GLOBAL COMPARISON PRINCIPLES FOR THE p-LAPLACE OPERATOR ON RIEMANNIAN MANIFOLDS

ILKKA HOLOPAINEN†, STEFANO PIGOLA, AND GIONA VERONELLI

Abstract. We prove global comparison results for the p-Laplacian on a p-parabolic manifold. These involve both real-valued and vector-valued maps with finite p-energy. Further L^q comparison principles in the non-parabolic setting are also discussed.

1. Introduction

Let $(M, \langle , \rangle)$ be a connected, m-dimensional, complete Riemannian manifold and let $p > 1$. Recall that the p-Laplacian of a real valued function $u : M \to \mathbb{R}$ is defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. A function $u \in W^{1,p}_{\text{loc}}(M)$ is said to be p-subsolution if $\Delta_p u \geq 0$ weakly on M. In case any bounded above, p-subsolution is necessarily constant we say that the manifold M is p-parabolic. It is known that p-parabolicity is related to volume growth properties of the underlying manifold. Accordingly, M is p-parabolic provided, for some $x \in M$,

$$\left(\frac{r}{\operatorname{vol}_m B_r(x)}\right)^{\frac{1}{p-1}} \notin L^1(+\infty), \tag{1}$$

where $B_r(x)$ denotes the metric ball centered at x, of radius $r > 0$, and vol_m is the m-dimensional Hausdorff measure. Thus, for instance, the standard Euclidean space $\mathbb{R}^m$ is p-parabolic if $m \leq p$. Condition (1) is quite natural in that it shares the quasi-isometry invariance of p-parabolicity. Moreover, it turns out that there are geometric situations where (1) is also necessary for M to be p-parabolic; see [5], [7] and references therein. On the other hand, it was established in [18], [16] and [6] that the most general volume growth condition ensuring p-parabolicity is that, for some $x \in M$,

$$\left(\frac{1}{\operatorname{vol}_{m-1}\partial B_r(x)}\right)^{\frac{1}{p-1}} \notin L^1(+\infty).$$

Now, suppose that M is p-parabolic, with $p \geq 2$. It is known, [13], that a smooth p-subharmonic function $u : M \to \mathbb{R}$ with finite p-energy $|\nabla u| \in L^p(M)$ must be constant. We shall show that this is nothing but a very special case of a genuine comparison principle for the p-Laplace operator.

Recall that, given a function $f \in L^1_{\text{loc}}(M)$ and a vector field $X \in L^1_{\text{loc}}(M)$, we say that $\operatorname{div} X \geq f$ weakly (or in the sense of distributions) on M if, for all

Date: December 1, 2009.
2000 *Mathematics Subject Classification.* 35B05, 31C45.
Key words and phrases. Nonlinear comparisons, finite p-energy, p-parabolicity.
†Work partially supported by the Academy of Finland, project 123633.

non-negative, compactly supported, smooth test functions φ, $0 \le \varphi \in C_c^\infty(M)$,

$$(\operatorname{div} X, \varphi) := -\int \langle X, \nabla\varphi \rangle \ge \int f\varphi. \tag{2}$$

In particular, if $X = |\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v$ for some real-valued functions $u, v \in W^{1,p}_{\mathrm{loc}}(M)$ and $f \equiv 0$, we have that the weak inequality $\Delta_p u \ge \Delta_p v$ means

$$\int \left\langle |\nabla u|^{p-2}\nabla u, \nabla\varphi \right\rangle \le \int \left\langle |\nabla v|^{p-2}\nabla v, \nabla\varphi \right\rangle, \tag{3}$$

for all $0 \le \varphi \in C_c^\infty(M)$. Note that, by standard density results and by dominated convergence, it is equivalent to require the validity of (2) and (3) for all $0 \le \varphi \in W^{1,p}_c(M)$ if $|\nabla u|, |\nabla v| \in L^p(M)$. Above $W^{1,p}_{\mathrm{loc}}(M)$ stands for the (local) Sobolev space of all functions $u \in L^p_{\mathrm{loc}}(M)$ whose weak (distributional) gradients also belong to $L^p_{\mathrm{loc}}(M)$. Furthermore, $W^{1,p}_c(M)$ is the closure of $C_c^\infty(M)$ in $W^{1,p}(M)$.

Theorem 1. *Let $(M, \langle, \rangle)$ be a connected, p-parabolic Riemannian manifold, with $p > 1$. Assume that $u, v \in W^{1,p}_{loc}(M) \cap C^0(M)$ satisfy*

$$\Delta_p u \ge \Delta_p v \text{ weakly on } M,$$

and

$$|\nabla u|, |\nabla v| \in L^p(M).$$

Then, $u = v + A$ on M, for some constant $A \in \mathbb{R}$.

