Economic factors and solvency

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Abstract

We study solvency of insurers in a practical model where in addition to basic insurance claims and premiums, economic factors like inflation, real growth and returns on the investments affect the capital developments of the companies. The objective is to give qualitative descriptions of risks by means of crude estimates for finite time ruin probabilities. In our setup, the economic factors have a dominant role in the estimates. In addition to this theoretical view, we will focus on applied interpretations of the results by means of discussions and examples.

Key words: Ruin probability, Inflation, Real growth, Investment, Cycle, Large deviation.

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1 Introduction

Solvency of insurance companies is one of the main concerns in actuarial practice and theory. In order to continue the business, the companies have to show a reasonable capacity to survive, that is, to meet their obligations. An appropriate requirement is that the survival probability within a given time horizon must be above a predescribed high level. For regulatory purposes, the time horizon is typically small, for example, one or two years. From the viewpoint of the company management, longer time horizons are also of interest.

To get quantitative estimates for the solvency position of the company, it is necessary to build up mathematical models for claims, premiums, returns on the investments etc. Practical models use to be complicated and therefore, simulation is a popular tool in the estimation of the survival probabilities. The purpose of the present paper is to take a more theoretical look at the problem. Our results should be understood as qualitative descriptions of risks associated with the company but not, for example, as competitors for simulation in the implementation of a solvency test. We will study a comprehensive model which is largely based on Pentikäinen and Rantala (1982). We also refer the reader to Pentikäinen et al. (1989) and Daykin et al. (1994) for further developments in modelling and for other practical aspects of actuarial problems. For empirical observations concerning causes of solvency problems, we refer to the report of The Conference of Insurance Supervisory Services of the Member States of the European Union (2002).

To describe our interest in detail, let $u > 0$ be the initial capital of the company, and let U_n be the capital at the end of the year n for $n = 1, 2, \ldots$. Instead of survival probabilities, it is equivalent to study ruin probabilities. Define the time of ruin T by

$$T = \begin{cases} \inf\{n \in \mathbb{N}\,;\, U_n < 0\} \\ \infty \text{ if } U_n \geq 0 \text{ for every } n. \end{cases} \tag{1.1}$$

We take the most common approach seen in theoretical studies by considering limits of ruin probabilities as u tends to infinity. The limiting procedure directs the focus to small probabilities which is motivated in solvency considerations. For appropriate fixed $x > 0$, we will show that in our model, the approximation

$$\mathbb{P}(T \leq x \log u) \approx u^{-R(x)} \tag{1.2}$$

can be justified with a specific parameter $R(x)$. The precise meaning of (1.2) is stated in Theorem 2.1 below. The time horizon in the estimate increases slowly with u and hence, our study may be viewed to focus on solvency questions within moderate time horizons.

Approximation (1.2) is theoretical in the sense that it is crude and asymptotic. However, the result is also of applied interest. We will assume in the paper that suitable variations can be expected in the numbers of claims and that the claim sizes are not very heavy tailed. In this setup, inflation, real growth of the business and the returns on the investments will completely determine $R(x)$. All these factors are connected with the general economy. The conclusion is that in our model, the economic factors determine the magnitude of the ruin probability while the affect of the basic insurance risks is less critical. To clarify this further, suppose that the company had a need to make its solvency position safer. This should be possible by cutting large losses in the investment side by means of appropriate options. The returns on the investments contribute the parameters $R(x)$, and we can expect that they would increase in the above change. Consequently, the magnitudes of ruin probabilities had a tendency to decrease. An alternative would be to cut large insurance claims by means of an excess of loss reinsurance contract. This should also decrease ruin probabilities, but the parameters $R(x)$ would remain unchanged. Hence, from the solvency point of view, the result indicates that a change in the investment strategy is more effective than a change in the reinsurance strategy.

Another application of approximation (1.2) is that it provides a quality control for nonasymptotic bounds for ruin probabilities. To explain this, suppose that it would be possible to show that

$$\mathbb{P}(T \leq x \log u) \leq \phi(x, u) \tag{1.3}$$

for every finite u where ϕ is a known function. These types of bounds are obviously of interest from the applied point of view, for example, in connection with solvency tests. To have a good upper bound for large initial capital, $\phi(x, u)$ should behave asymptotically similarly to $\mathbb{P}(T \leq x \log u)$, that is, we should have

$$\phi(x, u) \approx u^{-R(x)}. \tag{1.4}$$

If this is not the case then one can argue that the upper bound does not focus carefully on essential parts of the model, and consequently, relative errors are easily huge for large u. In this sense, (1.4) may be seen as a mimimal quality requirement for the upper bound.

In recent years, there has been a lot of interest in ruin probabilities for processes which include stochastic submodels for inflation and for the returns on the investments. It is generally understood that these factors have a crucial impact to ruin probabilities. An early observation in this direction is given by Schnieper (1983). Later on, Paulsen (1993) provides a general framework for many subsequent papers on the problem. We also refer the reader to Frolova et al. (2002) and Kalashnikov and Norberg (2002) which focus on the risks associated with the economic factors. A few of the papers in the area deals with finite time ruin probabilities. Approximation (1.2) as such has been studied in Nyrhinen (2001). The present paper is an extension since here we allow real growth and economic cycles in the model, and discuss other applied aspects related to the problem. Tang and Tsitsiashvili (2003) and (2004) consider ruin probabilities within a fixed time horizon. The papers focus on heavy tailed claim sizes. This leads to estimates where also insurance claims participate the parameters $R(x)$. The same is true in Nyrhinen (2007) even if the claim sizes in the paper are typically light-tailed. This may sound contradictory to the above discussion, but is explained by differences in the limiting procedures.

The paper is organized as follows. Main results and discussions are given in Section 2. Section 3 consists of the proofs.

2 Statement of results

We begin by describing the main variables and parameters of the model in our interest. Some variants and extensions will be discussed in Section 2.2 below. For the motivation and the background, we refer the reader to Pentikäinen and Rantala (1982).

Numbers of claims Associated with the year n, write

N_n = the accumulated number of claims occured in the years $1, \ldots, n$,
λ = the basic level of the mean of the number of claims in the year,
g_n = the rate of real growth,
q_n = the structure variable describing short term fluctuations in the numbers of claims
b_n = the variable describing cycles and other long term fluctuations in the numbers of claims.

Write further $N_0 = 0$ so that $N_n - N_{n-1}$ represents the number of claims occured in the year n. We assume that they have mixed Poisson distributions such that conditionally, given $b_1, \ldots, b_n$, $g_1, \ldots, g_n$ and $q_1, \ldots, q_n$, the variables $N_1 - N_0, \ldots, N_n - N_{n-1}$ are independent and $N_k - N_{k-1}$ has the Poisson distribution with the parameter

$$\lambda b_k (1 + g_1) \cdots (1 + g_k) q_k \tag{2.1}$$

for $k = 1, \ldots, n$. The exact formulation of the model is given in (2.8) below.

Total claim amounts Let

X_n = the total claim amount in the year n,
Z_j = the size of the jth claim in the inflation-free economy,
m_Z = the mean of the claim size in the inflation-free economy,
i_n = the rate of inflation in the year n.

