikoff, W.: Kleinian groups - geometrically finite and geometrically perverse. In “Geometry of group representations” id. by W.M. Goldman and A.R. Maid, Contemporary Mathematics 74, 1988, 1-50.

[G] Gehring, F.W.: Rings and quasiconformal mappings in space, Trans. Amber. Math. Sock. 103, 1962, 353-393.

[LV] Lehto, O, and Virtuosi, K.I.: Quasiconformal mappings in the plane, Springer 1973.

[M] Moore, R.L.: Concerning upper semi-continuous collections of continua. Trans. Amber. Math. Sock. 27, 1925, 418-428.

[S] Strebel, K.: Quadratic differentials, Springer 1984.

[T1] Tukia, P: Limits of Teichmüller mappings on trajectories, J. Anal. Math. 92 (2004), 137-189.

[T2] Tukia, P.: Teichmüller sequences on trajectories invariant under a Kleinian group. J. Anal. Math. 99(2006), 35-87.

[V] Väisälä,J.: Lectures on n-dimensional quasiconformal mappings. Springer Lecture Notes 229, 1971.

Department of Mathematics
P.O.Box 68 (Gustav Hällströmin kate 2b)
FIN-00014 University of Helsinki
Finland
e-mail: pekka.tukia@helsinki.fib