Simple examples show that both the p-parabolicity of M and the L^p-integrability of $|\nabla u|$ or $|\nabla v|$ are needed above. Indeed, let M be, for instance, the open unit ball in $\mathbb{R}^m$, u a constant function, and v a non-constant p-harmonic function in M (i.e. a continuous weak solution to $\Delta_p v = 0$), with $|\nabla v| \in L^p(M)$. Then M is non-p-parabolic for all $p > 1$ and the conclusion of Theorem 1 clearly fails. On the other hand, let M be the infinite cylinder $\mathbb{R} \times \mathbb{S}^{m-1}$ equipped with the product metric $ds^2 = dr^2 + d\vartheta^2$, where $d\vartheta^2$ is the standard metric of the sphere $\mathbb{S}^{m-1}$. Furthermore, let u be a constant function and $v(t, \vartheta) = t$. Now M is p-parabolic for all $p > 1$, u and v are p-harmonic in M, but the conclusion of Theorem 1 again fails.

To prove Theorem 1 we will introduce an inequality for the p-Laplacian which resembles a well known inequality for the mean curvature operator. A basic use of this inequality will enable us to get also the next result in the spirit of [12].

Theorem 2. *Let $(M, \langle, \rangle)$ be a complete Riemannian manifold. Let $u, v \in C^\infty(M)$ be such that*

$$\Delta_p u \ge \Delta_p v \text{ on } M$$

for some $p \ge 2$. Suppose there exist $q \ge 1$ and $s > p$ such that

$$\left(\int_{\partial B_t(o)} |u - v|^{q + \frac{1}{s-1}} (|\nabla u| + |\nabla v|)^{p - \frac{s}{s-1}} \right)^{1-s} \notin L^1(+\infty), \tag{4}$$

for some $o \in M$. Then either $u \equiv v + A$ for some constant $A \in \mathbb{R}$ or $u \le v$ on M.

Besides real-valued functions one is naturally led to consider manifold-valued maps. Several topological questions are related to the p-Laplacian of maps; [19],[15].

Recall that the p-Laplacian (or the p-tension field) of a map $u : M \to N$ between Riemannian manifolds is defined by

$$\Delta_p u = \operatorname{div}\left(|du|^{p-2}\, du\right).$$

Here, $du \in T^*M \otimes u^{-1}TN$ denotes the differential of u and the bundle $T^*M \otimes u^{-1}TN$ is endowed with its Hilbert-Schmidt scalar product $\langle , \rangle$. Moreover, $-\operatorname{div}$ stands for the formal adjoint of the exterior differential d with respect to the standard L^2 inner product on vector-valued 1-forms. Say that u is p-harmonic if $\Delta_p u = 0$. In [17], Schoen and Yau prove a general comparison principle for homotopic (2-)harmonic maps with finite (2-)energy into non-positively curved targets. They assume that the complete, non-compact manifold M has finite volume but the request that M is (2-)parabolic suffices, [13]. In this direction, comparisons for homotopic p-harmonic maps with finite p-energy into non-positively curved manifolds are far from being completely understood. Some progress in the special situation of a single map homotopic to a constant has been made in [13]. In this note, we focus our attention on the case $N = \mathbb{R}^n$. According to [13], it is clear that, if M is p-parabolic, then every p-harmonic map $u : M \to \mathbb{R}^n$ with finite p-energy $|du| \in L^p(M)$ must be constant. However, using the very special structure of $\mathbb{R}^n$, we are able to extend this conclusion, thus establishing a comparison principle for maps $u, v : M \to \mathbb{R}^n$ having the same p-Laplacian. In some sense, this can be considered as a further step towards the comprehension of the general comparison problem alluded to above.

Theorem 3. *Suppose that $(M, \langle , \rangle)$ is p-parabolic, with $p \geq 2$. Let $u, v : M \to \mathbb{R}^n$ be smooth maps satisfying*

$$\Delta_p u = \Delta_p v \text{ on } M, \tag{5}$$

and

$$|du|, |dv| \in L^p(M).$$

If $(M, \langle , \rangle)$ is p-parabolic then $u = v + A$, for some constant $A \in \mathbb{R}^n$.

Acknowledgement. The second and the third authors are indebted to A.G. Setti for valuable conversations during the preparation of the paper.

2. Main tools

In the proofs of Theorems 1 and 3 we will use two main ingredients: (a) a version for the p-Laplacian of a classical inequality for the mean-curvature operator, which will be also used in a final section to prove Theorem 2; (b) a global form of the divergence theorem in non-compact settings which inspires to a p-parabolicity criterion involving vector fields.

2.1. A key inequality.

The following basic inequality was discovered by Lindqvist, [9].

Lemma 4. *Let $(V, \langle , \rangle)$ be a finite dimensional, real vector space endowed with a positive definite scalar product and let $p > 1$. Then, for every $x, y \in V$ it holds*

$$|x|^p + (p-1)|y|^p - p|y|^{p-2}\langle x, y\rangle \geq C(p)\Psi(x, y),$$

where

$$\Psi(x,y) := \begin{cases} |x-y|^p & p \geq 2 \\ \frac{|x-y|^2}{(|x|+|y|)^{2-p}} & 1 < p < 2, \end{cases}$$

and $C(p)$ is a positive constant depending only on p.