We consider the model where

$$X_n = (1 + i_1) \cdots (1 + i_n) \sum_{j=N_{n-1}+1}^{N_n} Z_j.$$

Premiums For the year n, write

P_n = the premium income,
s = the safety loading coefficient,
c_n = the variable describing long term fluctuations in the premiums.

We take

$$P_n = (1 + s)\lambda m_Z c_n (1 + g_1) \cdots (1 + g_n)(1 + i_1) \cdots (1 + i_n). \tag{2.2}$$

The transition rule We next describe the development of the capital in time. Let

U_n = the capital at the end of the year n,
r_n = the rate of return on the investments in the year n.

Let $U_0 = u > 0$ be the deterministic initial capital of the company. We define

$$U_n = (1 + r_n)(U_{n-1} + P_n - X_n). \tag{2.3}$$

Technical specifications and assumptions We end the description by specifying the dependence structure and other technical features of the model. All the random variables below are assumed to be defined on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We begin by giving a detailed mathematical description for the total claim amounts in the inflation-free economy. For the year n, denote this quantity by V_n, that is,

$$V_n = \sum_{j=N_{n-1}+1}^{N_n} Z_j. \tag{2.4}$$

The distributions of N-variables depend on the b-, g- and q-variables. We take

$$(g, q), (g_1, q_1), (g_2, q_2), \ldots$$

to be an i.i.d. sequence of random vectors where the first one (g, q) is generic and is introduced for notational simplicity. We also assume that g and q are independent, and that the b-variables are independent of the g- and q-variables. We do not give a specific dependence structure for the sequence $\{b_n\}$. Instead of that, we just assume that

$$\mathbb{P}\left(b_n \in \left[\underline{b}, \bar{b}\,\right]\right) = 1 \quad \text{for every } n \tag{2.5}$$

where $\underline{b}$ and $\bar{b}$ are finite and positive constants. Let F_n^b be the joined distribution function of $(b_1, \ldots, b_n)$, and let F^g and F^q be the distribution functions of $1 + g$ and q, respectively. Let further F_n be the joined distribution function of the random vector

$$\xi_n := (b_1, \ldots, b_n, 1 + g_1, \ldots, 1 + g_n, q_1, \ldots, q_n). \tag{2.6}$$

Thus for every $y_1^b, \ldots, y_n^b, y_1^g, \ldots, y_n^g, y_1^q, \ldots, y_n^q \in \mathbb{R}$,

$$\begin{aligned} & F_n(y_1^b, \ldots, y_n^b, y_1^g, \ldots, y_n^g, y_1^q, \ldots, y_n^q) \\ = \quad & F_n^b(y_1^b, \ldots, y_n^b)\, F^g\left(y_1^g\right) \cdots F^g\left(y_n^g\right)\, F^q\left(y_1^q\right) \cdots F^q\left(y_n^q\right). \end{aligned} \tag{2.7}$$

By these specifications, we assume that for every $h_1, \ldots, h_n \in \mathbb{N} \cup \{0\}$ and for every Borel set $C \subseteq \mathbb{R}^{3n}$,

$$\begin{aligned} & \mathbb{P}\left(N_1 - N_0 = h_1, \ldots, N_n - N_{n-1} = h_n, \xi_n \in C\right) \\ = \quad & \int_C \prod_{k=1}^{n} e^{-\lambda y_k^b y_1^g \cdots y_k^g y_k^q} \frac{(\lambda y_k^b y_1^g \cdots y_k^g y_k^q)^{h_k}}{h_k!}\, dF_n(y_1^b, \ldots, y_n^b, y_1^g, \ldots, y_n^g, y_1^q, \ldots, y_n^q). \end{aligned} \tag{2.8}$$

The claim sizes $Z, Z_1, Z_2, \ldots$ are assumed to be i.i.d. (Z is again a generic variable). We also assume that they are independent of the numbers of claims in all respects. Let F^Z be the distribution function of Z, and let $\left(F^Z\right)^{h*}$ be the hth convolution power of F^Z. In precise terms, we assume that for every $h_1, \ldots, h_n \in \mathbb{N} \cup \{0\}$ and $y_1, \ldots, y_n \in \mathbb{R}$, and for every Borel set $C \subseteq \mathbb{R}^{3n}$,

$$\begin{aligned} & \mathbb{P}\left(V_1 \leq y_1, \ldots, V_n \leq y_n, N_1 - N_0 = h_1, \ldots, N_n - N_{n-1} = h_n, \xi_n \in C\right) \\ = \quad & \mathbb{P}\left(N_1 - N_0 = h_1, \ldots, N_n - N_{n-1} = h_n, \xi_n \in C\right) \prod_{k=1}^{n} \left(F^Z\right)^{h_k*}(y_k). \end{aligned} \tag{2.9}$$

Intuitively, $(V_1, \ldots, V_n)$ is a mixture of n-dimensional random vectors such that each of them has independent compound Poisson variables as components. We refer to Grandell (1997) for more information about mixed Poisson distributions and related topics.

Consider now the other parts of the model. We do not give a specific dependence structure for the fluctuation sequence $\{c_n\}$ associated with the premiums. Instead of that, we assume similarly to (2.5) that

$$\mathbb{P}\left(c_n \in [\underline{c}, \bar{c}]\right) = 1 \quad \text{for every } n \tag{2.10}$$

where $\underline{c}$ and $\bar{c}$ are finite and positive constants. We allow an arbitrary dependence structure between the c- and V-variables. As the model for inflation and the returns on the investments, we take

$$(i, r), (i_1, r_1), (i_2, r_2), \ldots \tag{2.11}$$

to be an i.i.d. sequence of random vectors, and these vectors are assumed to be independent of g- and V-variables.

Concerning the parameters of the model, we take λ, m_Z and s to be positive real numbers. For the supports of Z and q, we assume that

$$\mathbb{P}(Z \geq 0) = 1, \, \mathbb{P}(q > 0) = 1 \text{ and } \mathbb{P}\left(q > (1+s)\bar{c}/\underline{b}\right) > 0, \tag{2.12}$$

and for the supports of the economic factors that

$$\mathbb{P}(i > -1) = 1, \, \mathbb{P}(g > -1) = 1 \text{ and } \mathbb{P}(r > -1) = 1.$$

For the moments of the main variables, we assume that

$$\mathbb{E}\left((1+i)^{\alpha}\right), \mathbb{E}\left((1+g)^{\alpha}\right) \text{ and } \mathbb{E}\left((1+r)^{\alpha}\right)$$

are all finite for every $\alpha \in \mathbb{R}$, and that $\mathbb{E}\left(q^{\alpha}\right)$ and $\mathbb{E}\left(Z^{\alpha}\right)$ are finite for every $\alpha > 0$. Finally, assume that $\mathbb{E}(q) = 1$ and that $\mathbb{E}(\log(1+g)) \geq 0$.

2.1 Estimates for ruin probabilities

Let the model be as described in the first part of Section 2, and let the time of ruin T be as in (1.1). Recall that $U_0 = u$ is the initial capital. Our objective is to give the magnitude of the ruin probability $\mathbb{P}(T \leq x \log u)$ for appropriate values of x and for large u. The impact of the economic factors will come via the variable

$$Y = \frac{(1+i)(1+g)}{1+r}, \tag{2.13}$$

and in fact, Y will be in the key role in our considerations.