As a consequence, we deduce the validity of the next

Corollary 5. *In the above assumptions, for every $x, y \in V$, it holds*

$$\left\langle |x|^{p-2}\, x - |y|^{p-2}\, y, x-y \right\rangle \geq 2C(p)\Psi(x,y). \tag{6}$$

Proof. We start computing

$$\left\langle |x|^{p-2}\, x - |y|^{p-2}\, y, x-y \right\rangle = |x|^p + |y|^p - \langle x, y\rangle \left(|x|^{p-2} + |y|^{p-2}\right).$$

On the other hand, applying twice Lindqvist inequality with the role of x and y interchanged we get

$$p\left(|x|^p + |y|^p\right) \geq p\left(|x|^{p-2} + |y|^{p-2}\right)\langle x, y\rangle + 2C(p)\Psi(x,y).$$

Inserting into the above completes the proof. □

Remark 6. Inequality (6) can be considered as a version for the p-Laplacian of the classical Mikljukov-Hwang-Collin-Krust inequality; [11], [8], [1]. This latter states that, for every $x, y \in V$,

$$\left\langle \frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}}, x-y \right\rangle \geq \frac{\sqrt{1+|x|^2}+\sqrt{1+|y|^2}}{2} \left| \frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}} \right|^2,$$

equality holding if and only if $x = y$. This analogy suggests the validity of global comparison results, without any p-parabolicity asssumption, in the spirit of [12], as exemplified by Theorem 2. See Section 4.

2.2. **p-parabolicity and related properties.** As we mentioned in the introduction, a manifold M is p-parabolic if a Liouville type property holds for p-subsolutions that are bounded above. It is well known that this is just one of the several equivalent definitions of p-parabolicity; see [4]. For instance, and in view of future purposes, we recall the next

Theorem 7. *The manifold M is p-parabolic if and only if the (relative) p-capacity of any compact set K vanishes. This means that*

$$\inf \int_M |\nabla \varphi|^p = 0$$

where the infimum is taken over all compactly supported smooth functions φ satisfying $\varphi = 1$ on K.

A further very useful characterization of (non-)p-parabolicity involves special vector fields on the underlying manifold. It goes under the name of Kelvin-Nevanlinna-Royden criterion. In the linear setting $p = 2$ it was proved in a paper by T. Lyons and D. Sullivan, [10]. See also Theorem 7.27 in [14]. The following non-linear extension is due to Gol'dshtein and Troyanov, [2].

Theorem 8. *The manifold M is not p-parabolic if and only if there exists a vector field X on M such that:*

(a) $|X| \in L^{\frac{p}{p-1}}(M)$

(b) $\operatorname{div} X \in L^1_{loc}(M)$ *and* $\min(\operatorname{div} X, 0) = (\operatorname{div} X)_- \in L^1(M)$

(c) $0 < \int_M \operatorname{div} X \leq +\infty$.

Accordingly, if M is p-parabolic and X is a vector field satisfying (a') $|X| \in L^{\frac{p}{p-1}}(M)$, (b') $\operatorname{div} X \in L^1_{\mathrm{loc}}(M)$, and (c') $\operatorname{div} X \geq 0$ on M, then we must necessarily conclude that $\operatorname{div} X = 0$ on M. It is worth pointing out that, even if condition (b') is not satisfied, we can obtain a similar conclusion as shown in the next

Proposition 9. *Let $(M, \langle , \rangle)$ be a p-parabolic Riemannian manifold, $p > 1$. Let X be a vector field satisfying $|X| \in L^{\frac{p}{p-1}}(M)$ and*

$$\operatorname{div} X \geq f \geq 0$$

in the sense of distributions, for some $0 \leq f \in L^1_{loc}(M)$. Then

$$f \equiv 0.$$

Proof. Let $\{\Omega_j\}_{j=0}^{\infty}$ be an increasing sequence of precompact open sets with smooth boundaries such that $\Omega_j \nearrow M$. Let φ_j be the p-equilibrium potential of the condenser $C(\Omega_j, \overline{\Omega_0})$, namely

$$\int_M |\nabla \varphi_j|^p = \min \int_M |\nabla \varphi|^p$$

where the minimum is taken over all smooth φ compactly supported in Ω_j and satisfying $\varphi = 1$ on $\overline{\Omega_0}$. Then, φ_j solves the Dirichlet problem

$$\begin{cases} \Delta_p \varphi_j = 0 & \Omega_j \setminus \overline{\Omega_0} \\ \varphi_j = 1 & \text{on } \overline{\Omega_0} \\ \varphi_j = 0 & \text{on } \partial\Omega_j \end{cases}$$

and we have

$$\begin{aligned} 0 \leq \int_M \varphi_j f &\leq (\operatorname{div} X, \varphi_j) \\ &= -\int_M \langle X, \nabla \varphi_j \rangle \\ &\leq \left(\int_M |X|^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \left(\int_M |\nabla \varphi_j|^p \right)^{\frac{1}{p}}. \end{aligned} \tag{7}$$