Define the generating function $\Lambda : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ by

$$\Lambda(\alpha) = \log \mathbb{E}\left(Y^{\alpha}\right), \tag{2.14}$$

and let

$$\mathfrak{r} = \sup\{\alpha; \Lambda(\alpha) \leq 0\} \in [0, \infty]. \tag{2.15}$$

Let further $\Lambda^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be the convex conjugate of Λ, that is,

$$\Lambda^*(x) = \sup\{\alpha x - \Lambda(\alpha) \, ; \, \alpha \in \mathbb{R}\}.$$

It is well-known that both Λ and Λ^* are convex functions. We refer the reader to Rockafellar (1970) and Dembo and Zeitouni (1998) for the background.

Define the parameters μ and x_0 by

$$\mu = \begin{cases} 1/\Lambda'(\mathfrak{r}) \text{ if } \mathfrak{r} < \infty \text{ and } \Lambda'(\mathfrak{r}) \neq 0 \\ \infty \text{ otherwise,} \end{cases}$$

and

$$x_0 = \begin{cases} \inf\{1/\Lambda'(\alpha); \alpha > \mathfrak{r}\} \text{ if } \mathfrak{r} < \infty \\ \infty \text{ otherwise.} \end{cases}$$

For $x > 0$, write finally

$$R(x) = x\Lambda^*(1/x). \tag{2.16}$$

We next state the main results of the paper.

Lemma 2.1 *For the above parameters, we have $x_0 \leq \mu$, $R(x) = \infty$ for every $x \in (0, x_0)$, and $R(x) < \infty$ for every $x \in (x_0, \mu)$.*

Theorem 2.1 *Let $x \in (0, \mu) \setminus \{x_0\}$ be arbitrary. Then*

$$\lim_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(T \leq x \log u) = -R(x). \tag{2.17}$$

Theorem 2.1 describes the magnitude of the ruin probability. More precisely, if (2.17) holds for $x \in (x_0, \mu)$ then for a given $\varepsilon > 0$,

$$u^{-(R(x)+\varepsilon)} \leq \mathbb{P}(T \leq x \log u) \leq u^{-(R(x)-\varepsilon)}$$

for sufficiently large u. The parameter $R(x)$ only depends on the economic factors.

It is interesting to compare our model with the classical one where economic factors are not present. So let $i \equiv 0, g \equiv 0$ and $r \equiv 0$. Then $Y \equiv 1$, $\mathfrak{r} = \infty$, $\mu = \infty$ and $x_0 = \infty$. Thus limit (2.17) holds for every $x > 0$ with $R(x) = \infty$, and hence, the magnitudes of ruin probabilities are asymptotically smaller than in general in the present paper. As an extension of the classical model, suppose that inflation and real growth are not present, but that the return on the investments is always non-negative. Then $Y \leq 1$. By Theorem 2.1, we still have (2.17) for every $x > 0$ with $R(x) = \infty$.

By the above discussion, the company could have a motivation to adjust its strategy such that $Y \leq 1$ would hold. A problem here seems to be that it is difficult to control inflation. To illustrate this, take $g \equiv 0$, for simplicity. Then the target would be to have

$$\frac{1+i}{1+r} \leq 1. \tag{2.18}$$

In financial terms, this can be viewed as a superhedging against inflation by means of appropriate investments. It is not obvious that instruments can be found for the hedging, especially, because the company faces specific claim inflation instead of general inflation in the economy. We refer to Pentikäinen and Rantala (1982), Volume I, Section 2.5.

2.2 Discussion of conditions

The model we have studied is complicated but there is still applied motivation for generalizations. We briefly discuss in this section our conditions and some possible extensions.

Dependences between the years Dependences between consequtive years in the model are caused, for example, by inflation, real growth and the returns on the investments. However, we assumed in (2.11) and elsewhere that the corresponding rates in different years are independent. It would be natural to allow at least a Markovian dependence. This type of extension has been generally possible in classical models in the case where the state space of the underlying Markov chain is finite. We refer to Asmussen (2000). We believe that a similar extension is possible here, especially, since we only consider crude estimates for ruin probabilities.

Short term fluctuations We assumed for the structure variable q in (2.12) that

$$\mathbb{P}\left(q > (1+s)\bar{c}/\underline{b}\right) > 0. \tag{2.19}$$

Roughly speaking, the condition means that in any circumstances, the yearly profit $P_n - X_n$ is negative with a moderate probability. Without the assumption, positive long-term real growth could make the probabilities very small. Something like (2.19) seems to be necessary to end up to the conclusion of Theorem 2.1. We note, however, that it should be possible to relax the condition by specifying the fluctuation sequences $\{b_n\}$ and $\{c_n\}$ in more detail. Nevertheless, in the presence of real growth, also the short term fluctuation may be viewed as an essential risk factor in the model.

Heavy tailed claim sizes We assumed that $\mathbb{E}(Z^\alpha)$ is finite for every $\alpha > 0$. This excludes heavy tailed distributions as models for the claim sizes. If the assumption is relaxed then limit (2.17) may change but there are still chances to specify it. We refer to Nyrhinen (2005), Example 3.4.

Economic cycles Economic cycles may be included in the model by means of the b- and c-variables as it was described in the first part of Section 2. It is intuitively clear that cycles increase the risk of ruin in a short time horizon, especially, if a bad period is just starting. Still their impact is not seen in the moderate time horizon of Theorem 2.1. We believe, however, that cyclicity associated with the economic factors would affect the limits of the theorem.

Other variants Some further changes in the model could be motivated from the applied point of view. For example, in the definition of the premium in (2.2), it could be natural to replace the last inflation rate i_n by an estimate. Also in the transition rule for the capital in (2.3), alternative models could be used for the investment return on the profit $P_n - X_n$ of the current year. The proofs indicate that small changes in these directions would not affect the limits of Theorem 2.1.

2.3 Examples

We illustrate in this section Theorem 2.1 by means of three examples. It turns out that the crude description of the theorem is sufficient to confirm some intuitively natural viewpoints concerning the risk of ruin associated with the models in question. In each example, the risk will be measured by $R(x)$ for *small* x. It is interesting that for large x, the conclusions may be different. We prefer to focus on short time horizons since they are probably more relevant from the applied point of view.

We will consider some financial instruments in the examples. The reader is referred to Panjer et al. (1998) for the background.

Example 2.1 We will compare two investment strategies in a model where inflation and real growth are not present, that is, $i \equiv 0$ and $g \equiv 0$. Suppose that there are a stock and an associated put option available in the financial market. Let S_n be the value of the stock at the end of the year n, and let κS_n be the strike price of the put option associated with the year $n+1$ where $\kappa > 0$ is a constant. According to Theorem 2.1, assume that

$$\{S_{n+1}/S_n; n = 0, 1, 2, \ldots\}$$

is an i.i.d. sequence of random variables. The value of the option at the end of the year $n+1$ is

$$\max(\kappa S_n - S_{n+1}, 0).$$

We assume that the price of the option at the beginning of the year $n+1$ is $\pi(\kappa)S_n$ where $\pi(\kappa)$ is a constant. The above assumptions hold, for example, in the Black-Scholes model for the financial market.