Note that, by Theorem 7,

$$\int_M |\nabla \varphi_j|^p \to 0, \qquad \text{as } j \to \infty,$$

which implies that the RHS of (7) vanishes as $j \to \infty$. Moreover, by the comparison principle on precompact domains it follows that $0 \leq \varphi_j \leq 1$ is a non-decreasing sequence of functions pointwise converging to some $\varphi > 0$. Hence, taking limits in (7) and using monotone convergence,

$$0 \leq \int_M \varphi f \leq 0$$

and this latter gives $f \equiv 0$. □

Remark 10. Using an Ahlfors type characterization of p-parabolicity in terms of a boundary maximum principle for p-harmonic functions on generic domains we see that, in fact, $\varphi \equiv 1$.

3. Proofs of the finite-energy comparison principles

We are now in the position to prove the main results.

Proof (of Theorem 1). Fix any $x_0 \in M$, let $A = u(x_0) - v(x_0)$ and define Ω_A to be the connected component of the open set

$$\{x \in M : A - 1 < u(x) - v(x) < A + 1\}$$

which contains x_0. By standard topological arguments, $\Omega_A \neq \emptyset$ is a (connected) open set. Let $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ be the piece-wise linear function defined by

$$\alpha(t) = \begin{cases} 0 & t \leq A - 1 \\ (t - A + 1)/2 & A - 1 \leq t \leq A + 1 \\ 1 & t \geq A + 1. \end{cases}$$

Consider the vector field

$$X = \alpha \circ (u - v) \left\{ |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right\},$$

and note that, for a suitable constant $C > 0$,

$$|X|^{\frac{p}{p-1}} \leq C (|\nabla u|^p + |\nabla v|^p) \in L^1(M).$$

¿From now on we abbreviate $\alpha(u-v) = \alpha \circ (u-v)$, $\alpha'(u-v) = \alpha' \circ (u-v)$, etc. Since $\alpha(u-v) \in W^{1,p}_{\mathrm{loc}}(M)$ then, by assumption, for all functions $0 \leq \varphi \in C^\infty_c(M)$ we have

$$\begin{aligned} 0 &\geq \int \left\langle \nabla(\varphi\alpha(u-v)), |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right\rangle \\ &= \int \left\langle \nabla\varphi, \alpha(u-v) \left\{ |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right\} \right\rangle \\ &+ \int \varphi\alpha'(u-v) \left\langle \nabla u - \nabla v, |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right\rangle \\ &\geq -(\operatorname{div} X, \varphi) + 2C(p) \int \varphi\alpha'(u-v) \Psi(x,y) \end{aligned}$$

where in the last inequality we have used Corollary 5 and the fact that $\alpha' \geq 0$. Then

$$\operatorname{div} X \geq 2C(p)\alpha'(u-v) \Psi(x,y) \geq 0$$

in the sense of distributions and Proposition 9 yields

$$\alpha'(u-v) |\nabla u - \nabla v| = 0.$$

Since $\alpha'(u-v) \neq 0$ on Ω_A, we deduce

$$u - v \equiv A, \text{ on } \Omega_A.$$

It follows that the open set Ω_A is also closed. Since M is connected we must conclude that $\Omega_A = M$ and $u - v = A$ on M. □

Remark 11. In the above proof, inequality (6) is not used in its full strength. What we really need is that

$$\left\langle |\nabla u|^{p-2}\,\nabla u - |\nabla v|^{p-2}\,\nabla v, \nabla u - \nabla v\right\rangle > 0$$

whenever $\nabla u \neq \nabla v$. According to this observation, the same proof works with minor changes for more general operators such as the $\mathcal{A}$-Laplacian of [3] or the φ-Laplacian of [16]. In this latter case, $\varphi(t)$ is required to be increasing.

Proof (of Theorem 3). We suppose that either u or v is non-constant, for otherwise there's nothing to prove. Fix $q_0 \in M$. Set $C := u(q_0) - v(q_0) \in \mathbb{R}^n$ and introduce the radial function $r : \mathbb{R}^n \to \mathbb{R}$ defined as $r(x) = |x - C|$. For $T > 0$, consider the piecewise differentiable vector field X_T on M defined as

$$X_T(x) := \Big[dh_T|_{(u-v)(x)} \circ \big(|du(x)|^{p-2}du(x) - |dv(x)|^{p-2}dv(x)\big)\Big]^\sharp, \quad x \in M,$$

where $h_T \in C^1(\mathbb{R}^n, \mathbb{R})$ is the function

$$h_T(x) := \begin{cases} \frac{r^2(x)}{2} & \text{if } r(x) < T \\ Tr(x) - \frac{T^2}{2} & \text{if } r(x) \geq T \end{cases}$$

and $\sharp$ denotes the isomorphism defined by using the Riemannian metric as $\langle \omega^\sharp, V\rangle = \omega(V)$ for all differential 1-forms ω and vector fields V. We observe that $h_T \in C^2$ where $r(x) \neq T$ and that X_T is well defined since there exists a canonical identification

$$T_{(u-v)(q)}\mathbb{R}^n \cong T_{u(q)}\mathbb{R}^n \cong T_{v(q)}\mathbb{R}^n \cong \mathbb{R}^n.$$