Suppose first that the company invests all its money to the stock. Let ρ_s be the associated generic rate of return on the investments. Hence, $1 + \rho_s$ has the same distribution as S_{n+1}/S_n. Associated with this investment strategy, denote by Λ_s the function corresponding to (2.14). Thus

$$\Lambda_s(\alpha) = \log \mathbb{E}((1 + \rho_s)^{-\alpha}).$$

Let further $R_s(x)$ be the parameter corresponding to (2.16), that is, $R_s(x) = x\Lambda_s^*(1/x)$.

Consider an alternative investment strategy where the company cuts large losses in the investment side. This can be done by keeping always the numbers of the stocks and the options equal in the portfolio. Define the variable ρ_a according to

$$1 + \rho_a = \frac{\max(1 + \rho_s, \kappa)}{1 + \pi(\kappa)}.$$

Then ρ_a describes the rate of return associated with the strategy. Corresponding to (2.14) and (2.16), write

$$\Lambda_a(\alpha) = \log \mathbb{E}((1 + \rho_a)^{-\alpha}) \quad \text{and} \quad R_a(x) = x\Lambda_a^*(1/x).$$

Let's compare ruin probabilities related to the above two strategies. Under the natural assumption that

$$\mathbb{P}\left(1 + \rho_s < \frac{\kappa}{1 + \pi(\kappa)}\right) > 0,$$

there exists $\alpha_1 \geq 0$ such that $\Lambda_s(\alpha) \geq \Lambda_a(\alpha)$ for every $\alpha \geq \alpha_1$. See for example Bahadur and Zabell (1979), Theorem 2.4, and Rockafellar (1970), Corollary 26.4.1. It is easy to see that then for every $v \geq \Lambda_s'(\alpha_1)$,

$$\begin{aligned} \Lambda_s^*(v) &= \sup\{\alpha v - \Lambda_s(\alpha); \alpha \geq \alpha_1\} \\ &\leq \sup\{\alpha v - \Lambda_a(\alpha); \alpha \geq \alpha_1\} \leq \Lambda_a^*(v). \end{aligned}$$

Thus $R_s(x) \leq R_a(x)$ for small x, and the inequality is often strict. If this is the case then the ruin probability within the time horizon $[0, x \log u]$ has a tendency to be smaller when the alternative strategy with options is used.

Example 2.2 We will focus in this example on the correlation between inflation and the returns on the investments. Suppose that there are a stock and a risk-free asset available in the financial market. Let i be the generic rate of inflation as earlier. The rates of the returns on the stock are assumed to be i.i.d. Denote by ρ_s the generic rate of return. We assume that the pair $(\log(1+i), \log(1+\rho_s))$ has a two-dimensional normal distribution. Let (m_i, m_s) be the mean and

$$\Sigma = \begin{pmatrix} \sigma_i^2 & \sigma_{is} \\ \sigma_{is} & \sigma_s^2 \end{pmatrix}$$

the covariance matrix of the distribution. Assume further that the rate of return on the risk-free asset is a fixed constant ρ_f. Write in short $m_f = \log(1 + \rho_f)$.

We assume that $m_s > m_f > m_i$. This corresponds to the natural situation where in the mean, the returns on the investments suffice to compensate the affect of inflation, and the return on the stock is larger than the risk-free return. It is also natural to assume a positive correlation between inflation and the return on the stock. Hence, we take $\sigma_{is} > 0$.

Consider first the strategy where the company invests all its money to the stock. Let η be the variance of the variable $\log(1 + i) - \log(1 + \rho_s)$, that is,

$$\eta = \sigma_i^2 - 2\sigma_{is} + \sigma_s^2.$$

We assume that $\eta > 0$ which just excludes superhedging (2.18). Associated with this strategy, let Λ_s and R_s be the functions corresponding to Λ in (2.14) and R in (2.16). Then

$$\Lambda_s(\alpha) = (m_i - m_s)\alpha + \eta\alpha^2/2 \quad \text{and} \quad R_s(x) = \frac{x}{2\eta}\left(\frac{1}{x} - (m_i - m_s)\right)^2.$$

Consider an alternative strategy where the company invests its money to the risk-free asset only. Let R_a be the function corresponding to R in (2.16). Then

$$R_a(x) = \frac{x}{2\sigma_i^2}\left(\frac{1}{x} - (m_i - m_f)\right)^2.$$

Suppose first that $\sigma_s^2 - 2\sigma_{is} > 0$. It is easy to see that then $R_s(x) < R_a(x)$ for small $x > 0$. This indicates that by investing to the risk-free asset, the company ends up to smaller ruin probabilities than by investing to the stock. On the other hand, if $\sigma_s^2 - 2\sigma_{is} < 0$ then $R_a(x) < R_s(x)$ for small $x > 0$. This gives the signal that it is safer to invest to the risky asset in the case where the correlation between inflation and the return on the stock is high.

Example 2.3 We illustrate in this example the impact of real growth to ruin probabilities. Let's start with the model where $g \equiv 0$ so that

$$\Lambda(\alpha) = \log \mathbb{E}\left(\left(\frac{1+i}{1+r}\right)^{\alpha}\right). \tag{2.20}$$

If we add real growth to the model then we have to add $\log \mathbb{E}\left((1+g)^{\alpha}\right)$ to the right-hand side of (2.20). This makes $\Lambda(\alpha)$ larger for $\alpha \geq 0$ since we assumed that $\mathbb{E}(\log(1+g)) \geq 0$. Similarly to the previous examples, we conclude that by adding real growth to the model, ruin probabilities have a tendency to increase.

3 Proofs

We begin by recalling some basic facts from the theory of convex functions. They will be used throughout the proofs. The background can be found in Rockafellar (1970).

Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a convex function, that is,

$$f(at + (1-a)u) \leq af(t) + (1-a)f(u)$$

for every $t, u \in \mathbb{R}$ and $a \in (0,1)$. The convex conjugate f^* of f is a function $\mathbb{R} \to \mathbb{R} \cup \{\infty\}$ defined by

$$f^*(x) = \sup\{tx - f(t); t \in \mathbb{R}\}.$$

Also f^* is convex. Assume henceforth that $f(0) = 0$. Then $f^*(x) \geq 0$ for every x. If $x = f'(t_x)$ for some $t_x \in \mathbb{R}$ then $f^*(x) = t_x x - f(t_x)$. In particular, if $f'(0)$ exists then $f^*(f'(0)) = 0$. In this case, f^* attains its global minimum at $f'(0)$, and so f^* is increasing on $(f'(0), \infty)$. Assume further that f is differentiable on some interval $[t_0, \infty)$, and write $z = \lim_{t\to\infty} f'(t)$. Then

$$f^*(x) = \sup\{tx - f(t); t \geq t_0\} \tag{3.1}$$

for $x \geq f'(t_0)$. Further,

$$f^*(x) < \infty \text{ for } x \in (f'(t_0), z) \quad \text{and} \quad f^*(x) = \infty \text{ for } x > z. \tag{3.2}$$

Proof of Lemma 2.1 The result follows immediately from the convexity of Λ and from (3.2). □

Before the proof of Theorem 2.1, we will give an asymptotic result concerning the moments of compound Poisson distributions. The proof of the result will be given at the end of the section. Let $\mathcal{Z}, \mathcal{Z}_1, \mathcal{Z}_2, \ldots$ be an i.i.d. sequence of non-negative random variables, and assume that $\mathbb{P}(\mathcal{Z} > 0) > 0$. Let $\mathcal{N}_\nu$ be a Poisson distributed random variable with the parameter ν. Assume that $\mathcal{N}_\nu$ is independent of the $\mathcal{Z}$-variables so that

$$\mathcal{X}_\nu := \mathcal{Z}_1 + \cdots + \mathcal{Z}_{\mathcal{N}_\nu} \tag{3.3}$$

has a compound Poisson disrtibution.