We also observe that, by Sard theorem, for a.e. $T > 0$, the level set $\{|u - v - C| = T\}$ is a smooth (possibly empty) hypersurface, hence a set of measure zero. Thus, the vector field X_T is weakly differentiable and, for a.e. $T > 0$, the weak divergence of X_T is given by

$$\begin{aligned} \operatorname{div} X_T &= d(\tfrac{r^2}{2})|_{(u-v)} \circ (\Delta_p u - \Delta_p v) \\ &\quad + {}^M\operatorname{tr}\Big(\operatorname{Hess}(\tfrac{r^2}{2})|_{(u-v)}\big(du - dv, |du|^{p-2}du - |dv|^{p-2}dv\big)\Big) \end{aligned}$$

if $r(x) < T$ and

$$\begin{aligned} \operatorname{div} X_T &= d(Tr)|_{(u-v)} \circ (\Delta_p u - \Delta_p v) \\ &\quad + {}^M\operatorname{tr}\big(\operatorname{Hess}(Tr)|_{(u-v)}\big(du - dv, |du|^{p-2}du - |dv|^{p-2}dv\big)\big) \end{aligned}$$

if $r(x) \geq T$. In both cases the first term on the RHS vanishes by assumption. Moreover, by standard computations, we have $\operatorname{Hess}(r) = r^{-1}(\langle,\rangle_{\mathbb{R}^n} - dr \otimes dr)$ on $\mathbb{R}^n \setminus \{C\}$. Thus,

$$\begin{array}{ll} \operatorname{Hess}(\frac{r^2}{2}) = dr \otimes dr + r\operatorname{Hess}(r) = \langle,\rangle_{\mathbb{R}^n} & \text{if } r(x) < T, \\ \operatorname{Hess}(Tr) = T\operatorname{Hess}(r) = \frac{T}{r}(\langle,\rangle_{\mathbb{R}^n} - dr \otimes dr\rangle) & \text{if } r(x) \geq T. \end{array}$$

As a consequence, for $q \in M$ such that $r((u - v)(q)) < T$, by Corollary 5 we get

$$\operatorname{div} X_T = \big\langle du - dv, |du|^{p-2}du - |dv|^{p-2}dv\big\rangle \geq 2C(p)|du - dv|^p, \tag{8}$$

while, for $q \in M$ such that $r((u-v)(q)) \geq T$, it holds

(9)
$$\begin{aligned}\operatorname{div} X_T = & \frac{T}{r(u-v)} \left\langle du - dv, |du|^{p-2}du - |dv|^{p-2}dv \right\rangle \\ & - \frac{T}{r(u-v)} \left\langle dr|_{(u-v)}(du-dv), dr|_{(u-v)}(|du|^{p-2}du - |dv|^{p-2}dv) \right\rangle \\ \geq & \frac{T}{r(u-v)} 2C(p)|du-dv|^p - (|du|+|dv|)(|du|^{p-1}+|dv|^{p-1}) \\ \geq & \frac{T}{r(u-v)} 2C(p)|du-dv|^p - (|du|^p+|dv|^p+|du|^{p-1}|dv|+|dv|^{p-1}|du|) \\ \geq & \frac{T}{r(u-v)} 2C(p)|du-dv|^p - 2(|du|^p+|dv|^p),\end{aligned}$$

where we have used again Corollary 5 for the first term and Cauchy-Schwarz inequality, Young's inequality and the facts that $|dr| = 1$ and $r(u-v) \geq T$ for the second one. Let us now compute the $L^{\frac{p}{p-1}}$-norm of X_T. Since

$$\left||du|^{p-2}du - |dv|^{p-2}dv\right|^{\frac{p}{p-1}} \leq \left(|du|^{p-1}+|dv|^{p-1}\right)^{\frac{p}{p-1}} \leq 2^{\frac{1}{p-1}}\left(|du|^p+|dv|^p\right),$$

we have

$$\begin{aligned}\int_{\{|u-v-C|<T\}} |X_T|^{\frac{p}{p-1}} \leq & \int_{\{|u-v-C|)<T\}} |u-v-C|^{\frac{p}{p-1}} \left||du|^{p-2}du - |dv|^{p-2}dv\right|^{\frac{p}{p-1}} \\ \leq & \, T^{\frac{p}{p-1}} 2^{\frac{1}{p-1}} \left(\|du\|_p^p + \|dv\|_p^p\right) < +\infty\end{aligned}$$

and

$$\begin{aligned}\int_{\{|u-v-C|)>T\}} |X_T|^{\frac{p}{p-1}} \leq & \int_{\{|u-v-C|>T\}} T^{\frac{p}{p-1}} \left||du|^{p-2}du - |dv|^{p-2}dv\right|^{\frac{p}{p-1}} \\ \leq & \, T^{\frac{p}{p-1}} 2^{\frac{1}{p-1}} \left(\|du\|_p^p + \|dv\|_p^p\right) < +\infty.\end{aligned}$$