Lemma 3.1 *Assume that* $\mathbb{E}(\mathcal{Z}) < \infty$. *If* $\mathbb{E}(\mathcal{Z}^\alpha) < \infty$ *for* $\alpha > 0$ *then*

$$\lim_{\nu\to\infty} (\log \nu)^{-1} \log \mathbb{E}(\mathcal{X}_\nu^\alpha) = \alpha. \tag{3.4}$$

We now turn to the proof of Theorem 2.1. It is convenient to consider a discounted version of the process $\{U_n\}$. For $n \in \mathbb{N}$, write

$$A_n = \frac{1 + i_n}{1 + r_n}$$

and

$$B_n = V_n - (1+s)\lambda m_Z c_n (1+g_1)\cdots(1+g_n) \tag{3.5}$$

where V_n is as in (2.4). Let further

$$Y_n = \sum_{k=1}^{n} A_1 \cdots A_{k-1}(1+i_k)B_k.$$

By dividing each U_n by $(1+r_1)\cdots(1+r_n)$, it is seen that the time of ruin T can be expressed as

$$T = \begin{cases} \inf\{n \in \mathbb{N}\,;\, Y_n > u\} \\ \infty \text{ if } Y_n \le u \text{ for every } n. \end{cases} \tag{3.6}$$

Define the generating functions Λ_i, Λ_A and Λ_g by

$$\Lambda_i(\alpha) = \log \mathbb{E}\left((1+i)^\alpha\right), \tag{3.7}$$

$$\Lambda_A(\alpha) = \log \mathbb{E}\left(\left(\frac{1+i}{1+r}\right)^\alpha\right) \tag{3.8}$$

and

$$\Lambda_g(\alpha) = \log \mathbb{E}\left((1+g)^\alpha\right) \tag{3.9}$$

for $\alpha \in \mathbb{R}$. By our assumptions,

$$\Lambda = \Lambda_A + \Lambda_g. \tag{3.10}$$

Proof of Theorem 2.1 We begin by showing that for every $x \in (0, \mu)$,

$$\limsup_{u\to\infty} (\log u)^{-1} \log \mathbb{P}(T \le x \log u) \le -R(x). \tag{3.11}$$

Let V_n be as in (2.4), and let

$$\bar{Y}_n = 1 + \sum_{k=1}^{n} A_1 \cdots A_{k-1}(1+i_k)V_k. \tag{3.12}$$

Then $\bar{Y}_n \ge 1$, $\bar{Y}_n \ge Y_n$, and $\{\bar{Y}_n\}$ is an increasing process. Hence,

$$\mathbb{P}(T \le x \log u) \le \mathbb{P}\left(\bar{Y}_{\lceil x \log u \rceil} \ge u\right) \tag{3.13}$$

where $\lceil a \rceil$ denotes the smallest integer $\ge a$.

We will apply the Gärtner-Ellis theorem to the sequence $\{\log \bar{Y}_n\}$. We refer to Dembo and Zeitouni (1998) for the background. To apply the theorem, define the function $\Gamma : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\Gamma(\alpha) = \limsup_{n\to\infty} n^{-1} \log \mathbb{E}\left(\bar{Y}_n^\alpha\right).$$

Then Γ is convex. The first step is to show that

$$\Gamma(\alpha) \le \begin{cases} 0 \text{ for } \alpha \le \mathfrak{r} \\ \Lambda(\alpha) \text{ for } \alpha > \mathfrak{r}. \end{cases} \tag{3.14}$$

For $\alpha \le 0$, (3.14) holds since $\bar{Y}_n \ge 1$. Let now $\alpha > 0$, and let $\varepsilon > 0$ and $k \in \mathbb{N}$. By our model assumptions,

$$\mathbb{E}(V_k^\alpha) = \mathbb{E}\left(e^{-\lambda b_k(1+g_1)\cdots(1+g_k)q_k} \sum_{h=0}^{\infty} \frac{(\lambda b_k(1+g_1)\cdots(1+g_k)q_k)^h}{h!} \mathbb{E}((Z_1+\cdots+Z_h)^\alpha)\right).$$

Take $\mathcal{Z} = Z$ in Lemma 3.1 and choose large $M > 0$ such that $\mathbb{E}\left(\mathcal{X}_\nu^\alpha\right) \le \nu^{\alpha+\varepsilon}$ whenever $\nu > M$. Write

$$G_M = \{\lambda b_k(1+g_1)\cdots(1+g_k)q_k > M\}.$$

Then

$$\begin{aligned}\mathbb{E}(V_k^\alpha \mathbf{1}(G_M)) &\leq \mathbb{E}\left((\lambda b_k(1+g_1)\cdots(1+g_k)q_k)^{\alpha+\varepsilon}\right) \\ &\leq (\lambda\bar{b})^{\alpha+\varepsilon}\mathbb{E}\left(q^{\alpha+\varepsilon}\right)e^{k\Lambda_g(\alpha+\varepsilon)}.\end{aligned}$$

Concerning the complement of G_M, we have

$$\mathbb{E}(V_k^\alpha \mathbf{1}(G_M^c)) \leq e^M \mathbb{E}\left(\mathcal{X}_M^\alpha\right).$$

We assumed that $\mathbb{E}(\log(1+g)) \geq 0$ so that $\Lambda_g(\alpha+\varepsilon) \geq 0$. By the above estimates, there exists a constant C_1 such that

$$\mathbb{E}(V_k^\alpha) \leq C_1 e^{k\Lambda_g(\alpha+\varepsilon)}$$

for every $k \in \mathbb{N}$. By (3.10), there exists a constant C_2 such that

$$\mathbb{E}\left((A_1\cdots A_{k-1}(1+i_k)V_k)^\alpha\right) \leq C_2 e^{(k-1)\Lambda(\alpha)} e^{(k-1)(\Lambda_g(\alpha+\varepsilon)-\Lambda_g(\alpha))} \tag{3.15}$$

for every k.