Hence X_T is a weakly differentiable vector field with $|X_T| \in L^{\frac{p}{p-1}}(M)$ and $\operatorname{div} X_T \in L^1_{\mathrm{loc}}(M)$. To apply Theorem 8, it remains to show that $(\operatorname{div} X_T)_- \in L^1(M)$. By inequalities (8) and (9), we deduce that

$$\int_M \left|(\operatorname{div} X_T)_-\right| \leq 2 \int_{\{|u-v-C|>T\}} (|du|^p + |dv|^p) \leq 2(\|du\|_p^p + \|dv\|_p^p) < +\infty. \tag{10}$$

Then, the assumptions of Theorem 8 are satisfied and we get, for a.e. $T > 0$,

$$\int_M \operatorname{div} X_T \leq 0.$$

According to (10) we now choose a sequence $T_n \nearrow +\infty$ such that

$$\int_M \left|(\operatorname{div} X_{T_n})_-\right| \leq 2 \int_{\{|u-v-C|>T_n\}} (|du|^p + |dv|^p) < \frac{1}{n}.$$

As a consequence,

$$\begin{aligned}(11)\qquad \int_{\{|u-v-C|<T_n\}} 2C(p)|du-dv|^p &\leq \int_{\{|u-v-C|<T_n\}} (\operatorname{div} X_{T_n})_+ \\ &\leq \int_M (\operatorname{div} X_{T_n})_+ \\ &\leq -\int_M (\operatorname{div} X_{T_n})_- < \frac{1}{n}.\end{aligned}$$

Therefore, letting n go to $+\infty$, we obtain

$$\int_M C(p)|d(u-v)|^p = 0,$$

that is, $u - v \equiv u(q_0) - v(q_0) = C$ on M. □

4. Further comparison results without parabolicity

In this last section we give a proof of Theorem 2. Note that the techniques developed in [12] can be used to conclude further (e.g. L^∞) comparison results. We shall need the following lemma

Lemma 12. *Let $p \geq 2$. Then, for every $x, y \in \mathbb{R}^n$, it holds*

$$||x|^{p-2}x - |y|^{p-2}y| \leq (p-1)(|x|+|y|)^{p-2}|x-y|.$$

Proof. Set $E(x) := |x|^{p-2}x$. We start by computing

$$\begin{aligned}|\frac{d}{dt}E(tx+(1-t)y)| &\leq (p-1)|tx+(1-t)y|^{p-2}|x-y| \\ &\leq (p-1)(|x|+|y|)^{p-2}|x-y|,\end{aligned}$$

from which we obtain

$$\begin{aligned}|E(x)-E(y)| &= |\int_0^1 \frac{d}{dt}E(tx+(1-t)y)dt| \\ &\leq \int_0^1 |\frac{d}{dt}E(tx+(1-t)y)|dt \\ &\leq (p-1)(|x|+|y|)^{p-2}|x-y|.\end{aligned}$$

□

Proof (of Theorem 2). First of all, for the ease of notation, we set

$$E(\xi) := |\xi|^{p-2}\xi,\ \xi \in TM.$$

Suppose that $u - v$ is not constant and, by contradiction, assume that there exists a point $x_0 \in M$ such that $u(x_0) > v(x_0)$. Fix a real number $0 < \epsilon < (u(x_0) - v(x_0))/2$ and define Ω_ϵ to be the connected component of the open set $\{x \in M : u(x) - v(x) > \epsilon\}$ which contains x_0. Note that, necessarily, $u - v$ is not constant on Ω_ϵ. Indeed, otherwise, by standard topological arguments we would have $\Omega_\epsilon = M$ and $u - v$ would be constant on all of M. We choose a smooth, non-decreasing function λ such that $\lambda(t) = 0$ for every $t < 2\epsilon$ and $0 < \lambda(t) \leq 1$ for every $t > 2\epsilon$ and we define the vector field

$$X := \lambda(u-v)(u-v)^q \left(E(\nabla u) - E(\nabla v)\right).$$

We write B_R for $B_R(o)$ and $\partial/\partial r$ for the radial vector field centered at o. Applying the divergence theorem, Lemma 12 and Hölder inequality, we get

$$\begin{aligned}
&\int_{B_R\cap\Omega_\epsilon} \operatorname{div} X \\
&= \int_{\partial B_R\cap\Omega_\epsilon} \left\langle X, \frac{\partial}{\partial r}\right\rangle \\
&\leq \int_{\partial B_R\cap\Omega_\epsilon} |E(\nabla u)-E(\nabla v)|\lambda(u-v)(u-v)^q \\
&\leq (p-1)\int_{\partial B_R\cap\Omega_\epsilon} \lambda(u-v)(|\nabla u|+|\nabla v|)^{p-2}|\nabla u-\nabla v|(u-v)^q \\
&\leq (p-1)\left(\int_{\partial B_R\cap\Omega_\epsilon} F(u,v)\right)^{\frac{1}{s}} \\
&\times\left(\int_{\partial B_R\cap\Omega_\epsilon} \lambda(u-v)(|\nabla u|+|\nabla v|)^{\frac{(p-2)s}{s-1}}|\nabla u-\nabla v|^{\left(1-\frac{p}{s}\right)\frac{s}{s-1}}(u-v)^{\frac{sq-q+1}{s-1}}\right)^{\frac{s-1}{s}} \\
&\leq (p-1)\left(\int_{\partial B_R\cap\Omega_\epsilon} F(u,v)\right)^{\frac{1}{s}}\left(\int_{\partial B_R} |u-v|^{q+\frac{1}{s-1}}(|\nabla u|+|\nabla v|)^{p-\frac{s}{s-1}}\right)^{\frac{s-1}{s}},
\end{aligned}$$

where

$$F(u,v)=\lambda(u-v)|\nabla u-\nabla v|^p(u-v)^{q-1}$$

and, we recall, $s>p$. On the other hand, computing the divergence of X we obtain

$$\begin{aligned}
\int_{B_R\cap\Omega_\epsilon} \operatorname{div} X &= \int_{B_R\cap\Omega_\epsilon} \lambda'(u-v)(u-v)^q\left\langle E(\nabla u)-E(\nabla v),\nabla u-\nabla v\right\rangle \\
&+ q\int_{B_R\cap\Omega_\epsilon} (u-v)^{q-1}\lambda(u-v)\left\langle E(\nabla u)-E(\nabla v),\nabla u-\nabla v\right\rangle \\
&+ \int_{B_R\cap\Omega_\epsilon} (\Delta_p u-\Delta_p v)\,\lambda(u-v)(u-v)^q \\
&\geq 2qC(p)\int_{B_R\cap\Omega_\epsilon} F(u,v),
\end{aligned}$$

where, in the last inequality, we have used Corollary 5. It follows that

$$H(R)^s \leq C'\xi(R)H'(R), \tag{12}$$

where we have defined

$$\begin{aligned}
H(R) &:= \int_{B_R\cap\Omega_\epsilon} F(u,v)\geq 0; \\
\xi(R) &:= \left(\int_{\partial B_R} |u-v|^{q+\frac{1}{s-1}}(|\nabla u|+|\nabla v|)^{p-\frac{s}{s-1}}\right)^{s-1} \\
C' &:= (p-1)^s\left[2qC(p)\right]^{-s}.
\end{aligned}$$

Choose $r_1 >> 1$ such that $F(u,v)$ does not vanish identically on $B_{r_1}\cap\Omega_\epsilon$. According to (12) we have $\xi(R), H(R)>0$, for every $R\geq r_1$. Therefore, we can integrate (12) on $[r_1, r_2]$ to obtain

$$\left(\frac{C'}{s-1}\right)\frac{1}{H(r_1)^{s-1}} \geq \left(\frac{C'}{s-1}\right)\left(-H(r_2)^{1-s}+H(r_1)^{1-s}\right) \geq \int_{r_1}^{r_2}\frac{dt}{\xi(t)}. \tag{13}$$

Letting $r_2 \to \infty$, the RHS of (13) goes to infinity by assumption, and this force $H(r_1) = 0$ for all r_1. Hence

$$\nabla(u-v) \equiv 0 \text{ on } \Omega_\epsilon$$

proving that $u - v$ is constant on Ω_ϵ. Contradiction. □

Remark 13. Applying Hölder and reverse Hölder inequalities, we can see that condition (4) in Theorem 2 is implied by the stronger assumption

$$\left(\int^R \left\| |u-v|^{q+\frac{1}{s-1}} \right\|_{t,\partial B_r}^{-\frac{s-1}{z}} dr\right)^z \left(\int^R \left\|(|\nabla u|+|\nabla v|)^{p-\frac{s}{s-1}}\right\|_{\frac{t}{t-1},\partial B_r}^{\frac{s-1}{z-1}} dr\right)^{1-z} \nearrow \infty,$$

as $R \to \infty$, for some $t \in [1, +\infty]$ and $z \in (-\infty, 0) \cup (1, +\infty)$. Here $\|f\|_{t,\Omega}$ denotes the L^t norm of f on Ω. In particular we obtain that Theorem 2 holds if we replace (4) with either of the following set of assumptions:

4.i) $|\nabla u|, |\nabla v| \in L^\infty(M)$ and $\left[\int_{\partial B_r} |u-v|^{q+\frac{1}{s-1}}\right]^{1-s} \notin L^1(+\infty)$ for some $q \geq 1$ and $s > p$;