Consider separately the cases where $\alpha > \mathfrak{r}$ and $\alpha \in [0,\mathfrak{r}]$. Let first $\alpha > \mathfrak{r}$. Then $\Lambda(\alpha) > 0$ and $\Lambda_g(\alpha+\varepsilon) - \Lambda_g(\alpha) \geq 0$. For $\alpha > 1$, apply Minkowski's inequality to conclude that there exists a constant C such that for every n,

$$\mathbb{E}(\bar{Y}_n^\alpha) \leq C e^{n\Lambda(\alpha)} e^{n(\Lambda_g(\alpha+\varepsilon)-\Lambda_g(\alpha))}. \tag{3.16}$$

By the continuity of Λ_g, the estimate implies (3.14) for $\alpha > \mathfrak{r}$ in the case where $\alpha \geq 1$. A similar proof applies to the case where $\alpha \in (0,1)$. Instead of Minkowski's inequality, we now make use of the inequality

$$(x+y)^\alpha \leq x^\alpha + y^\alpha \tag{3.17}$$

for $x, y \geq 0$. Let now $\alpha \in [0,\mathfrak{r}]$. Then $\Lambda(\alpha) \leq 0$. We still have (3.15) which now implies that

$$\mathbb{E}\left((A_1\cdots A_{k-1}(1+i_k)V_k)^\alpha\right) \leq C_2 e^{(k-1)(\Lambda_g(\alpha+\varepsilon)-\Lambda_g(\alpha)+\varepsilon)}.$$

It follows as above that

$$\mathbb{E}(\bar{Y}_n^\alpha) \leq C e^{n(\Lambda_g(\alpha+\varepsilon)-\Lambda_g(\alpha)+\varepsilon)}.$$

Hence, $\Gamma(\alpha) \leq \Lambda_g(\alpha+\varepsilon) - \Lambda_g(\alpha) + \varepsilon$ so that (3.14) holds for $\alpha \in [0,\mathfrak{r}]$.

Let $\varepsilon > 0$. By the Gärtner-Ellis theorem,

$$\begin{aligned}&\limsup_{u\to\infty}(\log u)^{-1}\log \mathbb{P}\left(\bar{Y}_{\lceil x\log u\rceil} \geq u\right) \qquad (3.18)\\ \leq\ &\limsup_{u\to\infty}\frac{\lceil x\log u\rceil}{\log u}(\lceil x\log u\rceil)^{-1}\log\mathbb{P}\left(\frac{\log \bar{Y}_{\lceil x\log u\rceil}}{\lceil x\log u\rceil} \geq \frac{1}{x}-\varepsilon\right)\\ \leq\ &-x\inf\left\{\Gamma^*(v); v \geq \frac{1}{x}-\varepsilon\right\}.\end{aligned}$$

Now if $\mathfrak{r} = \infty$ then $\Gamma(\alpha) \leq 0$ for every $\alpha \geq 0$ and hence, $\Gamma^*(v) = \infty$ for every $v > 0$. Thus (3.13) and (3.18) imply (3.11). Assume that $\mathfrak{r} < \infty$. It follows from (3.1) and (3.14) that for $v > \Lambda'(\mathfrak{r})$,

$$\begin{aligned}\Gamma^*(v) &= \sup\{\alpha v - \Gamma(\alpha); \alpha \in \mathbb{R}\}\\ &\geq \sup\{\alpha v - \Lambda(\alpha)\mathbf{1}(\alpha > \mathfrak{r}); \alpha \in \mathbb{R}\} = \Lambda^*(v).\end{aligned}$$

Further, Λ^* is increasing on $(\Lambda'(0),\infty)$, and hence, on $(\Lambda'(\mathfrak{r}),\infty)$. Recall that $x<\mu$. By the above discussion, it is seen that for small ε, (3.18) is at most $-x\Lambda^*(1/x-\varepsilon)$. Finally, $x\neq x_0$ so that $x\Lambda^*(1/x-\varepsilon)$ tends to $R(x)$ as ε tends to zero. Thus (3.13) implies (3.11).

To complete the proof, we have to show that for every $x\in(x_0,\mu)$,

$$\liminf_{u\to\infty}(\log u)^{-1}\log\mathbb{P}(T\leq x\log u)\geq -R(x). \tag{3.19}$$

See Lemma 2.1. In particular, we can assume that $x_0<\mu$. This implies that Λ is strictly convex. Recall the definitions of Λ_i, Λ_A and Λ_g from (3.7), (3.8) and (3.9).

We will construct a subset of $\{T\leq x\log u\}$ which is large enough to lead to (3.19). Consider first the case where g is not identically zero. We assumed that $\mathbb{E}(\log(1+g))\geq 0$ so that $\Lambda_g(\alpha)>0$ for every $\alpha>0$. Define the continuous time processes

$$\{z_n^A(t);0<t<\infty\},\quad \{z_n^{A,i}(t);0<t<\infty\}\quad\text{and}\quad \{z_n^g(t);0<t<\infty\}$$

by

$$\begin{aligned} z_n^A(t) &= (\log A_1+\cdots+\log A_{\lceil tn\rceil})/n,\\ z_n^{A,i}(t) &= (\log A_1+\cdots+\log A_{\lceil tn\rceil-1}+\log(1+i_{\lceil tn\rceil}))/n \end{aligned}$$

and

$$z_n^g(t) = (\log(1+g_1)+\cdots+\log(1+g_{\lceil tn\rceil}))/n.$$

Fix $p>0$ and small $\varepsilon>0$, and let x_1 and x_2 be such that $0<x_1<x_2<x$. Write

$$\begin{aligned} \mathcal{H}_n^A(\varepsilon) &= \left\{\sup_{0<t\leq x}\left|z_n^A(t)-(1-p)t/x_1\right|\leq\varepsilon\right\},\\ \mathcal{H}_n^{A,i}(\varepsilon) &= \left\{\sup_{0<t\leq x}\left|z_n^{A,i}(t)-(1-p)t/x_1\right|\leq\varepsilon\right\} \end{aligned}$$

and

$$\mathcal{H}_n^g(\varepsilon) = \left\{\sup_{0<t\leq x}|z_n^g(t)-pt/x_1|\leq\varepsilon\right\}. \tag{3.20}$$

Write further

$$\mathcal{H}_n^B(\varepsilon)=\{B_k\geq\varepsilon(1+g_1)\cdots(1+g_k),\ \forall k\in[x_1n,xn]\}$$

where B_k is as in (3.5). By Mogulskii's theorem,

$$\liminf_{n\to\infty}n^{-1}\log\mathbb{P}\left(\mathcal{H}_n^A(\varepsilon/4)\right)\geq -x\Lambda_A^*\left(\frac{1-p}{x_1}\right). \tag{3.21}$$

We refer to Dembo and Zeitouni (1998) and Martin-Löf (1983). For $\alpha>0$, we have by Chebycheff's inequality,

$$\mathbb{P}\left(\log(1+i)/n>\varepsilon/4\right)\leq e^{-n\alpha\varepsilon/4+\Lambda_i(\alpha)}.$$

A similar estimate holds for the probability $\mathbb{P}\left(\log(1+i)/n<-\varepsilon/4\right)$ so that

$$\begin{aligned} &\limsup_{n\to\infty}n^{-1}\log\mathbb{P}\left(|\log(1+i_k)/n|>\varepsilon/4 \text{ for some } 1\leq k\leq\lceil xn\rceil\right)\\ \leq\ &\limsup_{n\to\infty}n^{-1}\log\left(\lceil xn\rceil\left(e^{-n\alpha\varepsilon/4+\Lambda_i(\alpha)}+e^{-n\alpha\varepsilon/4+\Lambda_i(-\alpha)}\right)\right)=-\alpha\varepsilon/4. \end{aligned}$$

By the same arguments, it is seen that

$$\limsup_{n\to\infty} n^{-1}\log\mathbb{P}\left(|\log A_k/n| > \varepsilon/4 \text{ for some } 1\le k\le \lceil xn\rceil\right) \le -\alpha\varepsilon/4.$$

Since α is arbitrary we conclude by (3.21) that

$$\liminf_{n\to\infty} n^{-1}\log\mathbb{P}\left(\mathcal{H}_n^{A,i}(\varepsilon)\right) \ge -x\Lambda_A^*\left(\frac{1-p}{x_1}\right). \tag{3.22}$$

Similarly, by making use of Mogulskii's theorem, it is seen that

$$\liminf_{n\to\infty} n^{-1}\log\mathbb{P}\left(\mathcal{H}_n^{g}(\varepsilon)\right) \ge -x\Lambda_g^*\left(\frac{p}{x_1}\right).$$

Let $\mathcal{X}_\nu$ be a compound Poisson variable as in (3.3), and take $\mathcal{Z}=Z$. Let $m_Z=\mathbb{E}(Z)$ as earlier. Fix $\nu_0>0$, and write

$$\gamma=\gamma(\nu_0)=\inf\{\mathbb{P}(\mathcal{X}_\nu\ge \nu m_Z);\nu\ge\nu_0\}. \tag{3.23}$$

Obviously, γ is strictly positive. Denote

$$\mathbb{R}_+^k=\{(y_1,\dots,y_k);y_1>0,\dots,y_k>0\}.$$

Corresponding to $\mathcal{H}_n^g(\varepsilon)$ in (3.20), define the subset of $\mathbb{R}_+^{\lceil xn\rceil}$ by

$$H_n^g(\varepsilon)=\left\{(y_1^g,\dots,y_{\lceil xn\rceil}^g)\in\mathbb{R}_+^{\lceil xn\rceil};\ \sup_{0<t\le x}|(\log y_1^g+\cdots+\log y_{\lceil tn\rceil}^g)/n-pt/x_1|\le\varepsilon\right\}.$$

Choose $a>(1+s)\bar{c}/\underline{b}$ such that $\mathbb{P}(q>a)>0$. This is possible by assumption (2.12). Recall the definition of the distribution function F_n from (2.6) and (2.7). By (2.8) and (2.9),

$$\begin{aligned}\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)\right) &\ge \mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)\cap\{q_k>a \text{ for every } \lceil x_1n\rceil\le k\le\lceil xn\rceil\}\right)\\ &\ge \int_{C_n}\prod_{k=\lceil x_1n\rceil}^{\lceil xn\rceil}\mathbb{P}\left(\mathcal{X}_{\lambda y_k^b y_1^g\cdots y_k^g y_k^q}\ge((1+s)\lambda m_Z\bar{c}+\varepsilon)y_1^g\cdots y_k^g\right)\\ &\quad dF_{\lceil xn\rceil}(y_1^b,\dots,y_{\lceil xn\rceil}^b,y_1^g,\dots,y_{\lceil xn\rceil}^g,y_1^q,\dots,y_{\lceil xn\rceil}^q)\end{aligned}$$

where

$$\begin{aligned}C_n &= \{(y_1^b,\dots,y_{\lceil xn\rceil}^b,y_1^g,\dots,y_{\lceil xn\rceil}^g,y_1^q,\dots,y_{\lceil xn\rceil}^q)\in\mathbb{R}_+^{3\lceil xn\rceil};\\ &\quad (y_1^g,\dots,y_{\lceil xn\rceil}^g)\in H_n^g(\varepsilon),\ y_k^q>a,\forall k\in[\lceil x_1n\rceil,\lceil xn\rceil]\}.\end{aligned}$$

Recall that $b_k\ge\underline{b}>0$ for every k. Take $\nu_0=1$ in (3.23) to see that for small ε and large n,

$$\begin{aligned}\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)\right) &\ge \int_{C_n}\prod_{k=\lceil x_1n\rceil}^{\lceil xn\rceil}\mathbb{P}\left(\mathcal{X}_{\lambda y_k^b y_1^g\cdots y_k^g y_k^q}\ge\lambda y_k^b y_1^g\cdots y_k^g y_k^q m_Z\right)\\ &\quad dF_{\lceil xn\rceil}(y_1^b,\dots,y_{\lceil xn\rceil}^b,y_1^g,\dots,y_{\lceil xn\rceil}^g,y_1^q,\dots,y_{\lceil xn\rceil}^q)\\ &\ge \gamma^{\lceil xn\rceil-\lceil x_1n\rceil+1}\,\mathbb{P}(q>a)^{\lceil xn\rceil-\lceil x_1n\rceil+1}\,\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\right).\end{aligned} \tag{3.24}$$

Consequently,

$$\begin{aligned}
&\liminf_{n\to\infty} n^{-1}\log\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)\right) \\
\geq\ & (x-x_1)\left(\log\gamma+\log\mathbb{P}\left(q>a\right)\right)-x\Lambda_g^*\left(\frac{p}{x_1}\right).
\end{aligned} \tag{3.25}$$

On the event $\mathcal{H}_n^{A,i}(\varepsilon)\cap\mathcal{H}_n^g(\varepsilon)\cap\mathcal{H}_n^B(\varepsilon)$, we have for large n,

$$Y_{\lceil x_1 n\rceil}\geq -D_1e^{n(1+2\varepsilon)}$$

and

$$Y_{\lceil x_2 n\rceil}-Y_{\lceil x_1 n\rceil}\geq D_2e^{n(x_2/x_1-2\varepsilon)}$$

where D_1 and D_2 are positive constants. Choose ε, x_1 and x_2 in an appropriate way to see that $Y_{\lceil x_2 n\rceil}>e^n$ for large n on the event. By (3.22) and (3.25),

$$\begin{aligned}
&\liminf_{u\to\infty}(\log u)^{-1}\log\mathbb{P}\left(T\leq x\log u\right) \\
\geq\ & \liminf_{u\to\infty}(\log u)^{-1}\log\mathbb{P}\left(Y_{\lceil x_2\lceil\log u\rceil\rceil}>u\right) \\
\geq\ & \liminf_{u\to\infty}(\log u)^{-1}\log\mathbb{P}\left(\mathcal{H}^{A,i}_{\lceil\log u\rceil}(\varepsilon)\cap\mathcal{H}^{g}_{\lceil\log u\rceil}(\varepsilon)\cap\mathcal{H}^{B}_{\lceil\log u\rceil}(\varepsilon)\right) \\
\geq\ & -x\left(\Lambda_A^*\left(\frac{1-p}{x_1}\right)+\Lambda_g^*\left(\frac{p}{x_1}\right)\right)+o(1)
\end{aligned} \tag{3.26}$$

where $o(1)$ tends to zero as x_1 tends to x from the left. Choose x_1 close to x such that $x_1\in(x_0,\mu)$. Then $\Lambda'(\alpha_1)=1/x_1$ for some $\alpha_1>\mathfrak{r}$ and hence,

$$\Lambda^*(1/x_1)=\alpha_1/x_1-\Lambda(\alpha_1).$$

We now choose p such that $\Lambda_g'(\alpha_1)=p/x_1$. Then $p>0$ and it is easy to see by (3.10) that

$$\Lambda^*\left(\frac{1}{x_1}\right)=\Lambda_A^*\left(\frac{1-p}{x_1}\right)+\Lambda_g^*\left(\frac{p}{x_1}\right).$$

By (3.26),

$$\liminf_{u\to\infty}(\log u)^{-1}\log\mathbb{P}\left(T\leq x\log u\right)\geq -x\Lambda^*\left(\frac{1}{x_1}\right)+o(1). \tag{3.27}$$

Now Λ^* is continuous at $1/x$ since $x\in(x_0,\mu)$. Thus (3.19) follows from (3.27).