4.ii) $|u - v| \in L^\infty(M)$ and $\left[\int_{\partial B_r} (|\nabla u| + |\nabla v|)^{p-\frac{s}{s-1}}\right]^{1-s} \notin L^1(+\infty)$ for some $s > p$;

4.iii) $|\nabla u|, |\nabla v| \in L^{(p-\frac{s}{s-1})t}(M)$, for some $s > p$ and $t > 1$, and

$$\left[\int_{\partial B_r} |u-v|^{\left(q+\frac{1}{s-1}\right)\frac{t}{t-1}}\right]^{\frac{(1-s)(t-1)}{s+t-1}} \notin L^1(+\infty)$$

for some $q \geq 1$;

4.iv) $|u - v| \in L^{(q+\frac{1}{s-1})t}(M)$, for some $s > p$, $q \geq 1$ and $t > 1$, and

$$\left[\int_{\partial B_r} (|\nabla u|+|\nabla v|)^{\left(p-\frac{s}{s-1}\right)\frac{t}{t-1}}\right]^{\frac{(1-s)(t-1)}{s+t-1}} \notin L^1(+\infty).$$

References

[1] P. Collin, R. Krust. Le problème de Dirichlet pour l'equation des surfaces minimales sur des domaines non bornés. Bull. Soc. Math. France, **119** (1991), 443-458.

[2] V. Gol'dshtein, M. Troyanov, *The Kelvin-Nevanlinna-Royden criterion for p-parabolicity.* Math Z. **232** (1999), 607–619.

[3] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear potential theory of degenerate elliptic equations.* Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1993.

[4] I. Holopainen, *Nonlinear potential theory and quasiregular mappings on Riemannian manifolds.* Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes, **74** (1990), 45 pp.

[5] I. Holopainen, *Volume growth, Green's function, and parabolicity of ends.* Duke Math. J. **97** (1999), 319–346.

[6] I. Holopainen, *Quasiregular mappings and the p-Laplace operator.* Contemp. Math. **338** Amer. Math. Soc., Providence, RI, 2003.
[7] I. Holopainen, P. Koskela, *Volume growth and parabolicity.* Proc. Amer. Math. Soc. **129** (2001), 3425–3435.
[8] J.-F. Hwang, *Comparison principles and Liouville theorems for prescribed mean curvature equations in unbounded domains.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **15** (1988), 341–355.
[9] P. Lindqvist, *On the equation* $\mathrm{div}(|\nabla u|^{p-2}\nabla u)+\lambda |u|^{p-2} u = 0$. Proc. Amer Math. Soc. **109** (1996), 157–164.
[10] T. Lyons, D. Sullivan, *Function theory, random paths and covering spaces.* J. Diff. Geom. **18** (1984), 229–323.
[11] V.M. Mikljukov. A new approach to Bernstein theorem and to related questions for equations of minimal surface type (in Russian). Mat. Sb. **108** (1979), 268-289; Engl. transl. in Math. USSR Sb., **36** (1980), 251-271.
[12] S. Pigola, M. Rigoli, A.G. Setti, *Some remarks on the prescribed mean curvature equation on complete manifolds.* Pacific J. Math. **206** (2002), no. 1, 195–217.
[13] S. Pigola, M. Rigoli, A.G. Setti, *Constancy of p-harmonic maps of finite q-energy into non-positively curved manifolds.* Math. Z. **258** (2008), 347–362.
[14] S. Pigola, M. Rigoli, A.G. Setti, *Vanishing and finiteness results in geometric analysis: a generalization of the Bochner technique.* Progress in Mathematics **266** (2008), Birkhäuser.
[15] S. Pigola, G. Veronelli, *On the homotopy class of maps with finite p-energy into non-positively curved manifolds.* To appear in Geometriae Dedicata.
[16] M. Rigoli, A.G. Setti, *Liouville type theorems for φ-subharmonic functions.* Rev. Mat. Iberoamer. **17** (2001), 471–450.
[17] R. Schoen, S.T. Yau, *Compact group actions and the topology of manifolds with nonpositive curvature.* Topology **18** (1979), 361–380.
[18] M. Troyanov, *Parabolicity of manifolds.* Siberian Adv. Math. **9** (1999), no. 4, 125–150.
[19] S.W. Wei, *Representing homotopy groups and spaces of maps by p-harmonic maps.* Indiana Math. J. **47** (1998), 625–669.

Department of Mathematics and Statistics, P.O. Box 68, FIN-00014 University of Helsinki, Finland.
E-mail address: ilkka.holopainen@helsinki.fi

Dipartimento di Fisica e Matematica, Università dell'Insubria - Como, via Valleggio 11, I-22100 Como, Italy.
E-mail address: stefano.pigola@uninsubria.it

Dipartimento di Matematica, Università degli Studi di Milano, via Saldini 50, I-20133 Milano, Italy.
E-mail address: giona.veronelli@unimi.it