Consider finally the case where $g\equiv 0$. We now choose $p=0$ in the definitions of the sets $\mathcal{H}_n^A(\varepsilon)$ and $\mathcal{H}_n^g(\varepsilon)$. Then $\mathbb{P}\left(\mathcal{H}_n^g(\varepsilon)\right)=1$ for every n. Similarly to the case $p>0$, it is seen by choosing $\nu_0=\lambda\underline{b}a$ in (3.23) that

$$\begin{aligned}
&\liminf_{u\to\infty}(\log u)^{-1}\log\mathbb{P}\left(T\leq x\log u\right) \\
\geq\ & -x\Lambda_A^*\left(\frac{1}{x_1}\right)+o(1)=-x\Lambda^*\left(\frac{1}{x_1}\right)+o(1).
\end{aligned}$$

This implies (3.19). □

Proof of Lemma 3.1 Define the function $L_{\mathcal{N}}:(0,\infty)\to\mathbb{R}$ by

$$L_{\mathcal{N}}(t)=\limsup_{\nu\to\infty}(\log\nu)^{-1}\log\mathbb{E}\left(\mathcal{N}_\nu^t\right). \tag{3.28}$$

We first show that $L_{\mathcal{N}}(t) = t$ and that (3.28) holds as the limit for each t.

We have $\mathbb{E}(\mathcal{N}_\nu) = \nu$, and by convention, let $\mathbb{E}(\mathcal{N}_\nu^0) = 1$. It is easy to see that then

$$\mathbb{E}\left(\mathcal{N}_\nu^k\right) = \sum_{h=0}^{k-1} \binom{k-1}{h} \nu \mathbb{E}\left(\mathcal{N}_\nu^h\right) \tag{3.29}$$

for $k = 2, 3, \ldots$. This shows that

$$\lim_{\nu \to \infty} (\log \nu)^{-1} \log \mathbb{E}\left(\mathcal{N}_\nu^t\right) = t \tag{3.30}$$

for every $t \in \mathbb{N}$. Let $t \in (0,1)$. Apply Jensen's inequality to conclude that

$$\mathbb{E}\left(\mathcal{N}_\nu^t\right) \leq \mathbb{E}\left(\mathcal{N}_\nu\right)^t = \nu^t$$

so that $L_{\mathcal{N}}(t) \leq t$. Now by Hölder's inequality, $L_{\mathcal{N}}$ is convex so that necessarily, $L_{\mathcal{N}}(t) = t$ for every $t > 0$. It remains to show that (3.28) holds as the limit. Assume on the contrary that there would exist a sequence $\nu_j \to \infty$ and $t_0 > 0$ such that

$$\lim_{j \to \infty} (\log \nu_j)^{-1} \log \mathbb{E}\left(\mathcal{N}_{\nu_j}^{t_0}\right) < t_0. \tag{3.31}$$

Write

$$\underline{L}_{\mathcal{N}}(t) = \limsup_{j \to \infty} (\log \nu_j)^{-1} \log \mathbb{E}\left(\mathcal{N}_{\nu_j}^t\right) \tag{3.32}$$

for $t > 0$. By the first part of the proof, $\underline{L}_{\mathcal{N}}(t) = t$ for every $t \in \mathbb{N}$, and $\underline{L}_{\mathcal{N}}(t) \leq t$ for every $t \in (0,1)$. By (3.31), $\underline{L}_{\mathcal{N}}(t_0) < t_0$. This is a contradiction since also $\underline{L}_{\mathcal{N}}$ is convex. It follows that (3.28) holds as the limit for every $t > 0$.

Consider now (3.4). Let first $\alpha \geq 1$. By Minkowski's inequality,

$$\begin{aligned} \mathbb{E}\left(\mathcal{X}_\nu^\alpha\right) &= \sum_{h=1}^{\infty} e^{-\nu} \frac{\nu^h}{h!} \mathbb{E}\left((\mathcal{Z}_1 + \cdots + \mathcal{Z}_h)^\alpha\right) \\ &\leq \sum_{h=1}^{\infty} e^{-\nu} \frac{\nu^h}{h!} h^\alpha \mathbb{E}\left(\mathcal{Z}^\alpha\right) = \mathbb{E}\left(\mathcal{N}_\nu^\alpha\right) \mathbb{E}\left(\mathcal{Z}^\alpha\right). \end{aligned} \tag{3.33}$$

By (3.33) and Jensen's inequality,

$$\begin{aligned} \mathbb{E}\left(\mathcal{X}_\nu^\alpha\right) &\geq \sum_{h=1}^{\infty} e^{-\nu} \frac{\nu^h}{h!} h^\alpha \mathbb{E}\left(\mathcal{Z}\right)^\alpha \\ &= \mathbb{E}\left(\mathcal{N}_\nu^\alpha\right) \mathbb{E}\left(\mathcal{Z}\right)^\alpha. \end{aligned} \tag{3.34}$$

By the above estimates and the first part of the proof, (3.4) holds if $\alpha \geq 1$.

Let now $\alpha \in (0,1)$. By (3.33) and Jensen's inequality,

$$\begin{aligned} \mathbb{E}\left(\mathcal{X}_\nu^\alpha\right) &\leq \sum_{h=1}^{\infty} e^{-\nu} \frac{\nu^h}{h!} h^\alpha \mathbb{E}\left(\mathcal{Z}\right)^\alpha \\ &= \mathbb{E}\left(\mathcal{N}_\nu^\alpha\right) \mathbb{E}\left(\mathcal{Z}\right)^\alpha. \end{aligned}$$

To get an appropriate lower bound, let $M > 0$ be such that $\mathbb{P}(\mathcal{Z} \in (0, M)) > 0$, and let $\underline{\mathcal{Z}}_k = \min(\mathcal{Z}_k, M)$ for $k \in \mathbb{N}$. Write

$$\underline{\mathcal{X}}_\nu = \underline{\mathcal{Z}}_1 + \cdots + \underline{\mathcal{Z}}_{N_\nu}.$$

Then also $\underline{\mathcal{X}}_\nu$ has a compound Poisson disrtibution. Write

$$\underline{L}_{\mathcal{X}}(t) = \limsup_{\nu\to\infty}(\log\nu)^{-1}\log\mathbb{E}\left(\underline{\mathcal{X}}_\nu^t\right) \tag{3.35}$$

for $t > 0$. By the first part of the proof, $\underline{L}_{\mathcal{X}}(t) = t$ for $t \geq 1$, and $\underline{L}_{\mathcal{X}}(t) \leq t$ for $t \in (0,1)$. Also $\underline{L}_{\mathcal{X}}$ is convex so that by making use of arguments similar to the first part of the proof, it is seen that $\underline{L}_{\mathcal{X}}(t) = t$ for every $t > 0$, and further, that (3.35) holds as the limit for every t. The desired lower bound now follows since $\mathcal{X}_\nu \geq \underline{\mathcal{X}}_\nu$. □

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